Today in Physics 217: begin electrostatics

- Electric charge and electrostatics
- Units and dimensions
- Coulomb's Law
- Four examples of the use of Coulomb's Law to calculate the electric field.

The classic hand-to-the-van-de-Graaf experiment.

Foundation of electrostatics

Matter is composed of compact particles that carry electric charge:

- **Electron**: no spatial structure seen; your basic point object.

- **Proton**: has substructure, but on extremely small scale.

- The charge on electron and proton are observed to be equal and opposite, within an accuracy less than one part in 10^{20}. Nowadays the **quantum** of charge is taken to be exactly $4.803204270\times10^{-10}$ esu or $1.602176634\times10^{-19}$ coulombs, CGS or SI units respectively.

- The electron is said to have negative charge, the proton positive charge.

    - The signs we use are a historical accident (and the first permanent mark on science made by an American); the electron could just as well have been defined as the carrier of positive charge.

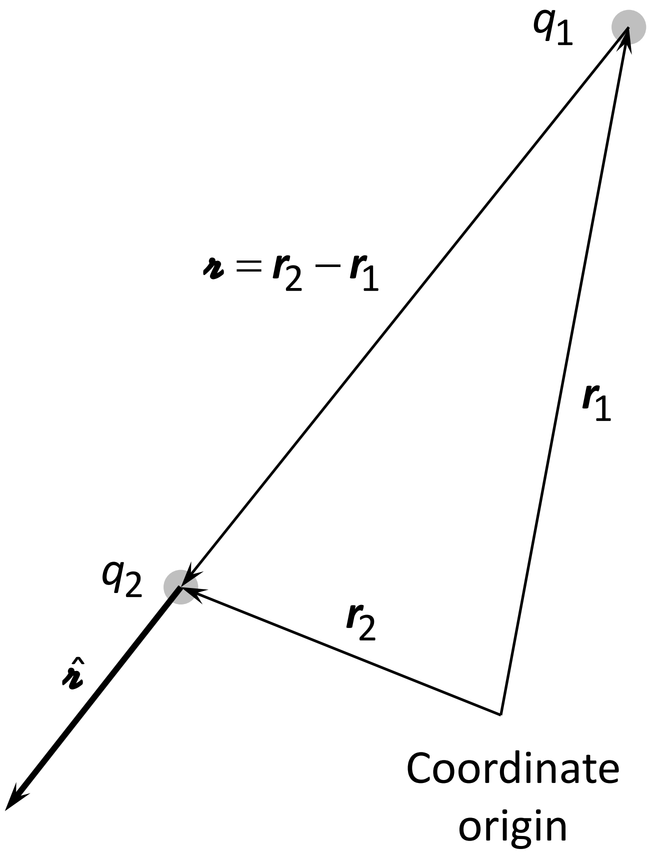
Foundation of electrostatics (continued)

Five memorable facts about electric charge, determined experimentally, e.g. in the Millikan oil-drop experiment:

- Charge is a **scalar** quantity.
- **Conservation**: the algebraic sum of charges (net charge) of an isolated system never changes.
- **Quantization**: all charged bodies have charges that are integer multiples of the quantum of charge.
- **Coulomb's law**: the force between two charges q_1 and q_2 is given by

$$\boldsymbol{F}_2 = K\frac{q_1 q_2}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

Note the symbols, same as used by Griffiths, which we will use henceforth.



Foundation of electrostatics (continued)

- **Superposition**. The force on a test charge Q, located at $\boldsymbol{r}$, by an assembly of point charges q_i, is the vector sum of the forces on Q by the individual charges:

$$\boldsymbol{F}(\boldsymbol{r}) = K\frac{Qq_1}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + K\frac{Qq_2}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \ldots$$

$$= Q\sum_{i=1}^{N} K\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i \equiv Q\boldsymbol{E}(\boldsymbol{r})$$

Electric field

Electrostatics is the use of this last expression, or more often its consequences, to derive $\boldsymbol{E}$ from a given distribution of q_i, or infer a distribution of q_i given $\boldsymbol{E}$.

Units and dimensions

- In Griffiths, as in most undergraduate textbooks and half of the current edition of Jackson, Coulomb's law and all the rest of the expressions of electromagnetism are written in SI units, in which $K = 1/4\pi\varepsilon_0$:

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} \quad , \qquad \varepsilon_0 = 8.85\times10^{-12}\ \mathrm{coul}^2\ \mathrm{Nt}^{-1}\ \mathrm{m}^{-2}.$$

- In Purcell, as I hear some used in PHYS 142, and as used most often by most professional physicists and astronomers, Gaussian (a.k.a. CGS) units are used, in which K = 1:

$$\boldsymbol{F} = \frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

  because the use of CGS units is advantageous in every topic of electromagnetism except electronics or materials. **You must soon be fluent in both systems**, not just conversant with both. I'll lecture and write homework solutions always in CGS, to get you used to it. (Giving final results in both systems.)

- Electric charge doesn't simply have different **units** in the two systems; it has different **dimensions**.

Units and dimensions (continued)

- Consider, for example, the Lorentz force equation and the Maxwell equations in vacuum.

