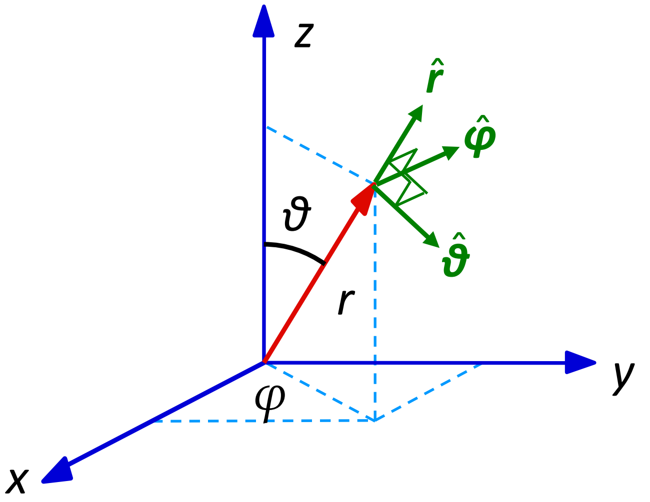
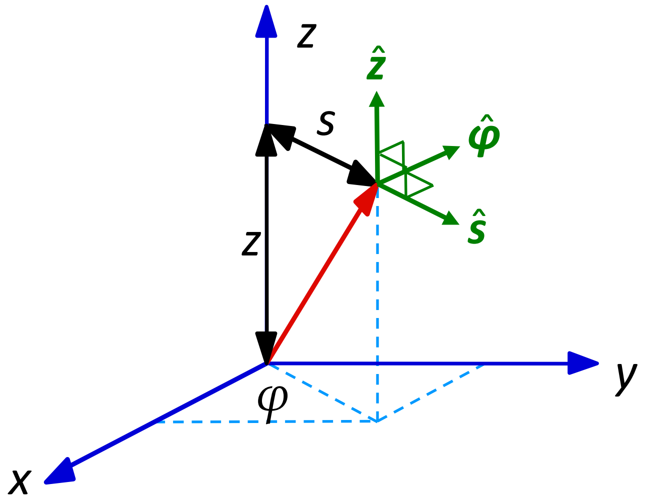
Today in Physics 217: vector calculus in curvilinear coordinates

- Vector derivatives in curvilinear coordinate systems: spherical and cylindrical coordinates.
- The Dirac delta function.
- Divergence and curl, fields and potentials, and the Helmholtz theorem.





Homework 1 is due today. Upload it to your Box folder.

Homework 2 is on the course website; due next Tuesday.

Curvilinear coordinates

Coordinate systems, which are probably all familiar to you but you'll need to get back in practice:

- Cartesian coordinates: used to describe systems with rectilinear symmetry, and as the default for systems without any apparent symmetry.

- Curvilinear coordinates: used to describe systems with symmetry. We will often find spherical symmetry or axial symmetry in the problems we will do this semester, and will thus use

  - Spherical coordinates

  - Cylindrical coordinates

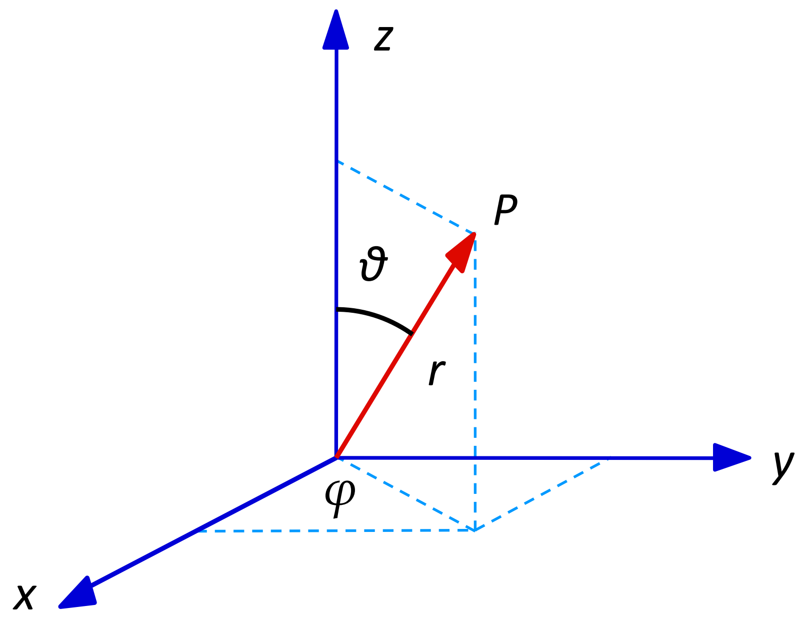
  Both always displayed on Cartesian axes.

- There are other curvilinear coordinate systems (e.g. ellipsoidal) which have special virtues, but we won't get to use them this semester.

Spherical coordinates

The location of a point P can be defined by specifying the following three parameters:

- Radius r: distance of P from the origin.
- Polar angle ϑ: angle between the position vector of P and the z axis. (Like 90° – latitude.)
- Azimuthal angle φ: angle between the projection of the position vector P and the x axis. (Like longitude.)



