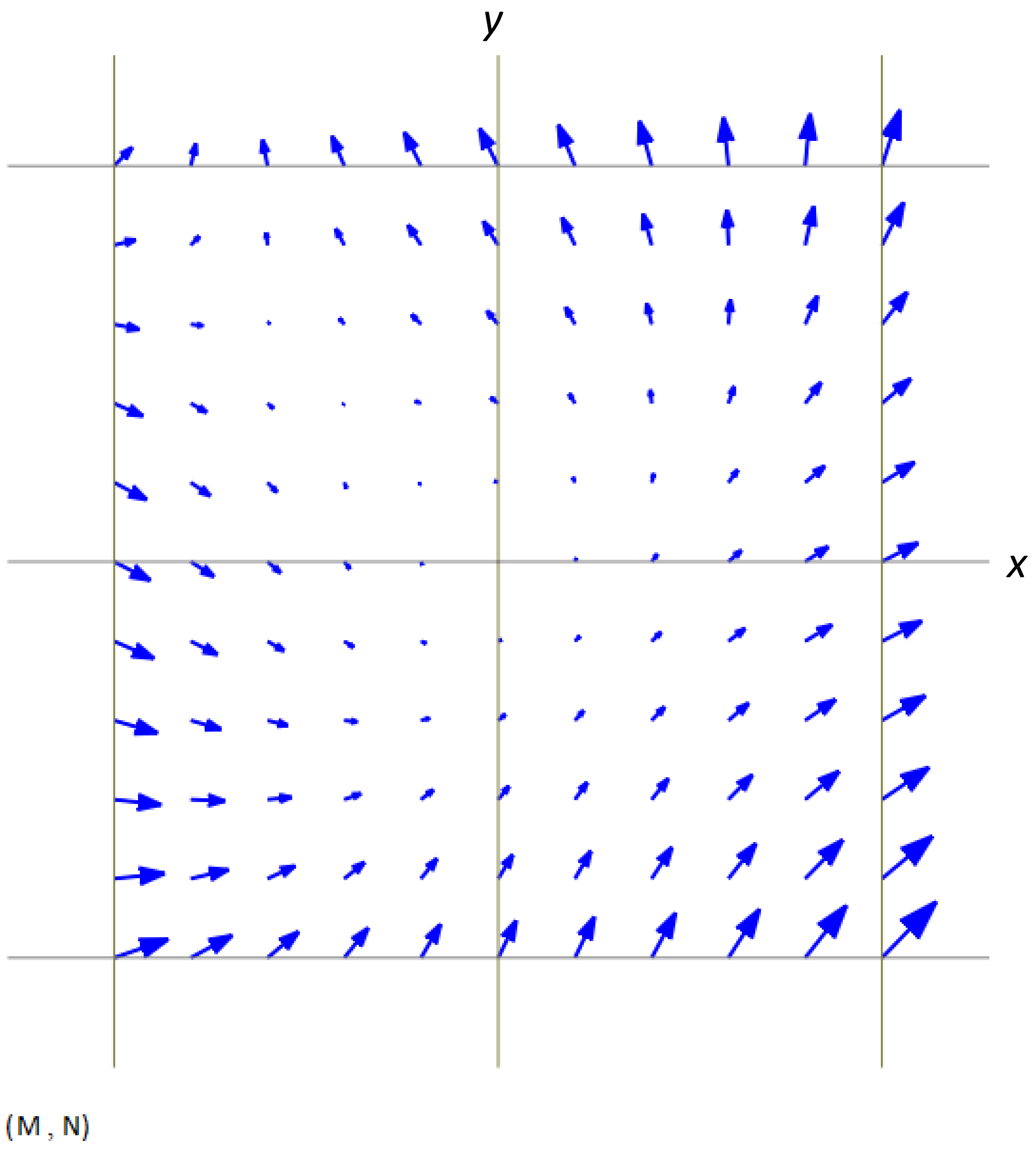
Today in Physics 217

- Vectors and vector operations
- Vector components and coordinate systems
- Dyadics and other second-rank tensors
- Vector derivatives; product rules



Vector operations

As we all know:

- Adding two vectors produces a third vector:

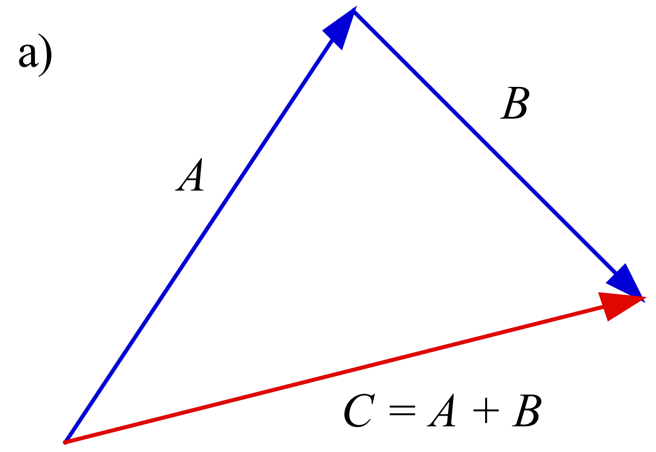
$$\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{C}$$

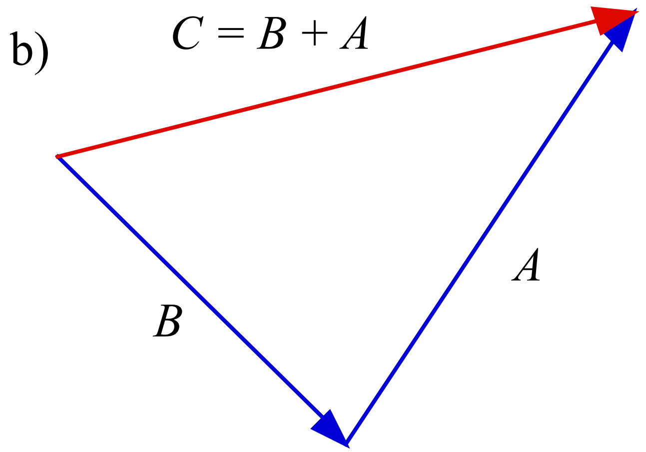
- Vector addition is commutative:

$$\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{C}=\boldsymbol{B}+\boldsymbol{A}$$