CGS:

$$\boldsymbol{F} = q\boldsymbol{E} + q\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\rho \qquad \nabla\cdot\boldsymbol{B} = 0$$

$$\nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} \qquad \nabla\times\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}$$

• $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same dimensions.
• Factors of c remind one that $\boldsymbol{B}$ can be considered a to be a relativistic effect of $\boldsymbol{E}$, and that they lead to a wave equation with phase speed c.

SI:

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B}$$

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla\cdot\boldsymbol{B} = 0$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \qquad \nabla\times\boldsymbol{B} = \frac{\mu_0}{4\pi}\boldsymbol{J} + \varepsilon_0\mu_0\frac{\partial\boldsymbol{E}}{\partial t}$$

• Leads to potentials and currents in the conventional units of electronics, volts and amps; presence of ε_0 and μ_0 evocative of material properties.

Electric charge distributions

- The quantum of electric charge is so small that in macroscopic systems we can treat charge as a continuous quantity, so that Coulomb's law is

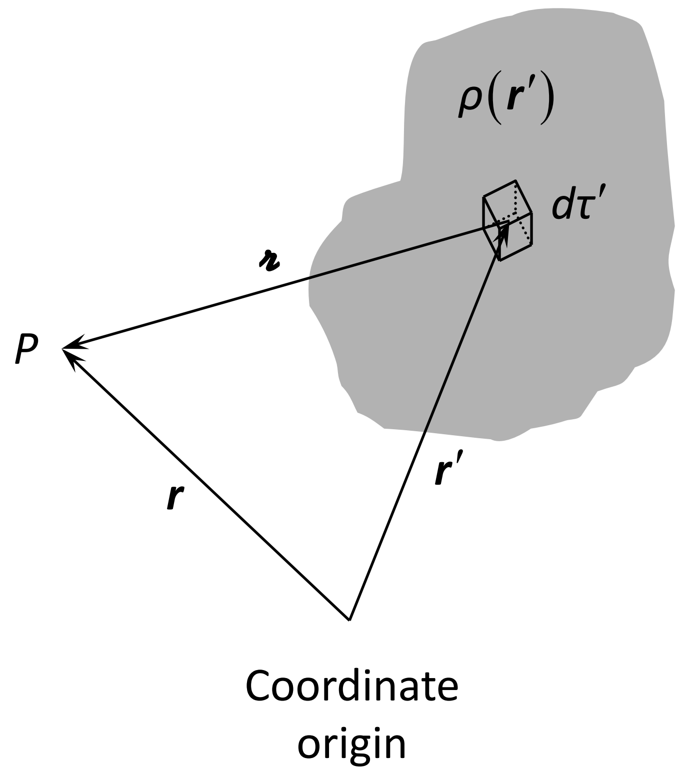
$$\boldsymbol{E}(\boldsymbol{r}) = \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dq$$

and we speak of charge **density**:

$$dq = \lambda(\boldsymbol{r}')d\ell' \qquad \lambda = \text{ charge/unit length for 1-D charges}$$
$$= \sigma(\boldsymbol{r}')da' \qquad \sigma = \text{ charge/unit area for 2-D charges}$$
$$= \rho(\boldsymbol{r}')d\tau' \qquad \rho = \text{ charge/unit volume for 3-D charges}$$

- As Griffiths does, I'll try always to use ***r***′ to point at elements of the charge distribution, and ***r*** for the **test** point *P* at which ***E*** is calculated.

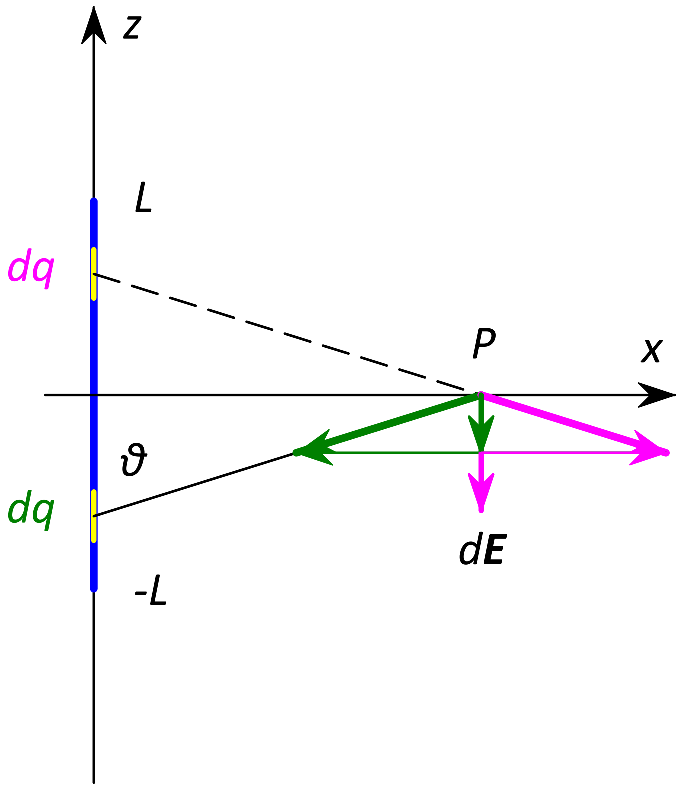
- Note again that the displacement between charge element and *P*, $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$, is equal to neither ***r*** nor ***r***′ unless one chooses the coordinate system to make it so.