Spherical coordinates (continued)

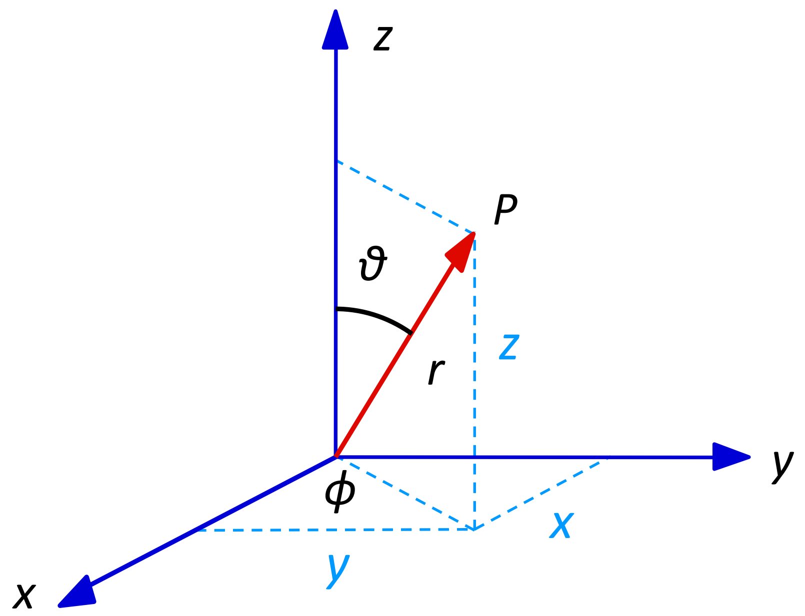
- The Cartesian coordinates of P are related to the spherical coordinates by

$$x = r\sin\vartheta\cos\varphi \qquad r = \sqrt{x^2+y^2+z^2}$$

$$y = r\sin\vartheta\sin\varphi \qquad \vartheta = \arctan\left(\sqrt{x^2+y^2}\big/z\right)$$

$$z = r\cos\vartheta \qquad \varphi = \arctan\left(y/x\right)$$

  as one can see easily by pondering the trigonometry of the diagram at right.



- The unit vectors of spherical coordinate systems are not constant: their direction changes when the position of point P changes.

Spherical coordinates (continued)

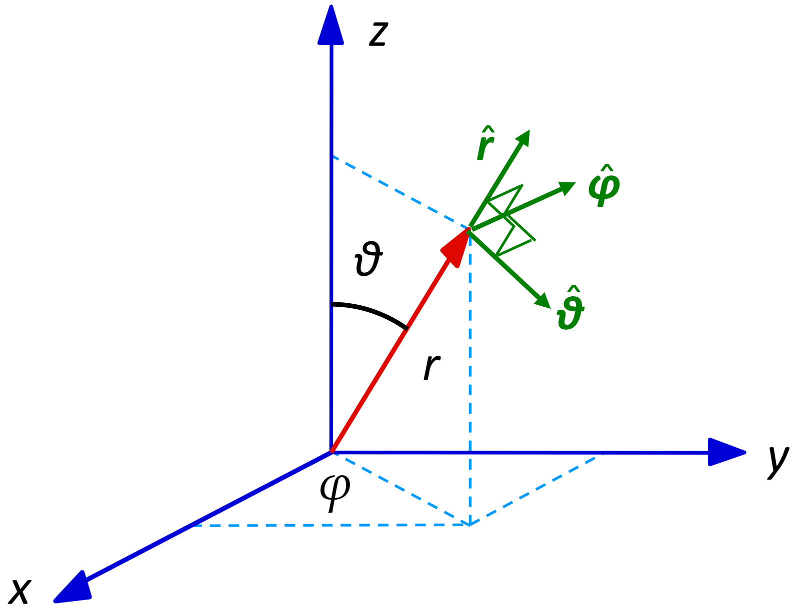
- In Cartesian coordinates, an infinitesimal displacement from point *P* is equal to

$$d\boldsymbol{\ell} = \hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}dy + \hat{\boldsymbol{z}}dz \quad .$$

- In spherical coordinates, an infinitesimal displacement from point P is equal to

$$d\boldsymbol{\ell} = \hat{\boldsymbol{r}}dr + \hat{\boldsymbol{\vartheta}}rd\vartheta + \hat{\boldsymbol{\varphi}}r\sin\vartheta d\varphi \quad ,$$

where $\hat{\boldsymbol{r}}$ is parallel to *r*, $\hat{\boldsymbol{\vartheta}}$ is perpendicular to $\hat{\boldsymbol{r}}$ and lies in the r-z plane, $\hat{\boldsymbol{\varphi}}$ is perpendicular to this plane, and **all point in the direction in which their corresponding coordinates increase**.