- Vector subtraction is equivalent to adding the opposite of a vector:

$$\boldsymbol{A}-\boldsymbol{B}=\boldsymbol{A}+\left(-\boldsymbol{B}\right)$$

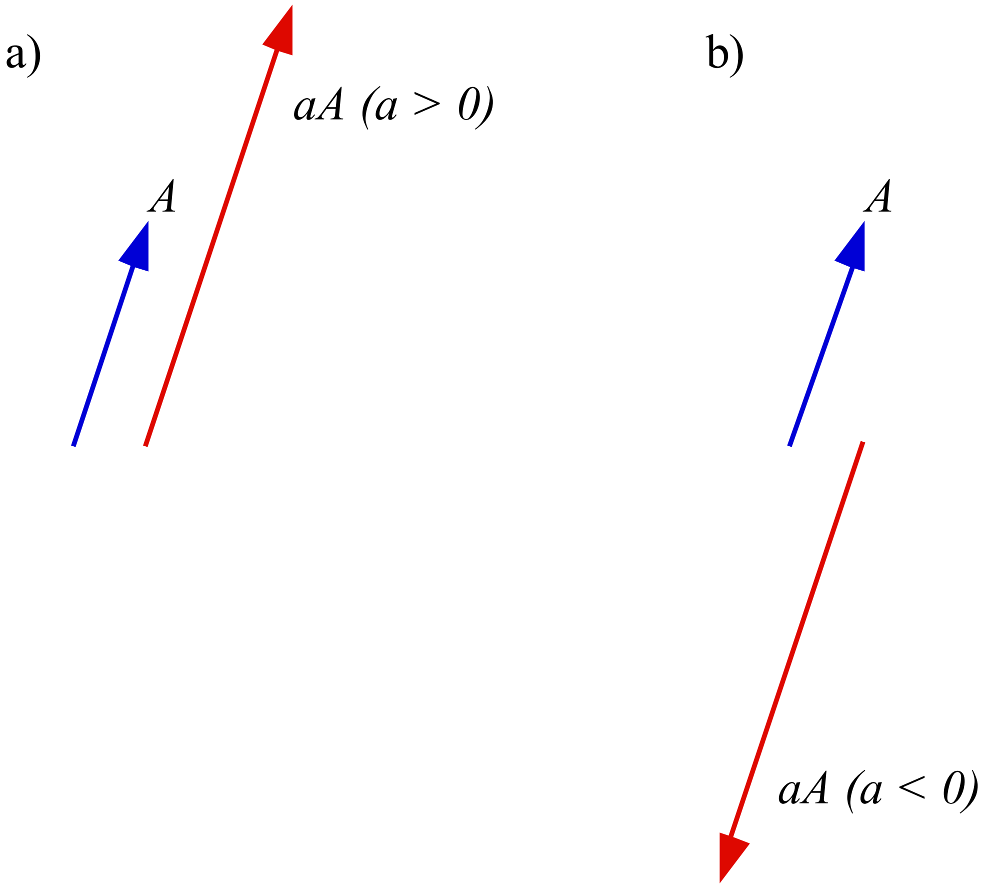




Vector operations (continued)

- The result of vector multiplication by a scalar is a vector.
- The magnitude of the resulting vector is the product of the magnitude of the scalar and the magnitude of the vector.
- The direction of the resulting vector is the same as the direction of the original vector if $a > 0$ and opposite to the direction of the original vector if $a < 0$.
- Scalar multiplication is distributive:

$$a(\boldsymbol{A}+\boldsymbol{B}) = a\boldsymbol{A} + a\boldsymbol{B}$$



Vector operations (continued)

The dot product (a.k.a. scalar, or inner, product):

- The results of the dot product is a scalar:

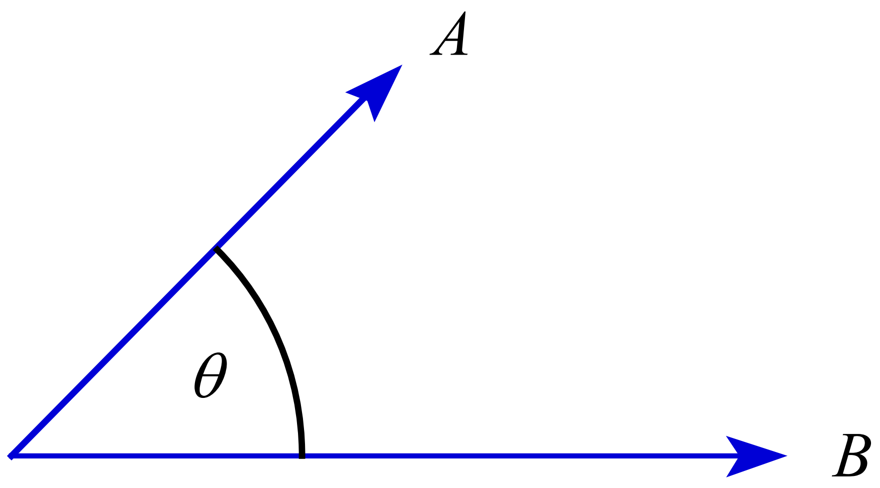
$$\boldsymbol{A}\cdot\boldsymbol{B}=|\boldsymbol{A}||\boldsymbol{B}|\cos\vartheta=AB\cos\vartheta$$