Examples: calculation of *E* from charge distributions, using Coulomb's law

If there is no global symmetry (e.g. spherical, axial, …) to the layout of *P* and the charge distribution, Coulomb's Law may be the best or only path to ***E***. One usually finds other symmetries to exploit, though, as in the following.

Consider a finite linear charge, with density $\lambda(\mathbf{r}') = -\lambda_0 \frac{z'^2}{L^2}, -L \leq z' < 0, \lambda_0 \frac{z'^2}{L^2},$ $0 \leq z' \leq L, = 0$ otherwise. Find ***E*** at $\mathbf{r} = (x \quad 0 \quad 0)$.



- As always, start by drawing a good diagram, as at right. The two halves of the line are symmetrically, oppositely charged; the total charge is zero.
- Identify a representative charge element *dq*. Here is one from each side: $dq = \lambda_0 z'^2 dz' / L^2$, $dq = -\lambda_0 (-z')^2 dz' / L^2$.
- Look for symmetries to exploit. Here, we note that the *x* components for *d**E*** from the two charge elements at ±*z'* are equal and opposite while their *z* components are equal, parallel, and, if $\lambda_0 \gtrless 0$, point along $\pm\hat{\mathbf{z}}$.

Calculation of *E* from charge distributions, using Coulomb's law (cont'd)

- Clarify the geometry of the components you keep. Here, note that the *z* components of $d\boldsymbol{E}$ contain factors of

$$\cos\vartheta = \frac{z'}{\mathscr{r}} = \frac{z'}{\sqrt{x^2 + z'^2}} \quad .$$

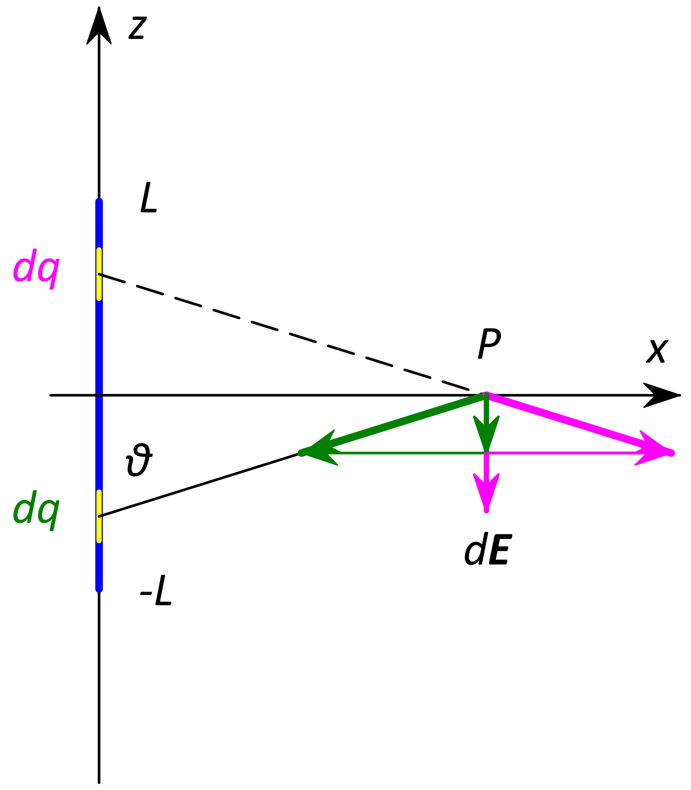
- Now write out Coulomb's Law and carry out the integration:

$$\boldsymbol{E} = \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dq = -2\frac{\lambda_0 \hat{\boldsymbol{z}}}{L^2} \int_0^L dz' \frac{z'^3}{\left(x^2 + z'^2\right)^{3/2}} \quad .$$

Note $\times 2$ and integration over just the upper half of the line.

x components are ignored, as they cancel. The factor of 2 is from integrating only over the upper half of the line, as they give identical contributions to the *z* component.

- Substitute: $u = x^2 + z'^2$, $du = 2z'dz'$, $x^2 \le u \le x^2 + L^2$, giving...