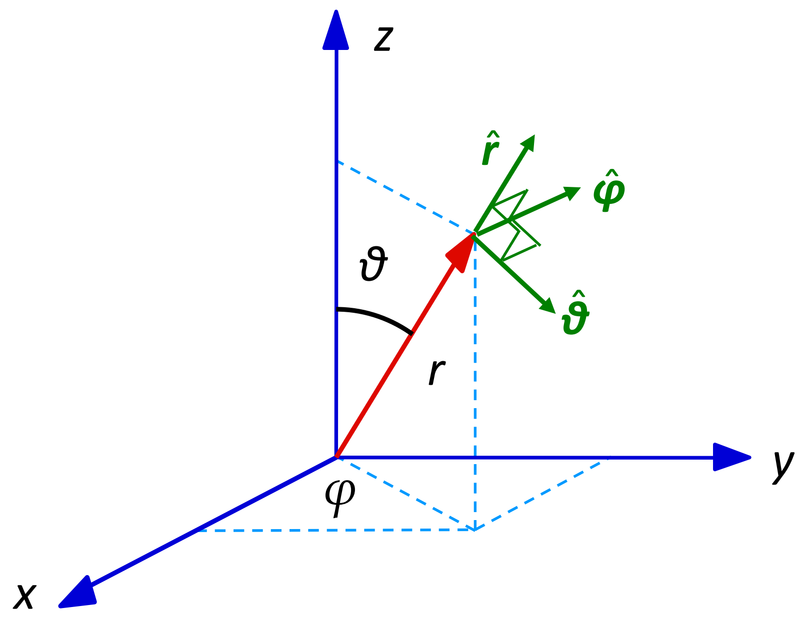


Example: transformation of unit vectors

Griffiths problem 1.38: Express the spherical-coordinate unit vectors in terms of the Cartesian ones.

The only difficulty with doing so is that the unit vectors are different at every point in space, so we study the geometry of very small displacements about the position vector ***r*** (at right):

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = r\sin\vartheta\cos\varphi\hat{\mathbf{x}} + r\sin\vartheta\sin\varphi\hat{\mathbf{y}} + r\cos\vartheta\hat{\mathbf{z}}$$



- An infinitesimal displacement ***dr*** along ***r*** points in the direction that the radial unit vector must point. Using the chain rule,

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial r}dr = dr\sin\vartheta\cos\varphi\hat{\mathbf{x}} + dr\sin\vartheta\sin\varphi\hat{\mathbf{y}} + dr\cos\vartheta\hat{\mathbf{z}} \quad .$$

- This displacement has magnitude

$$|d\mathbf{r}| = dr\left(\sin^2\vartheta\cos^2\varphi + \sin^2\vartheta\sin^2\varphi + \cos^2\vartheta\right)^{1/2} = dr.$$

Example: transformation of unit vectors (continued)

- Thus the unit vector in the direction of ***dr*** is

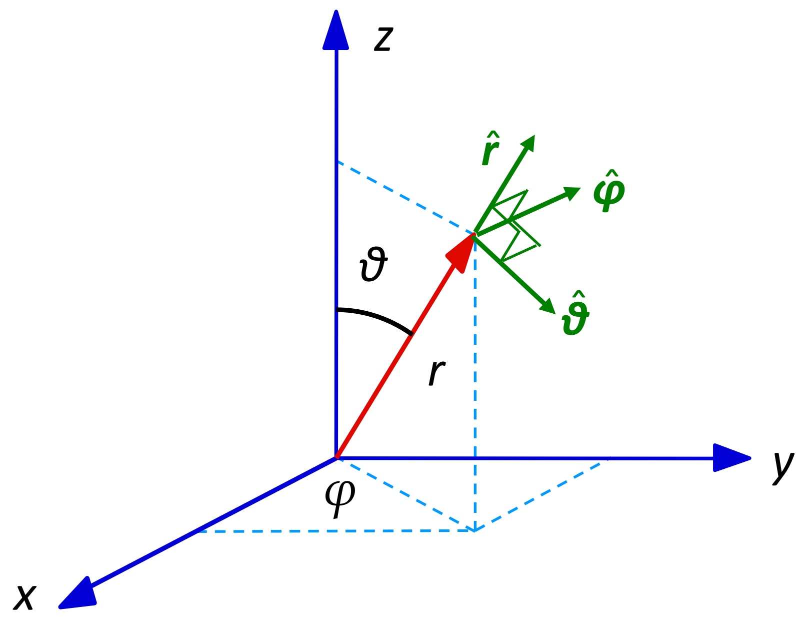
$$\boxed{\hat{\boldsymbol{r}} = d\boldsymbol{r}/|d\boldsymbol{r}| = \sin\vartheta\cos\varphi\hat{\boldsymbol{x}} + \sin\vartheta\sin\varphi\hat{\boldsymbol{y}} + \cos\vartheta\hat{\boldsymbol{z}}} \quad .$$