- The dot product is commutative:

$$\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{B}\cdot\boldsymbol{A}$$

- The dot product is distributive:

$$\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}$$

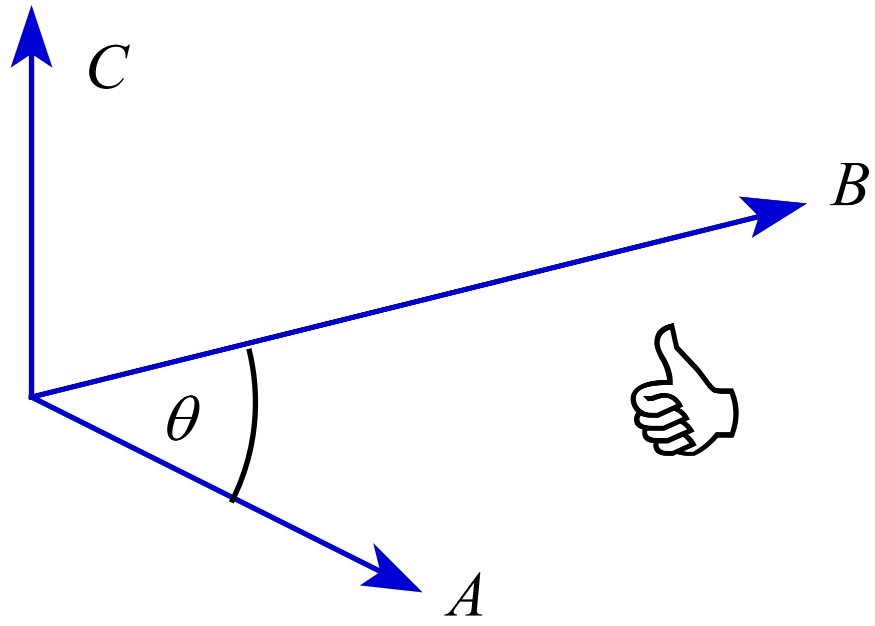


Vector operations (continued)

The cross product (a.k.a. vector product):

- The result of the cross product is a vector-**like** object perpendicular to the two original vectors.
  - Magnitude: $|\boldsymbol{C}| = |\boldsymbol{A} \times \boldsymbol{B}| = AB\sin\vartheta$
  - Direction: use right-hand rule.
  - Vector-like: see below, under **pseudovectors**.
- The cross product is **not** commutative: $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$
- The cross product is distributive:

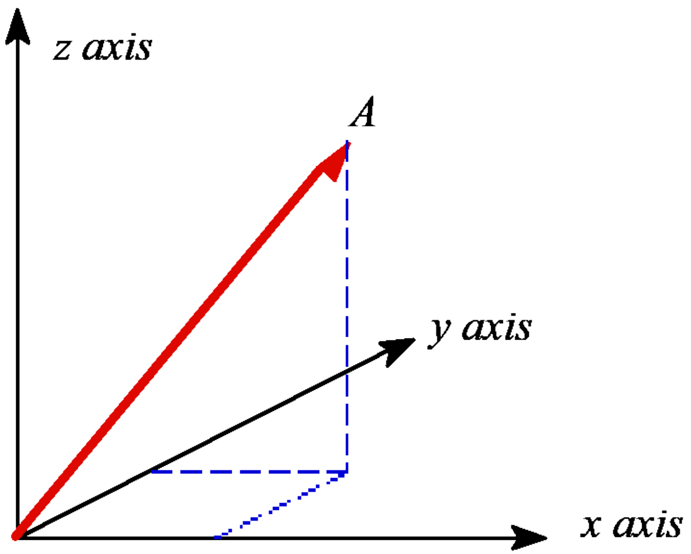
$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C}$$



Vector components

As we all *also* know, a vector can be identified by specifying its three Cartesian components and using the unit vectors of a Cartesian coordinate system:

$$\boldsymbol{A} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}$$



- To add vectors, add like components.
- To multiply a vector by a scalar, multiply each component.
- To calculate the dot product of two vectors, multiply like components and add:

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$$

- To calculate the cross product of two vectors, evaluate the determinant

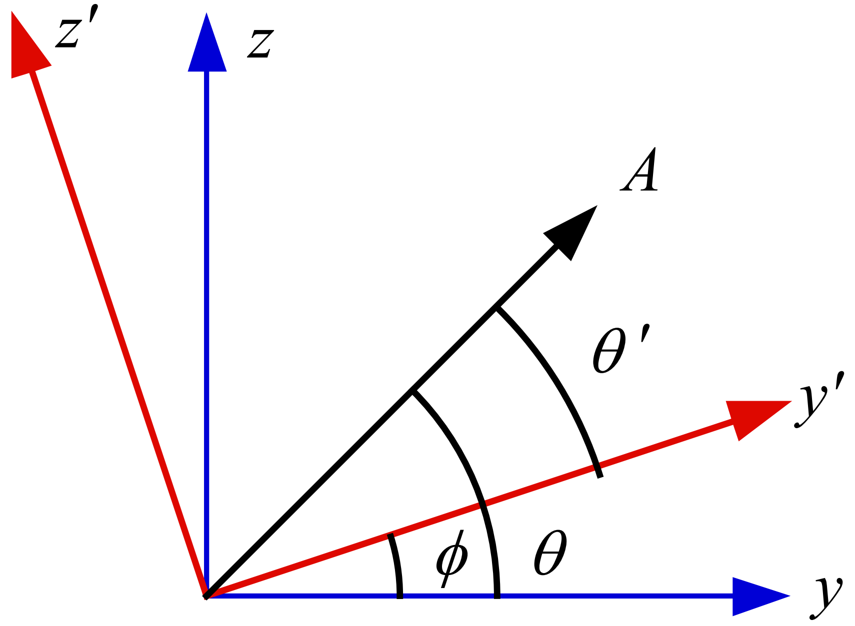
$$\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Vector transformation

Behavior under coordinate transformation is what really defines what are vectors and what are not.

As we all know,

- The components of a vector depend on the choice of the coordinate system.
- Different coordinate system will produce different components for the same vector.
- The choice of coordinate system being used can significantly change the complexity of problems in electrodynamics.