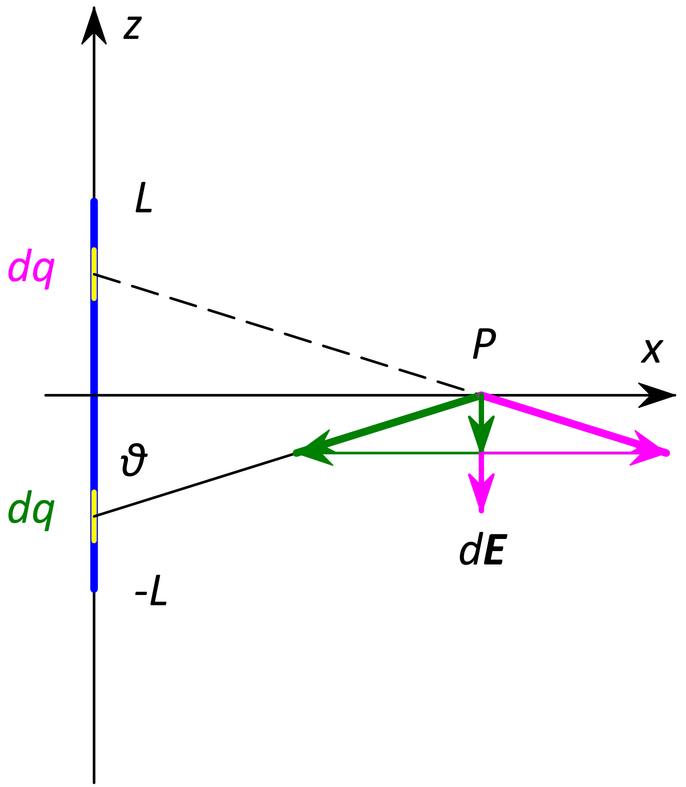


Calculation of *E* from charge distributions, using Coulomb's law (cont'd)

$$\boldsymbol{E} = -\frac{\lambda_0 \hat{\boldsymbol{z}}}{L^2}\int_{x^2}^{L^2+x^2} du \frac{u - x^2}{u^{3/2}} = \frac{\lambda_0}{L^2}\int_{x^2}^{L^2+x^2}\left(u^{-1/2} - x^2 u^{-3/2}\right)du$$

$$= -\frac{\lambda_0 \hat{\boldsymbol{z}}}{L^2}\left(2u^{1/2}\Big|_{x^2}^{L^2+x^2} + 2x^2 u^{-1/2}\Big|_{x^2}^{L^2+x^2}\right) = \frac{2\lambda_0 \hat{\boldsymbol{z}}}{L^2}\left(2x - \sqrt{L^2+x^2} - \frac{x^2}{\sqrt{L^2+x^2}}\right)$$

$$= \boxed{\frac{2\lambda_0 \hat{\boldsymbol{z}}}{L^2}\left(2x - \frac{L^2+2x^2}{\sqrt{L^2+x^2}}\right)} \quad .$$



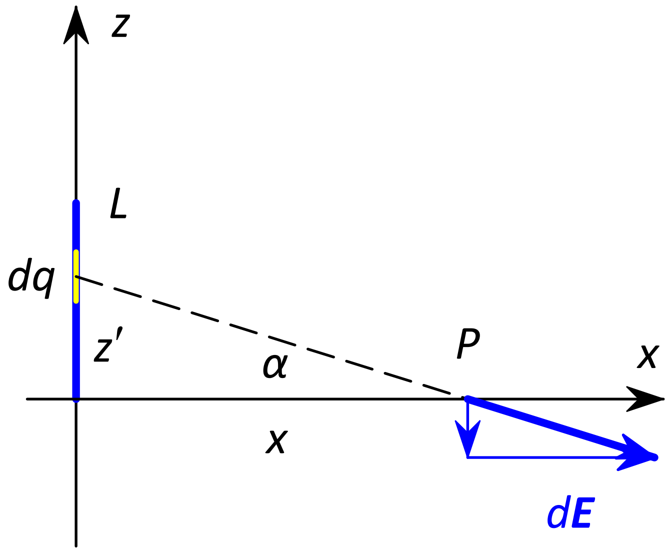
Check: if $x/L \gg 1$, $|\boldsymbol{E}| \to 0$, as it should since the total charge is zero.

$$|\boldsymbol{E}| = \frac{2\lambda_0}{L}\left(\frac{2x}{L} - \frac{1 + \frac{2x^2}{L^2}}{\sqrt{1 + \frac{x^2}{L^2}}}\right) \xrightarrow[x/L \gg 1]{} \frac{2\lambda_0}{L}\left(\frac{2x}{L} - \frac{\frac{2x^2}{L^2}}{\frac{x}{L}}\right) = 0 \quad .$$

Calculation of *E* from charge distributions, using Coulomb's law (cont'd)

No guarantee of symmetry, though.

A uniform line charge q lies between the origin and $z = L$. Calculate $\boldsymbol{E}$ along the x axis.



- Define $\lambda = q/L$ and $dq = \lambda dz'$. Directions in the diagram at right presume $q > 0$.
- Note $\cos\alpha = \dfrac{x}{\mathscr{r}} = \dfrac{x}{\sqrt{x^2 + z'^2}}$, $\sin\alpha = \dfrac{z'}{\sqrt{x^2 + z'^2}}$.
- Thus

$$\boldsymbol{E} = \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dq = \lambda \int_0^L dz' \left(\frac{x\hat{\boldsymbol{x}}}{\left(x^2 + z'^2\right)^{3/2}} - \frac{z'\hat{\boldsymbol{z}}}{\left(x^2 + z'^2\right)^{3/2}} \right) .$$

Calculation of *E* from charge distributions, using Coulomb's law (cont'd)