- Polar angle next, following the same pattern, reusing the position vector ***r*,** and noting that we want the unit vector to point in the direction of increasing ϑ: away from the +*z* axis, as at right:

$$d\boldsymbol{r} = \frac{\partial\boldsymbol{r}}{\partial\vartheta}d\vartheta = d\vartheta r\cos\vartheta\cos\varphi\hat{\boldsymbol{x}} + d\vartheta r\cos\vartheta\sin\varphi\hat{\boldsymbol{y}} - d\vartheta r\sin\vartheta\hat{\boldsymbol{z}} \quad ,$$

$$|d\boldsymbol{r}| = rd\vartheta\left(\cos^2\vartheta\cos^2\varphi + \cos^2\vartheta\sin^2\varphi + \sin^2\vartheta\right)^{1/2} = rd\vartheta \quad ,$$

$$\boxed{\hat{\boldsymbol{\vartheta}} = \frac{d\boldsymbol{r}}{|d\boldsymbol{r}|} = \cos\vartheta\cos\varphi\hat{\boldsymbol{x}} + \cos\vartheta\sin\varphi\hat{\boldsymbol{y}} - \sin\vartheta\hat{\boldsymbol{z}} \quad .}$$



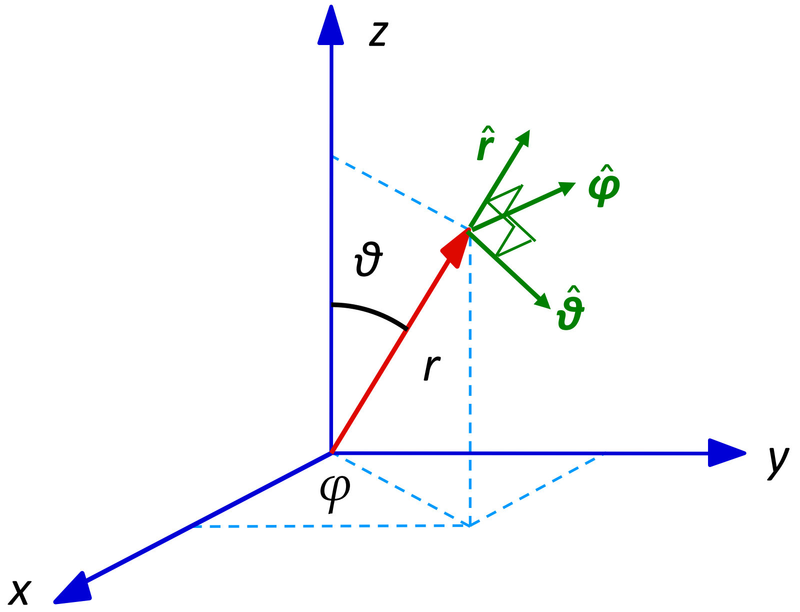
Example: transformation of unit vectors (continued)

- And then azimuthal angle, again making sure the unit vector points in the direction of increasing φ: counterclockwise in the *x*-*y* plane:

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial \varphi} d\varphi = -r d\varphi \sin\vartheta \sin\varphi \hat{\boldsymbol{x}} + r d\varphi \sin\vartheta \cos\varphi \hat{\boldsymbol{y}} \quad ,$$

$$|d\boldsymbol{r}| = r d\varphi \left(\sin^2\vartheta \sin^2\varphi + \sin^2\vartheta \cos^2\varphi \right)^{1/2} = r \sin\vartheta d\varphi \quad ,$$

$$\boxed{\hat{\boldsymbol{\varphi}} = \frac{d\boldsymbol{r}}{|d\boldsymbol{r}|} = -\sin\varphi \hat{\boldsymbol{x}} + \cos\varphi \hat{\boldsymbol{y}} \quad .}$$



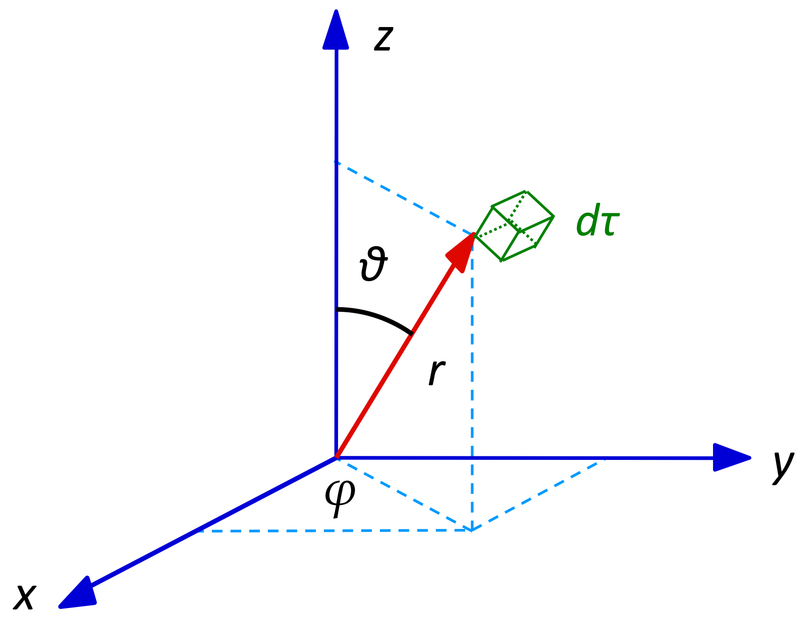
Spherical coordinates (continued)