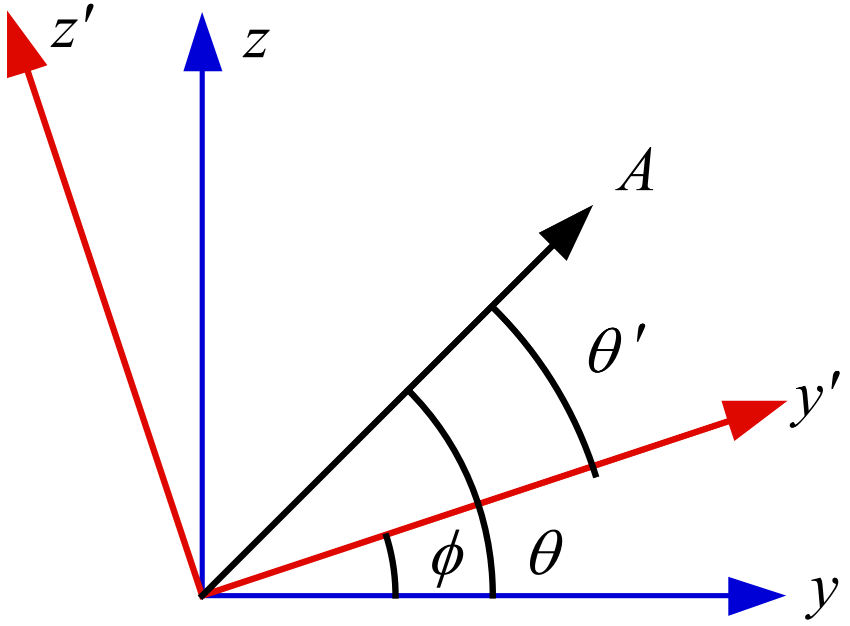
Vector transformation (continued)

- The coordinates of vector ***A*** in coordinate system S are related to the coordinates of vector ***A*** in coordinate system S':

$$\begin{pmatrix} A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

- The rotation considered here leaves the *x* axis untouched. The *x* coordinate of vector ***A*** will thus not change:

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \equiv \overleftrightarrow{\boldsymbol{R}} \cdot \boldsymbol{A}$$



Vector transformation (continued)

- Coordinate transformation resulting from a rotation around an **arbitrary** axis can be represented in Cartesian 3-D as:

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} R_{xx}A_x + R_{xy}A_y + R_{xz}A_z \\ R_{yx}A_x + R_{yy}A_y + R_{yz}A_z \\ R_{zx}A_x + R_{zy}A_y + R_{zz}A_z \end{pmatrix}$$

or, more compactly, with *x* denoted as 1, *y* as 2, *z* as 3: $A_i' = \sum_{j=1}^{3} R_{ij} A_j$

Vector transformation (continued)

- The rotation matrix $\overleftrightarrow{\boldsymbol{R}}$ is an example of a **unitary** transformation: one that does not change the magnitude of the object on which it operates:

$$\boldsymbol{A}' = \overleftrightarrow{\boldsymbol{R}} \cdot \boldsymbol{A} \quad \text{and} \quad A' = A.$$

as you will show in Problem 7 on this week's homework.

- If $\overleftrightarrow{\boldsymbol{R}}$ is unitary, then

$$\sum_{i=1}^{3} R_{ij} R_{ik} = \delta_{jk} \quad ,$$

where

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \ , \\ 0 & \text{otherwise} \ ; \end{cases}$$

the Kronecker delta, as you will make plausible in Problem 7 of this week's homework.

Vector transformation (continued)

- Corollary: under coordinate inversion, $x' \to -x$, $y' \to -y$, $z' \to -z$, vectors are inverted: $\boldsymbol{A}' \to -\boldsymbol{A}$.

- But a cross product of two vectors is not inverted under inversion: $\boldsymbol{A}' \times \boldsymbol{B}' \to \boldsymbol{A} \times \boldsymbol{B}$, as you will show in this week's homework.

- We therefore call the cross product of two vectors something else: a **pseudovector**, or **axial vector**. Pseudovectors transform under rotation like vectors, but under coordinate inversion exhibit **parity** that is different from that of vectors.

Dyadics

Dyadics, also known as **outer products**, are another kind of vector-multiplication result: $\overleftrightarrow{\boldsymbol{C}} = \boldsymbol{AB}$.

- Instead of having one component, like a dot product, or three like a cross product, dyadics have **nine** generally independent components.

- **In Cartesian 3-D representation**, dyadics can be decomposed into multiples of nine independent **dyads**, themselves dyadic products of the unit vectors in the manner of $\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}$ or $\hat{\boldsymbol{z}}\hat{\boldsymbol{x}}$:

$$\overleftrightarrow{\boldsymbol{C}} = A_x B_x \hat{\boldsymbol{x}}^2 + A_x B_y \hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + A_x B_z \hat{\boldsymbol{x}}\hat{\boldsymbol{z}} + A_y B_x \hat{\boldsymbol{y}}\hat{\boldsymbol{x}} + A_y B_y \hat{\boldsymbol{y}}^2 + A_y B_z \hat{\boldsymbol{y}}\hat{\boldsymbol{z}} + A_z B_x \hat{\boldsymbol{z}}\hat{\boldsymbol{x}} + A_z B_y \hat{\boldsymbol{z}}\hat{\boldsymbol{y}} + A_z B_z \hat{\boldsymbol{z}}^2 \quad ,$$

  and thus, in terms of the vectors which make them, have the matrix representation

$$\overleftrightarrow{\boldsymbol{C}} = \begin{bmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{bmatrix} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \begin{bmatrix} B_x & B_y & B_z \end{bmatrix} = \boldsymbol{AB}^T \quad .$$

(but not usually written as a transpose in E&M)

Dyadics (continued)

- Instead of having a magnitude, a dyadic $\overleftrightarrow{\boldsymbol{C}}$ has two invariants, $\left|\overleftrightarrow{\boldsymbol{C}}\right|$ and $\left\langle\overleftrightarrow{\boldsymbol{C}}\right\rangle$, called the **cusp** and the **rotation vector**. In Cartesian 3-D, for the dyadic with elements C_{ij},

$$\left|\overleftrightarrow{\boldsymbol{C}}\right| = \sum_{i=1}^{3} C_{ii} \quad , \qquad \left\langle\overleftrightarrow{\boldsymbol{C}}\right\rangle = \hat{\boldsymbol{x}}\left(C_{yz} - C_{zy}\right) + \hat{\boldsymbol{y}}\left(C_{zx} - C_{xz}\right) + \hat{\boldsymbol{z}}\left(C_{xy} - C_{yx}\right) \quad .$$

- Dyadics commute in addition and scalar multiplication: $\overleftrightarrow{\boldsymbol{C}} + \overleftrightarrow{\boldsymbol{D}} = \overleftrightarrow{\boldsymbol{D}} + \overleftrightarrow{\boldsymbol{C}}$, $a\overleftrightarrow{\boldsymbol{C}} = \overleftrightarrow{\boldsymbol{C}}a$.