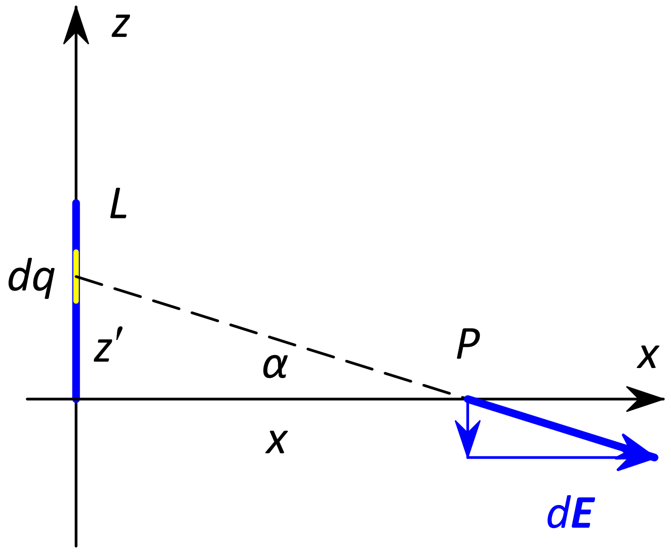
- The second integral yields to the same substitution used in the last example:

$$u = x^2 + z'^2,\ du = 2z'dz',\ x^2 \le u \le x^2 + L^2.$$

- The first integral needs a trig substitution, taking a cue from the diagram:

$$\tan\alpha = \frac{z'}{x},\ dz' = x\frac{d}{d\alpha}(\tan\alpha)d\alpha = x\sec^2\alpha d\alpha,$$

$$0 \le \alpha \le \arctan\left(\frac{L}{x}\right) \quad .$$



- So

$$\boldsymbol{E} = \lambda x^2\hat{\boldsymbol{x}} \int_0^{\arctan(L/x)} \frac{\sec^2\alpha}{\left(x^2 + x^2\tan^2\alpha\right)^{3/2}} d\alpha - \frac{\lambda\hat{\boldsymbol{z}}}{2}\int_{x^2}^{x^2+L^2}\frac{du}{u^{3/2}}$$

$$= \frac{\lambda\hat{\boldsymbol{x}}}{x}\int_0^{\arctan(L/x)} \frac{\sec^2\alpha}{\sec^3\alpha\left(\cos^2\alpha + \sin^2\alpha\right)^{3/2}} d\alpha - \frac{\lambda\hat{\boldsymbol{z}}}{2}\int_{x^2}^{x^2+L^2}\frac{du}{u^{3/2}}$$

Calculation of *E* from charge distributions, using Coulomb's law (cont'd)

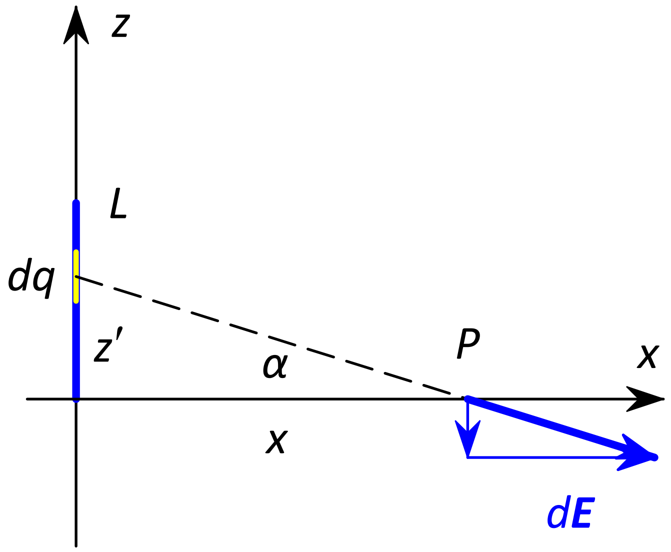
$$\boldsymbol{E} = \frac{\lambda\hat{\boldsymbol{x}}}{x}\int_0^{\arctan(L/x)} \cos\alpha\, d\alpha - \frac{\lambda\hat{\boldsymbol{z}}}{2}\int_{x^2}^{x^2+L^2} \frac{du}{u^{3/2}}$$

$$= \frac{\lambda\hat{\boldsymbol{x}}}{x}\sin\alpha\Big|_0^{\arctan(L/x)} + \lambda\hat{\boldsymbol{z}}\frac{1}{\sqrt{u}}\Bigg|_{x^2}^{x^2+L^2} = \frac{\lambda\hat{\boldsymbol{x}}}{x}\sin\left(\arctan\frac{L}{x}\right) + \lambda\hat{\boldsymbol{z}}\left(\frac{1}{\sqrt{x^2+L^2}} - \frac{1}{x}\right)$$

$$= \boxed{\frac{\lambda\hat{\boldsymbol{x}}}{x}\frac{L}{\sqrt{x^2+L^2}} + \lambda\hat{\boldsymbol{z}}\left(\frac{1}{\sqrt{x^2+L^2}} - \frac{1}{x}\right)} \; .$$

- Check $x \gg L$:

$$\boldsymbol{E} \to \frac{\lambda\hat{\boldsymbol{x}}}{x}\frac{L}{x} + \lambda\hat{\boldsymbol{z}}\left(\frac{1}{x} - \frac{1}{x}\right) = \frac{q}{x^2}\hat{\boldsymbol{x}} \; . \quad ✓✓$$