- In a Cartesian coordinates, an infinitesimal volume element at point *P* has volume $d\tau$ given by

$$d\tau = dx\,dy\,dz \quad .$$

- In a spherical coordinates, an infinitesimal volume element at point *P* has

$$\begin{aligned} d\tau &= d\ell_r d\ell_\vartheta d\ell_\varphi \\ &= r^2 \sin\vartheta\, dr d\vartheta d\varphi \quad . \end{aligned}$$



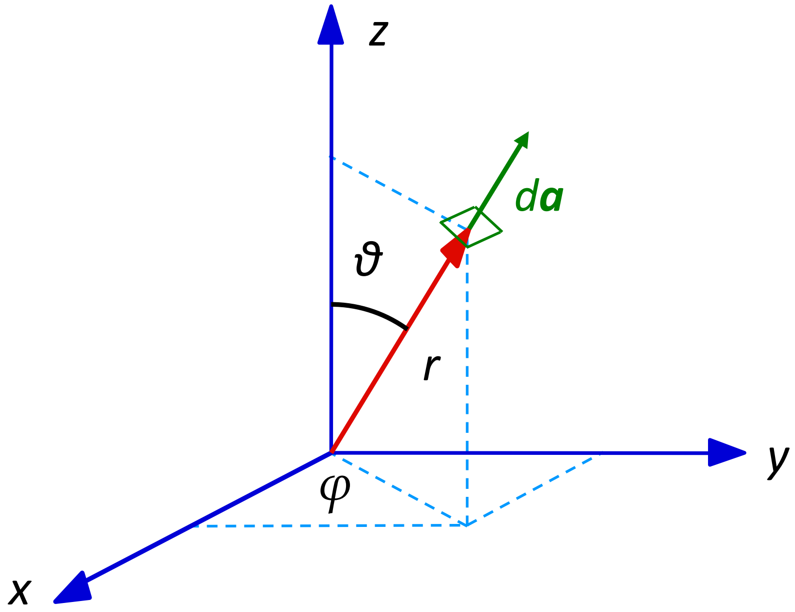
Spherical coordinates (continued)

- In Cartesian coordinates, an infinitesimal area element on a (cardinal) plane through point *P* is

$$d\mathbf{a} = \hat{\mathbf{z}}dxdy \quad , \text{ or}$$
$$= \hat{\mathbf{x}}dydz \quad , \text{ or}$$
$$= \hat{\mathbf{y}}dzdx \quad .$$

In spherical coordinates, the infinitesimal area element on a **sphere** through point *P* is

$$d\mathbf{a} = \hat{\mathbf{r}}d\ell_{\vartheta}d\ell_{\varphi} = \hat{\mathbf{r}}r^2 \sin\vartheta d\vartheta d\varphi \quad .$$



Vector derivatives in spherical coordinates

What if you want to express a vector derivative in spherical coordinates? (Or someone asks you to, on an exam …)

- Start from the Cartesian-coordinate version, and first use the chain rule to transform the derivatives, e.g.

$$\left(\nabla T\right)_x = \frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \vartheta}\frac{\partial \vartheta}{\partial x} + \frac{\partial T}{\partial \varphi}\frac{\partial \varphi}{\partial x} \quad .$$

- Use the coordinate definitions to reduce the remaining derivatives and eliminate all Cartesian coordinates, e.g.

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \sin\vartheta\cos\varphi \quad .$$

- Transform the unit vectors, as we did on pages 6-8.

- Then multiply the whole mess out and simplify.

- This is tedious, and takes hours, but is instructive and highly recommended, even though it’s not on the homework explicitly. See also Appendix A in Griffiths.

Vector derivatives in spherical coordinates (continued)

- The following operations will be encountered frequently enough that they're even written on the inside front cover of Griffiths:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial T}{\partial \vartheta}\hat{\boldsymbol{\vartheta}} + \frac{1}{r\sin\vartheta}\frac{\partial T}{\partial \varphi}\hat{\boldsymbol{\varphi}}$$

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta v_\vartheta\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi}\left(v_\varphi\right)$$