- They do not commute in scalar (inner) products: in general, $\boldsymbol{A} \cdot \overleftrightarrow{\boldsymbol{C}} \neq \overleftrightarrow{\boldsymbol{C}} \cdot \boldsymbol{A}$.

- However, to each dyadic $\overleftrightarrow{\boldsymbol{C}}$ corresponds a **conjugate** $\overleftrightarrow{\boldsymbol{C}}^*$ such that $\boldsymbol{A} \cdot \overleftrightarrow{\boldsymbol{C}} = \overleftrightarrow{\boldsymbol{C}}^* \cdot \boldsymbol{A}$ and $\overleftrightarrow{\boldsymbol{C}} \cdot \boldsymbol{A} = \boldsymbol{A} \cdot \overleftrightarrow{\boldsymbol{C}}^*$.

- In Cartesian 3-D representation, the conjugate is the same as the transpose: $\left(\overleftrightarrow{\boldsymbol{C}}^*\right)_{ji} = \left(\overleftrightarrow{\boldsymbol{C}}\right)_{ij}$.

Second-rank tensors and dyadics

More generally,

- Vectors are **first-rank tensors**: three independent components represented by a column matrix in Cartesian 3-D.

- An object $\overleftrightarrow{\boldsymbol{T}}$ with nine independent components that transforms under rotations as

$$T_{ij}{}' = \sum_{k=1}^{3}\sum_{l=1}^{3} R_{ik}R_{jl}T_{kl}$$

  is called a **second-rank** tensor. In 3-D, **dyadics are second-rank tensors**.

- Inner products of second-rank tensors and vectors have a vector result: $\boldsymbol{P} = \overleftrightarrow{\boldsymbol{T}} \cdot \boldsymbol{A}$, $\boldsymbol{Q} = \boldsymbol{A} \cdot \overleftrightarrow{\boldsymbol{T}}$.

- The dyadic form and its algebraic properties is useful as a coordinate-system-independent expression of second-rank tensors, especially as they can represent operators, such as (see below)

$$\overleftrightarrow{\boldsymbol{C}} = \boldsymbol{\nabla A}, \ \overleftrightarrow{\boldsymbol{C}}{}^* = \boldsymbol{A\nabla}.$$

Differential vector calculus

As we all know:

- *df/dx* provides us with information on how quickly a function of one variable, *f*(*x*), changes.
- For instance, when the argument changes by an infinitesimal amount, from *x* to *x*+*dx*, *f* changes by *df*, given by

$$df = \left(\frac{df}{dx}\right)dx$$

- In three dimensions, the function *f* is in general a function of *x*, *y*, and *z*: *f*(*x, y, z*). And for an infinitesimal change in *f,*

$$df = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z}\right)dz$$

$$= \left(\left(\frac{\partial f}{\partial x}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial f}{\partial y}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial f}{\partial z}\right)\hat{\boldsymbol{z}}\right)\cdot\left((dx)\hat{\boldsymbol{x}} + (dy)\hat{\boldsymbol{y}} + (dz)\hat{\boldsymbol{z}}\right)$$

$$\equiv \boldsymbol{\nabla} f \cdot d\boldsymbol{\ell} \quad .$$

Differential vector calculus (continued)

- The vector derivative operator, $\nabla = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}$, produces a vector when it operates on scalar function *f*(*x*,*y*,*z*).
- **Ñ** is a vector, as we can see from its behavior under coordinate rotations: $(\nabla f)' = \overleftrightarrow{\boldsymbol{R}} \cdot \nabla f$, but its magnitude is not a number: it is an operator.
- There are three kinds of vector derivatives, corresponding to the three kinds of multiplication possible with vectors:
  - **Gradient**, the analogue of multiplication by a scalar: ∇f.
  - **Divergence**, like the scalar (dot) product: $\nabla \cdot \boldsymbol{v}$.
  - **Curl**, which corresponds to the vector (cross) product: $\nabla \times \boldsymbol{v}$.
- The gradient of a scalar function, $\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial f}{\partial y}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial f}{\partial z}\right)\hat{\boldsymbol{z}}$, points in the direction of maximum increase of *f* (i.e. "uphill"), and the magnitude of the gradient gives the slope of *f* in the direction of maximum increase.

Divergence

- The scalar product of the vector derivative operator and a vector function is called the divergence of the vector function:

$$\boldsymbol{\nabla} \cdot v = \left(\hat{\boldsymbol{x}} \frac{\partial}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial}{\partial z} \right) \cdot \left(v_x \hat{\boldsymbol{x}} + v_y \hat{\boldsymbol{y}} + v_z \hat{\boldsymbol{z}} \right) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

- The divergence of a vector function is a scalar.

- What is the divergence?