Calculation of *E* from charge distributions, using Coulomb's law (cont'd)

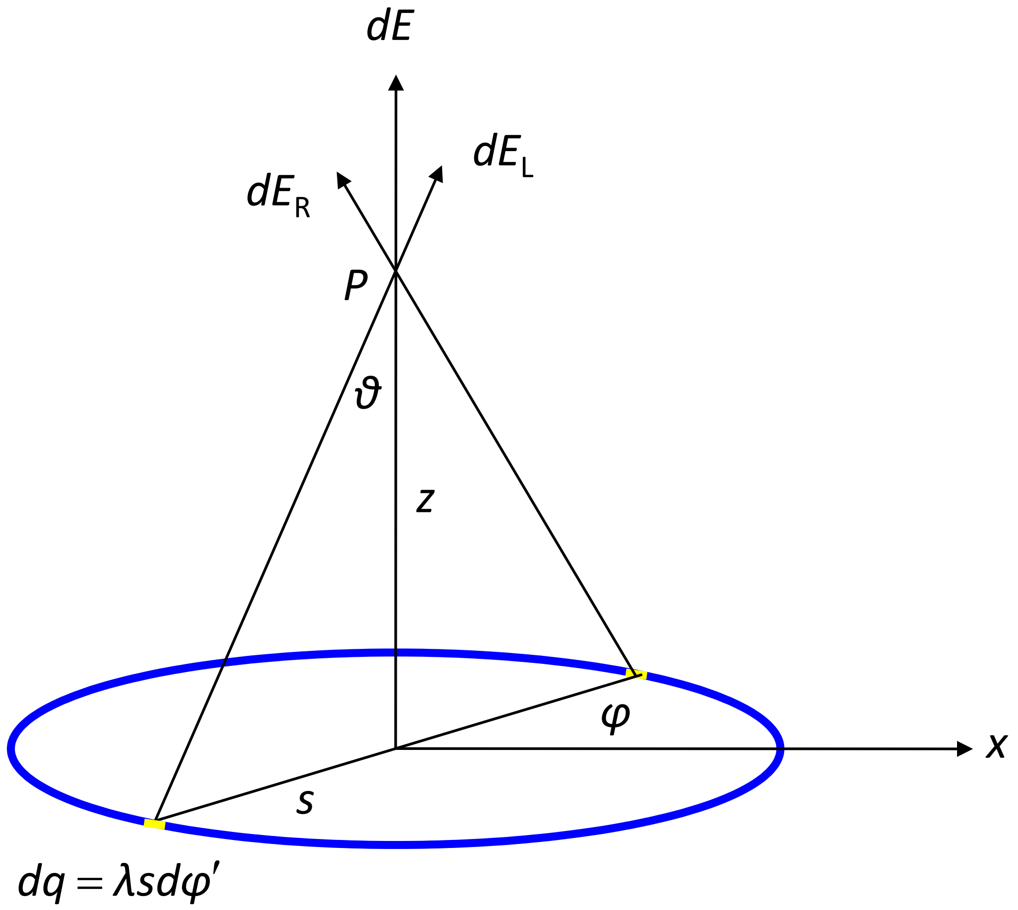
I'm sure you've done this one before:

Find ***E*** for distance *z* above the center of a circular loop, of radius *s,* which carries a uniform charge per unit length *λ*.

- Define $dq = \lambda s d\varphi'$, note $\cos\vartheta = \frac{z}{\sqrt{s^2+z^2}}$.
- Observe that the horizontal (*x,* y) components of ***dE*** from charge elements on opposite sides of the loop cancel, and vertical components add.

$$\boldsymbol{E} = \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dq = \frac{\lambda s z \hat{\mathbf{z}}}{\left(s^2+z^2\right)^{3/2}} \int_0^{2\pi} d\varphi' = \boxed{\frac{2\pi\lambda s z \hat{\mathbf{z}}}{\left(s^2+z^2\right)^{3/2}}} \ .$$

- Check $z \gg s$: $\boldsymbol{E} = \frac{(2\pi\lambda s) z\hat{\mathbf{z}}}{z^3} = \frac{q}{z^2}$. ✓✓