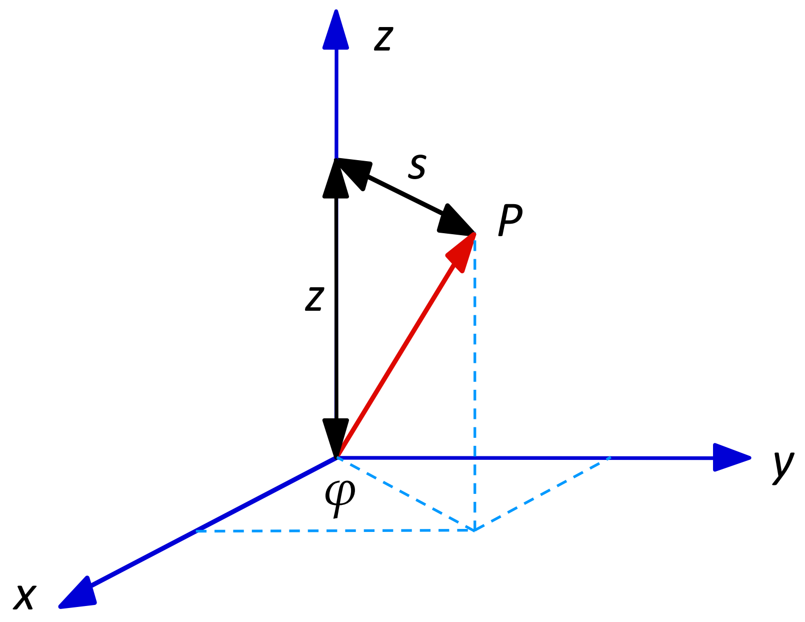
$$\nabla \times \boldsymbol{v} = \frac{1}{r\sin\vartheta}\left(\frac{\partial}{\partial \vartheta}\left(\sin\vartheta v_\phi\right) - \frac{\partial}{\partial \varphi}\left(v_\vartheta\right)\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{1}{\sin\vartheta}\frac{\partial}{\partial \varphi}\left(v_r\right) - \frac{\partial}{\partial r}\left(r v_\varphi\right)\right)\hat{\boldsymbol{\vartheta}}$$
$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r v_\vartheta\right) - \frac{\partial}{\partial \vartheta}\left(v_r\right)\right)\hat{\boldsymbol{\varphi}}$$

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial T}{\partial \vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 T}{\partial \varphi^2}$$

Cylindrical coordinates

Spherical coordinates are useful mostly for spherically symmetric situations. In problems involving symmetry about just one axis (axisymmetry), cylindrical coordinates are used:

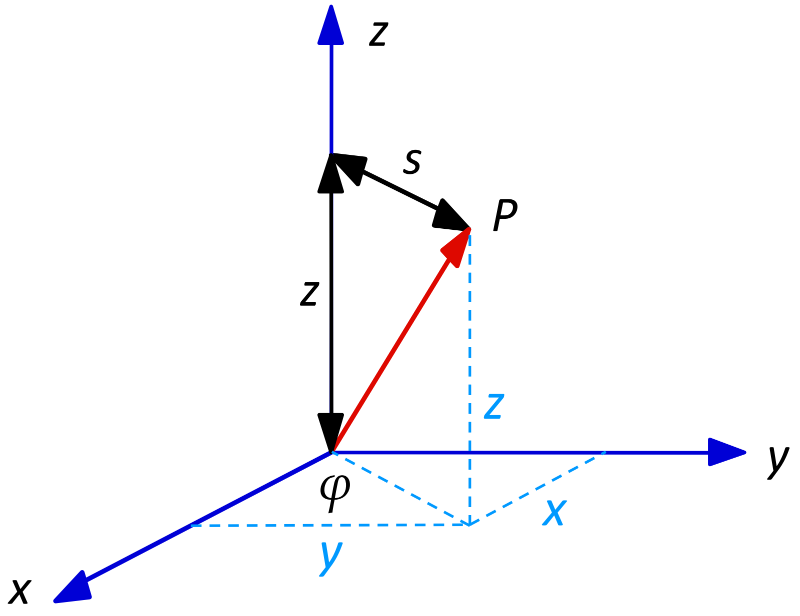
- The radius s: distance of P from the z axis.
- The azimuthal angle φ: angle between the projection onto the x-y plane of the position vector P and the x axis. Same as the spherical coordinate of the same name.
- The z coordinate: component of the position vector P along the z axis. Same as the Cartesian z.



Cylindrical coordinates (continued)

- The Cartesian coordinates of P are related to the cylindrical coordinates by

$$
\begin{aligned}
x &= s\cos\varphi & \qquad s &= \sqrt{x^2+y^2} \\
y &= s\sin\varphi & \qquad \varphi &= \arctan(y/x) \\
z &= z & \qquad z &= z
\end{aligned}
$$