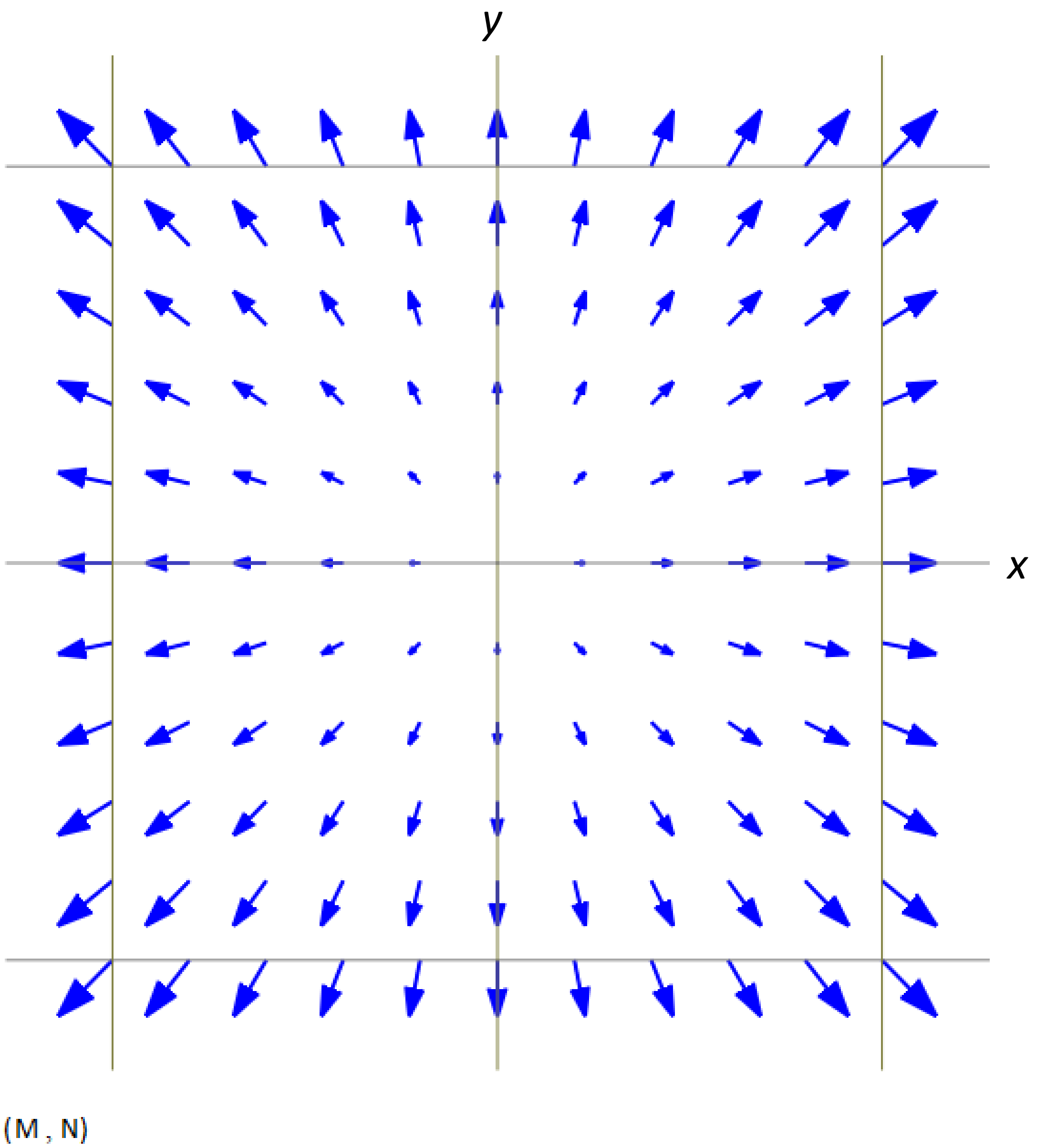
  - If two objects following the direction specified by the vector function increase their separation, the divergence of the vector function is positive.

  - If their separation decreases, the divergence of the vector function is negative.

A function with constant divergence

$$\boldsymbol{v}(x,y) = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} \quad ,$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 2 \quad .$$



Curl

- In Cartesian 3-D, the curl of a vector function $\boldsymbol{v}$ is

$$\nabla \times \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\boldsymbol{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\boldsymbol{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\boldsymbol{z}}$$

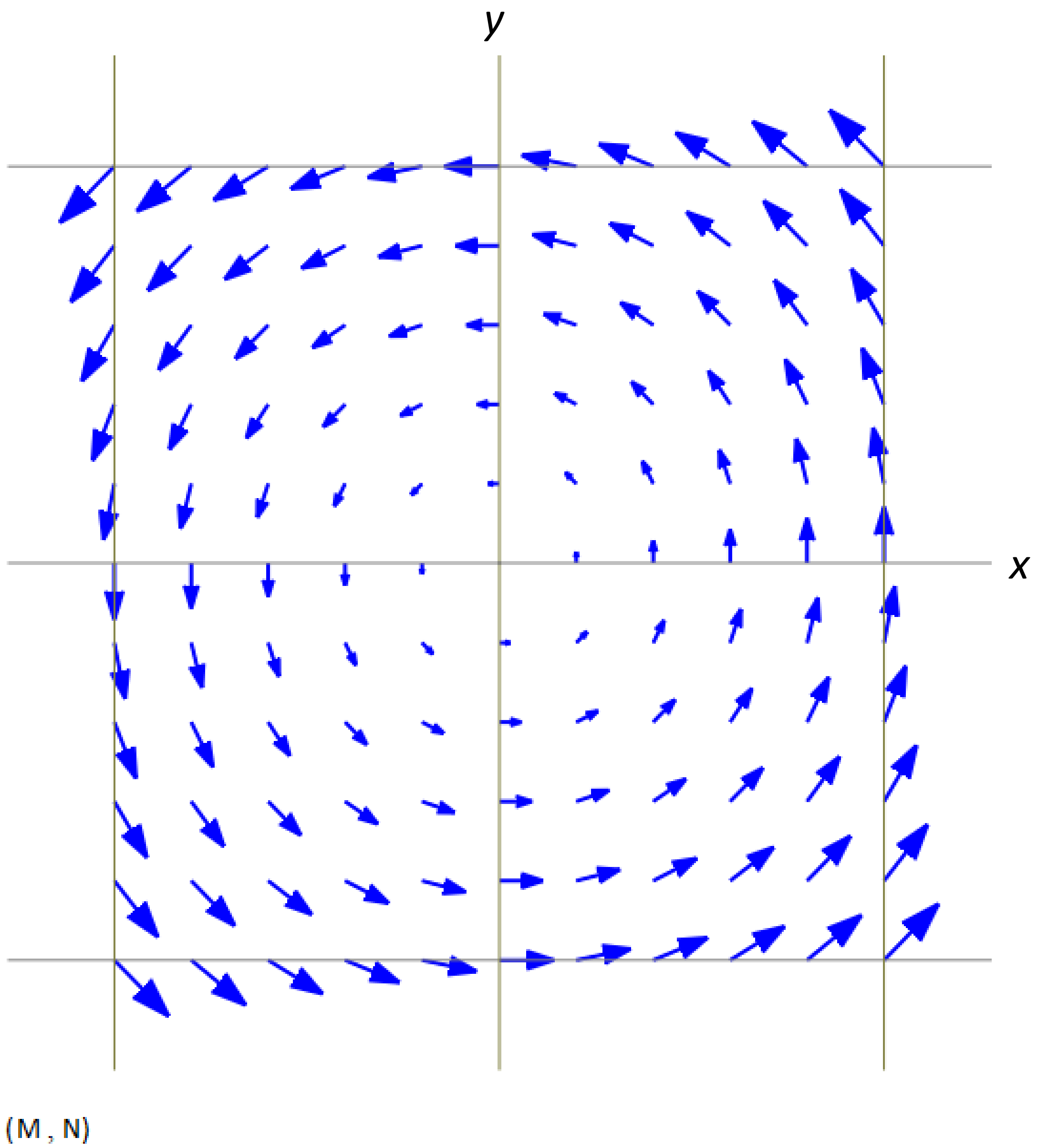
  and is itself a vector. To be precise: if $\boldsymbol{v}$ is a vector function, its curl is a **pseudovector function**, as above.