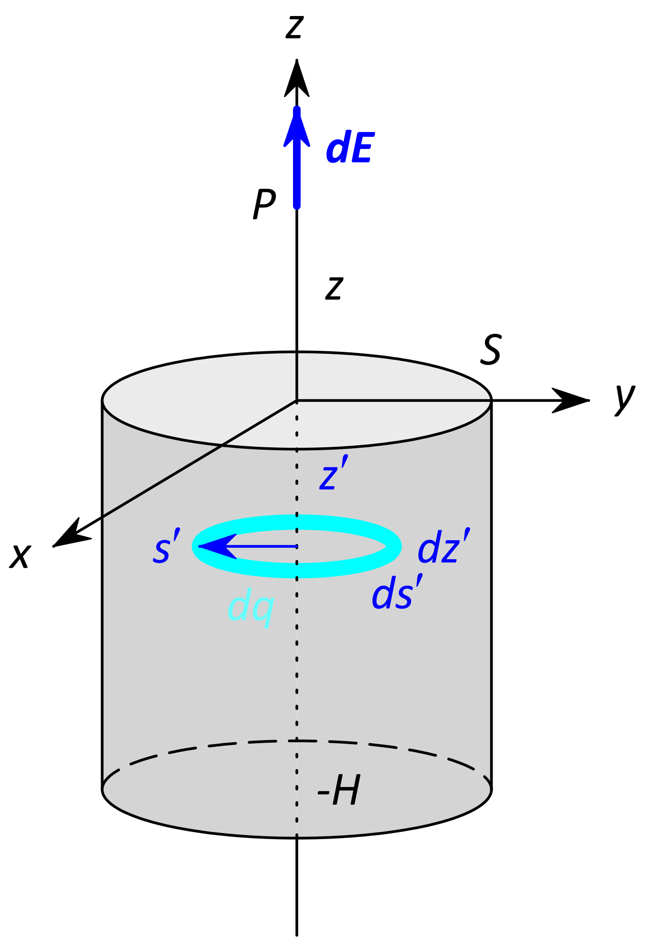


Calculation of *E* from charge distributions, using Coulomb's law (cont'd)

Use convenient and appropriate charge elements as you find them.

Find ***E*** a distance *z* along the axis from a cylinder with uniform charge density *ρ*, radius *S*, and height *H*.

- A diagram is at right; a cylindrical-coordinate-system origin is arbitrarily placed at top center of the cylinder.
- Analogy with the previous example provides a good charge element, a ring with radius *s* and thickness $ds'\times dz'$: $dq = 2\pi\rho s' ds' dz'$. At *P*, ***dE*** from this element points along $+\hat{\boldsymbol{z}}$, as we just showed.
- And thus $\boldsymbol{dE} = \dfrac{2\pi\rho s'(z-z')ds'dz'}{\left(s'^2+(z-z')^2\right)^{3/2}}\hat{\boldsymbol{z}}$.
- The integration bounds are $0 \le s' \le S,\ -H \le z' \le 0$.



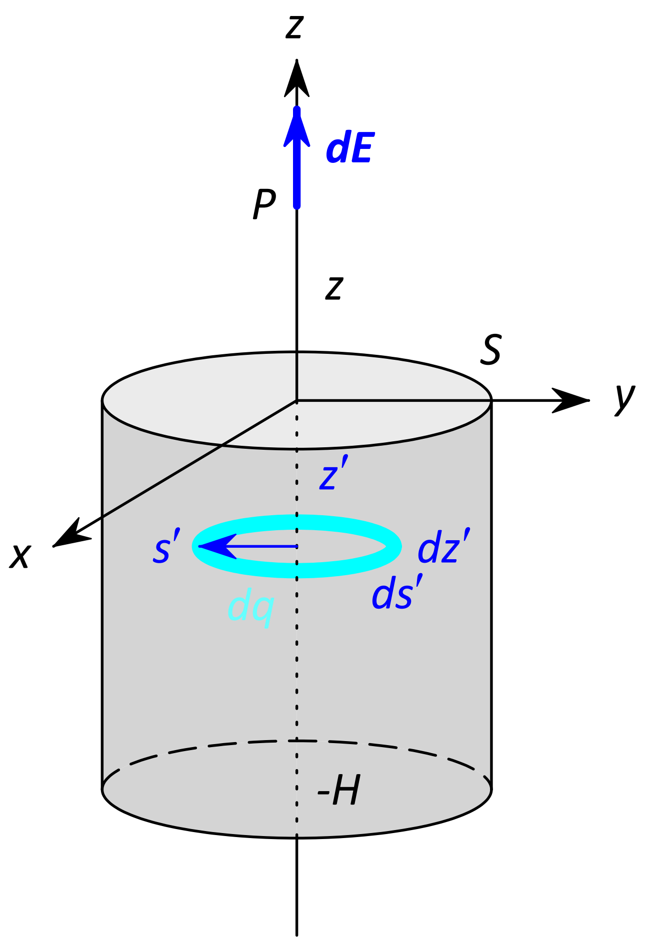
Calculation of *E* from charge distributions, using Coulomb's law (cont'd)

- Now carry out the integration. Substitute $u = s'^2 + (z - z')^2$, $du = 2s'ds'$, $(z - z')^2 \le u \le S^2 + (z - z')^2$, giving

$$\mathbf{E} = 2\pi\rho\hat{\mathbf{z}} \int_{-H}^{0} (z - z') \int_{0}^{S} \frac{s'ds'dz'}{\left(s'^2 + (z - z')^2\right)^{3/2}} = \pi\rho\hat{\mathbf{z}} \int_{-H}^{0} (z - z')dz' \int_{(z-z')^2}^{S^2+(z-z')^2} \frac{du}{u^{3/2}}$$

$$= \pi\rho\hat{\mathbf{z}} \int_{-H}^{0} (z - z')dz' \left[-\frac{2}{u^{1/2}} \right]_{(z-z')^2}^{S^2+(z-z')^2} = 2\pi\rho\hat{\mathbf{z}} \int_{-H}^{0} \left(1 - \frac{(z - z')}{\sqrt{S^2 + (z - z')^2}} \right) dz'$$

$$= 2\pi\rho H\hat{\mathbf{z}} - 2\pi\rho\hat{\mathbf{z}} \int_{-H}^{0} \frac{(z - z')}{\sqrt{S^2 + (z - z')^2}} dz' \quad .$$