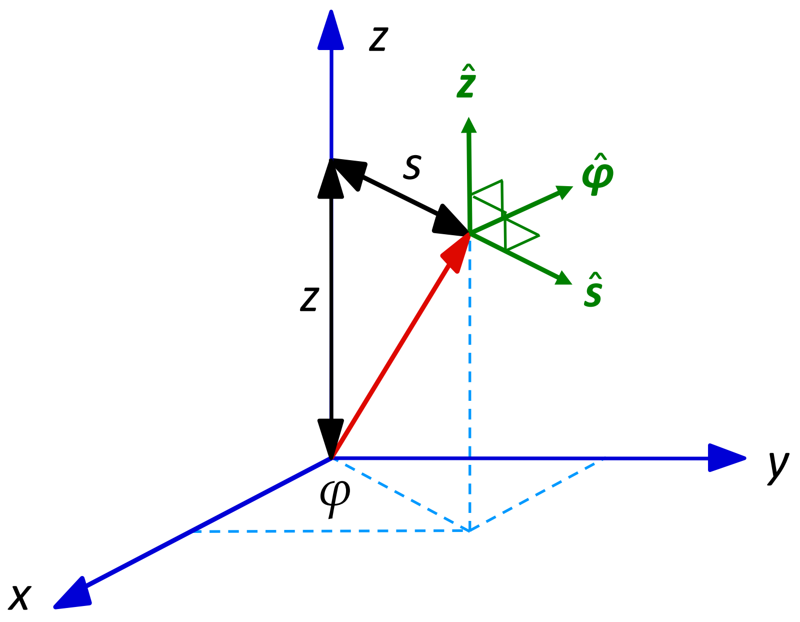


- Like those of spherical coordinates, the unit vectors of cylindrical-coordinate systems are not uniform; their direction changes when the position of point *P* moves.

Cylindrical coordinates (continued)

In cylindrical coordinates, as can be shown in the same pattern as spherical coordinates:

- Unit vectors
$$\hat{\boldsymbol{s}} = \hat{\boldsymbol{x}}\cos\varphi + \hat{\boldsymbol{y}}\sin\varphi$$
$$\hat{\boldsymbol{\varphi}} = -\hat{\boldsymbol{x}}\sin\varphi + \hat{\boldsymbol{y}}\cos\varphi$$
$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$$
- Infinitesimal displacement $d\boldsymbol{\ell} = \hat{\boldsymbol{s}}ds + \hat{\boldsymbol{\varphi}}rd\varphi + \hat{\boldsymbol{z}}dz$
- Infinitesimal volume element $d\tau = sdsd\varphi dz$
- Infinitesimal area element $d\boldsymbol{a} = \hat{\boldsymbol{z}}sdsd\varphi$ (top of cylinder),
$= \hat{\boldsymbol{s}}sd\varphi dz$ (cylinder wall).



Cylindrical coordinates (continued)

- The more common vector derivatives

$$\nabla T = \frac{\partial T}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial T}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial T}{\partial z}\hat{\boldsymbol{z}}$$

$$\nabla \cdot \boldsymbol{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial}{\partial \varphi}(v_\varphi) + \frac{\partial}{\partial z}(v_z)$$

$$\nabla \times \boldsymbol{v} = \left(\frac{1}{s}\frac{\partial}{\partial \varphi}(v_z) - \frac{\partial}{\partial z}(v_\varphi)\right)\hat{s} + \left(\frac{\partial}{\partial z}(v_s) - \frac{\partial}{\partial s}(v_z)\right)\hat{\boldsymbol{\varphi}} + \frac{1}{s}\left(\frac{\partial}{\partial s}(s v_\varphi) - \frac{\partial}{\partial \varphi}(v_s)\right)\hat{\boldsymbol{z}}$$

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}$$

The Dirac delta function

- In problem 1.16, on this week's homework, you will show that the vector function

$$\boldsymbol{v}(r,\varphi,\vartheta) = \frac{1}{r^2}\hat{\boldsymbol{r}}$$

  has divergence that is zero except at the origin, where it's infinite.

- However, you will also show in problem 1.39 that the integral of the divergence of this function, over any sphere centered at the origin, is neither zero nor infinity, but instead is

$$\int_{\text{sphere}} \nabla\cdot\boldsymbol{v}\,d\tau = \oint_{\substack{\text{sphere's}\\\text{surface}}} \boldsymbol{v}\cdot d\boldsymbol{a} = 4\pi$$

- This turns out to be an extremely useful set of characteristics, so we frequently use the divergence of this function, and give it a special name:

$$\delta^3(\boldsymbol{r}) = \frac{1}{4\pi}\nabla\cdot\left(\frac{\hat{\boldsymbol{r}}}{r^2}\right)$$ Delta function

The Dirac delta function (continued)

- With this definition, the delta function $\delta^3(\boldsymbol{r})$ is zero everywhere except r = 0, at which it is infinite, and the integral over any volume V that contains the origin is unity:

$$\int_V \delta^3(\boldsymbol{r})\,d\tau = 1 \quad .$$

Because the integral is dimensionless, $\delta^3(\boldsymbol{r})$ itself has dimensions of $(1/\text{length})^3$.

- The one-dimensional analogue of this function is also called a delta function:

$$\int_{-a}^{a} \delta(x)\,dx = 1 \quad \text{if } a \neq 0.$$

It has dimensions 1/length, and is related to the other one by

$$\delta^3(\boldsymbol{r}) = \delta^3(x,y,z) = \delta(x)\delta(y)\delta(z) \quad .$$

The Dirac delta function (continued)

Why is the delta function useful?