- What is the curl?
  - The curl of a vector function evaluated at a certain point is a measure of how much the vector function's direction wraps around that point.
  - If there were nearby objects moving in the direction of the function, they would circulate about that point, if the curl were nonzero.

A function with constant curl

$$\boldsymbol{v}(x,y) = -y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}} \quad ,$$

$$\nabla \times \boldsymbol{v} = 2\hat{\boldsymbol{z}} \quad .$$



A function with constant curl and divergence

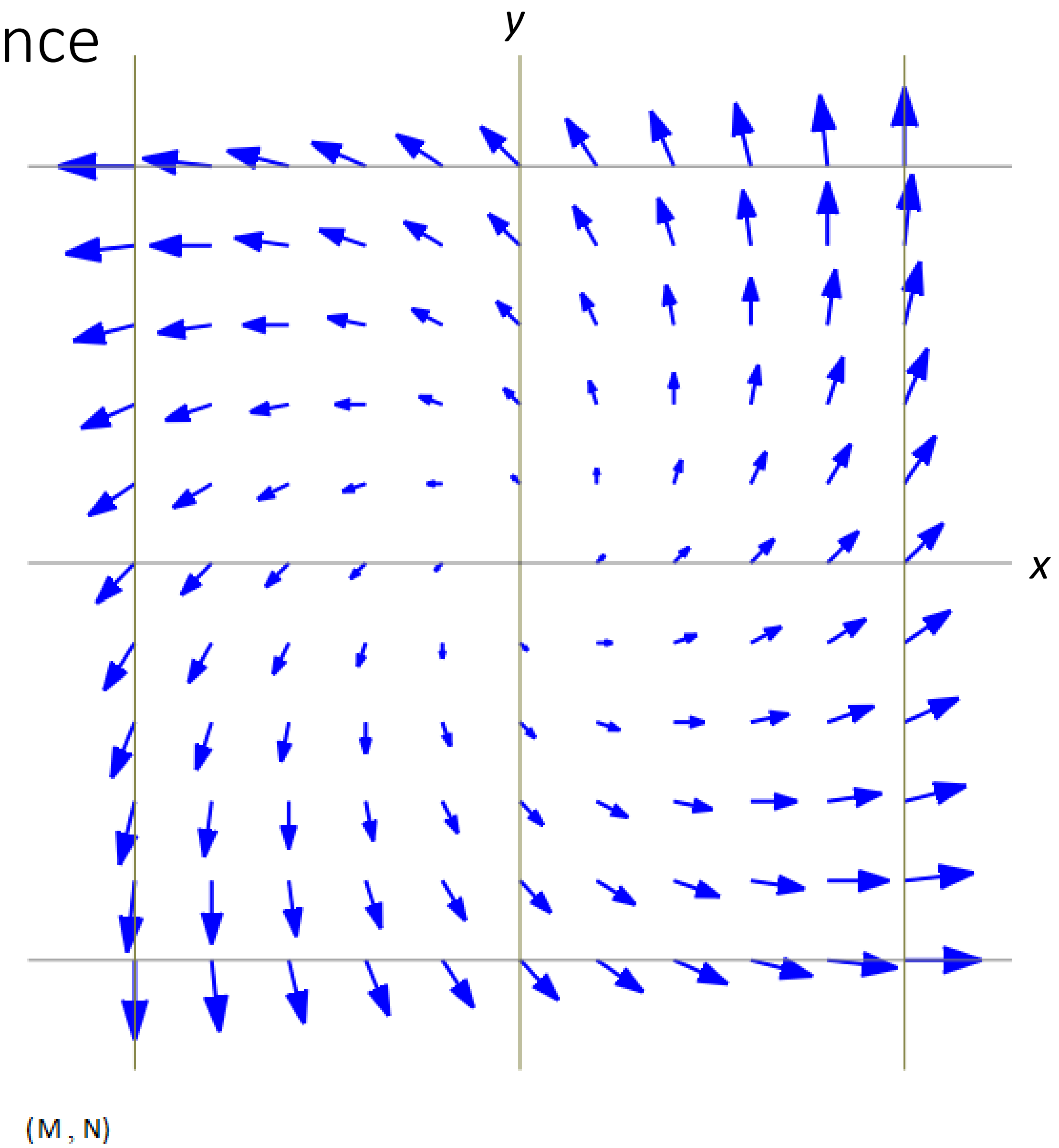
The two previous functions had nonzero divergence and zero curl, or vice versa. The sum of the two functions,

$$\boldsymbol{v}(x,y)=(x-y)\hat{\boldsymbol{x}}+(x+y)\hat{\boldsymbol{y}} \quad ,$$

has (constant) nonzero divergence and curl:

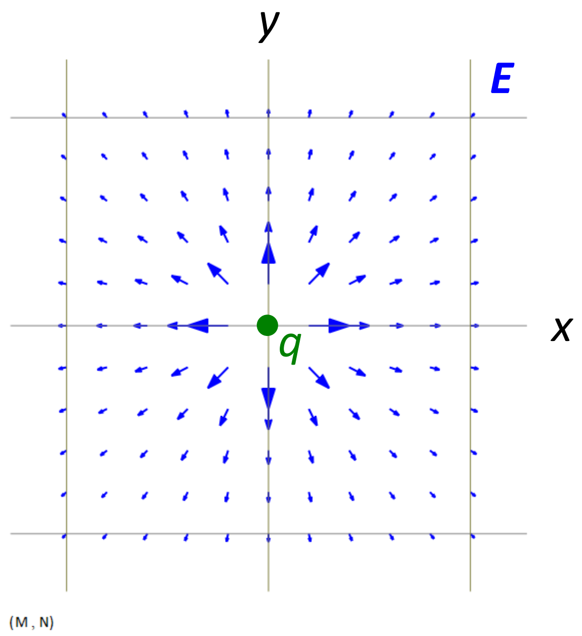
$$\nabla\cdot\boldsymbol{v}=2$$

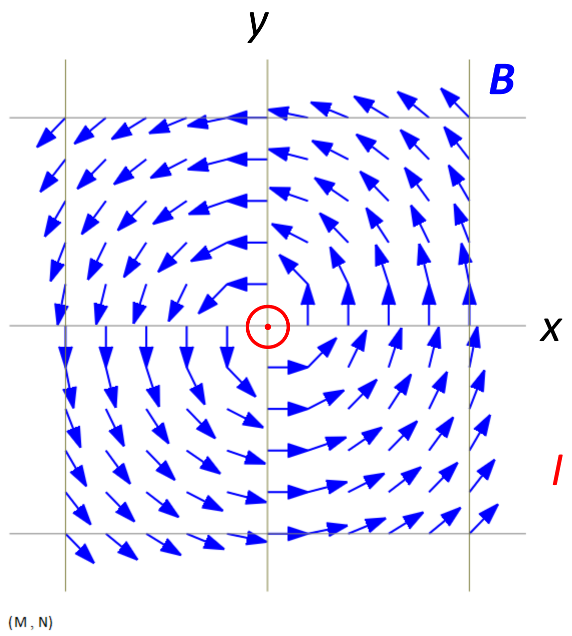
$$\nabla\times\boldsymbol{v}=2\hat{\boldsymbol{z}}$$



Why are div and curl important in E&M?

Consider the electric field from a point charge, and the magnetic field from a constant current in a long straight wire:





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- Nonzero divergence of ***E*** indicates the presence of charge.
- Nonzero curl of ***B*** indicates the presence of current.
- These vector derivatives point to the sources of the ***E*** and ***B*** fields.

Product rules for vector first derivatives

- The following product rules involving the vector products will be used frequently:

$$\nabla(fg) = f \cdot \nabla g + g \cdot \nabla f$$

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A}$$

$$\nabla\cdot(f\boldsymbol{A}) = f(\nabla\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot\nabla f$$

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$$

$$\nabla\times(f\boldsymbol{A}) = f(\nabla\times\boldsymbol{A}) + \boldsymbol{A}\times\nabla f$$

$$\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}(\nabla\cdot\boldsymbol{A})$$

- You'll also find them on the inside front cover of Griffiths, and will prove some of them yourself on the homework.

Vector second derivatives

There are five possibilities for second derivatives involving Ñ:

$$\nabla\cdot(\nabla f) \qquad \nabla\times(\nabla f) \qquad \nabla(\nabla\cdot\boldsymbol{v})$$
$$\nabla\cdot(\nabla\times\boldsymbol{v}) \quad \nabla\times(\nabla\times\boldsymbol{v})$$

- The divergence of a gradient is called the **Laplacian**, denoted Ñ²:

$$\nabla^2 f \equiv \nabla\cdot(\nabla f) = \left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x}+\hat{\boldsymbol{y}}\frac{\partial}{\partial y}+\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\boldsymbol{x}}\frac{\partial f}{\partial x}+\hat{\boldsymbol{y}}\frac{\partial f}{\partial y}+\hat{\boldsymbol{z}}\frac{\partial f}{\partial z}\right)$$
$$= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

Soon you'll be good friends with this operator.

Vector second derivatives (continued)

- The curl of a gradient is always zero, as you'll show in this week's homework:

$$\nabla \times (\nabla f) = 0$$

- The gradient of a divergence,

$$\nabla (\nabla \cdot \boldsymbol{v})$$

  appears frequently in the equations of fluid mechanics, but it never lasts long in the equations of electrodynamics.

- The divergence of a curl is always zero, as you'll also show in this week's homework:

$$\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$$

Vector second derivatives (continued)

- The curl of a curl of a vector function can be expressed in terms of the Laplacian and the gradient of the divergence of the vector function:

$$\boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \boldsymbol{v} \right) = \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \boldsymbol{v} \right) - \nabla^2 \boldsymbol{v}$$

  so it’s not really different in nature from the other four.

- Note that the Laplacian can indeed operate on scalar or vector functions, but unlike the operation of Ñ on a vector function, it does not result in a dyadic.