- It's a nice way mathematically to express compact entities such as point charges, when we have to describe them with differential equations. For instance, the charge density – electric charge per unit volume – of a point charge q can be written as (Problem 1.47a):

$$\rho(\boldsymbol{r}) = q\delta^3(\boldsymbol{r}) \quad .$$

- It's absurdly easy to integrate expressions containing delta functions, because the integrand is zero everywhere except on the delta function's "spike:"

$$\int_V f(\boldsymbol{r})\delta^3(\boldsymbol{r})d\tau = \begin{cases} f(0) & \text{if } V \text{ contains the origin,} \\ 0 & \text{if not.} \end{cases}$$

$$\int_{-a}^{a} f(x)\delta(x)dx = \begin{cases} f(0) & \text{if } a \neq 0, \\ 0 & \text{if not.} \end{cases}$$

The Dirac delta function (continued)

- Examples, from Griffiths problem 1.44 and 1.49.

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

$$\int_{2}^{6}\left(3x^2-2x-1\right)\delta(x-3)dx = 20$$

$$\int_{0}^{5}\cos x\,\delta(x-\pi)dx = \cos(\pi) = -1$$

$$\int_{0}^{3} x^3\delta(x+1)dx = 0$$

$$\int_{-\infty}^{\infty}\ln(x+3)\delta(x+2) = \ln 1 = 0$$

$$\int_V e^{-r}\left(\nabla\cdot\frac{\hat{\boldsymbol{r}}}{r^2}\right)d\tau = \int_V e^{-r}\left(4\pi\delta^3(\boldsymbol{r})\right)d\tau$$

$$= 4\pi e^0 = 4\pi \quad \text{if } V \text{ contains the origin}$$

$$= 0 \quad \text{otherwise}$$

Curl, divergence, fields and potentials

- As you saw in last week's homework: if a vector field $\boldsymbol{F}$ has zero curl, then it is the gradient of a **scalar potential** U, a function like the electrostatic potential V you have seen in PHYS 122 or 142.

$$\nabla \times \boldsymbol{F} = 0 \quad \Rightarrow \quad \boldsymbol{F} = -\nabla U \quad .$$

- $\boldsymbol{F}$ does not uniquely specify U; any scalar independent of x, y, z could be added to U and its gradient would not change. But any among this family of functions specifies the field $\boldsymbol{F}$ uniquely.

- Similarly, as you also showed: if a vector field $\boldsymbol{F}$ has zero divergence, then it is the curl of a **vector potential**, $\boldsymbol{W}$.

$$\nabla \cdot \boldsymbol{F} = 0 \quad \Rightarrow \quad \boldsymbol{F} = \nabla \times \boldsymbol{W} \quad .$$

  $\boldsymbol{W}$ would be like the magnetic vector potential $\boldsymbol{A}$, which is not usually introduced in PHYS 122 or 142, but will appear in association with the magnetic field $\boldsymbol{B}$ when we get to magnetostatics.

- Again $\boldsymbol{F}$ does not uniquely specify $\boldsymbol{W}$, as any gradient could be added to it without changing its curl. But any among this family of functions specify $\boldsymbol{F}$ uniquely.

Curl, divergence, fields and potentials (continued)

We mention potentials now because div, curl, fields and potentials are linked in three theorems we can use this semester, the first practically right away:

1. These conditions are equivalent*, for curl-less fields:

a. $\nabla \times \boldsymbol{F} = 0$ everywhere.

b. $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{F} \cdot d\boldsymbol{\ell}$ is path-independent.

c. $\oint \boldsymbol{F} \cdot d\boldsymbol{\ell} = 0.$

d. $\boldsymbol{F} = -\nabla U.$

2. These conditions are equivalent*, for divergenceless fields:

a. $\nabla \cdot \boldsymbol{F} = 0$ everywhere.

b. $\int_C \boldsymbol{F} \cdot d\boldsymbol{a}$ is boundary-independent.

c. $\oint \boldsymbol{F} \cdot d\boldsymbol{a} = 0.$

d. $\boldsymbol{F} = \nabla \times \boldsymbol{W}.$

* ***F*** satisfies one if and only if it satisfies all the others)

3. If $\nabla \cdot \boldsymbol{F}$ and $\nabla \times \boldsymbol{F}$ are specified, if $\nabla \cdot \boldsymbol{F}$ and $\nabla \times \boldsymbol{F} \to 0$ as $r \to \infty$ faster than $1/r^2$, and if $\boldsymbol{F} \to 0$ faster than $1/r$, then $\nabla \cdot \boldsymbol{F}$ and $\nabla \times \boldsymbol{F}$ uniquely determine ***F*** *via*

$\boldsymbol{F} = -\nabla U + \nabla \times \boldsymbol{W}.$

Helmholtz's field theorem

- You will demonstrate the equivalence of 1-2 a-d in workshop this week, in the form of Griffiths problems 1.51 and 1.52.

- It will be quicker for us to prove the Helmholtz theorem later this semester, after more experience with potentials.