- Substitute $f = z - z'$, $df = -dz'$, $z + H \ge f \ge z$:

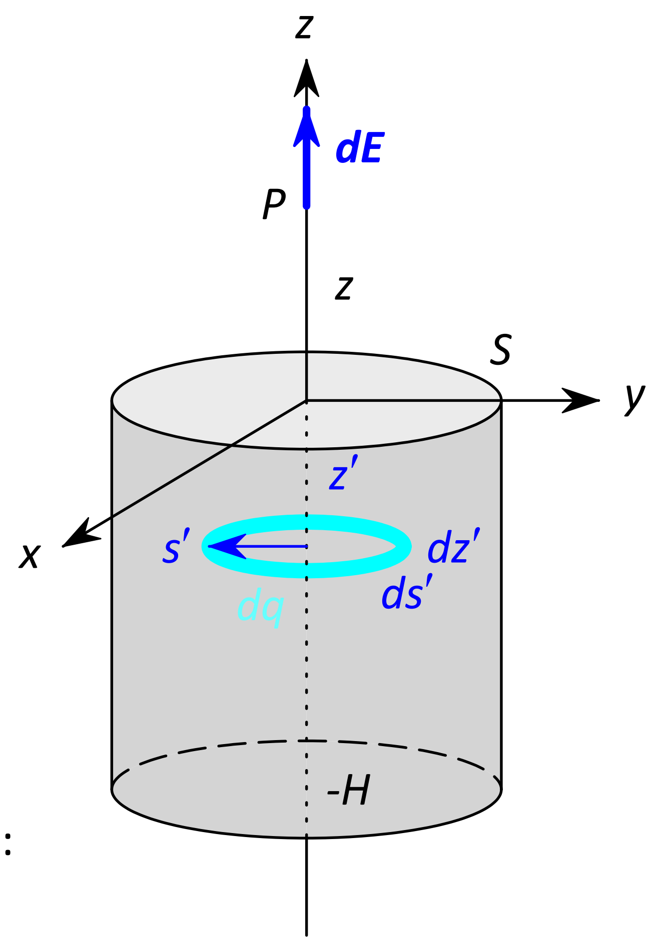
Calculation of *E* from charge distributions, using Coulomb's law (cont'd)

$$\mathbf{E} = 2\pi\rho H\hat{\mathbf{z}} + 2\pi\rho\hat{\mathbf{z}}\int_{z+H}^{z}\frac{f}{\sqrt{S^2+f^2}}df \quad .$$

- Then $g = S^2 + f^2$, $dg = 2fdf$, $S^2 + (z+H)^2 \geq g \geq S^2 + z^2$ again:

$$\mathbf{E} = 2\pi\rho H\hat{\mathbf{z}} + \pi\rho\hat{\mathbf{z}}\int_{S^2+(z+H)^2}^{S^2+z^2}\frac{dg}{\sqrt{g}} = 2\pi\rho H\hat{\mathbf{z}} - 2\pi\rho\hat{\mathbf{z}}\sqrt{g}\Big|_{S^2+z^2}^{S^2+(z+H)^2}$$

$$= 2\pi\rho H\hat{\mathbf{z}} - 2\pi\rho\hat{\mathbf{z}}\left(\sqrt{S^2+(z+H)^2} - \sqrt{S^2+z^2}\right) \quad .$$



- Check for $z \gg S, H$: first note that the total charge of the cylinder is $Q = \rho V = \rho\pi S^2 H$:

$$\mathbf{E} = \frac{2Q}{S^2}\hat{\mathbf{z}} - \frac{2Qz}{S^2H}\hat{\mathbf{z}}\left(\sqrt{\frac{S^2}{z^2}+\left(1+\frac{H}{z}\right)^2} - \sqrt{\frac{S^2}{z^2}+1}\right) \quad .$$

Calculation of *E* from charge distributions, using Coulomb's law (cont'd)

- Now use the binomial expansion, $(1+x)^a \cong 1 + ax + \frac{a(a-1)}{2}x^2 + \ldots,\ |x| \ll 1,$ to find a **third**-order approximation for **E** in the limit $S/z,\ H/z \ll 1$, and see lots of terms cancel. Ignore terms of order higher than third $\left(\text{e.g. } \propto 1/z^4\right)$.

$$\boldsymbol{E} = \frac{2Q}{S^2}\hat{\boldsymbol{z}} - \frac{2Qz}{S^2H}\hat{\boldsymbol{z}}\left(\sqrt{1+\left(2\frac{H}{z}+\frac{S^2}{z^2}+\frac{H^2}{z^2}\right)} - \sqrt{1+\frac{S^2}{z^2}}\right)$$

$$\cong \frac{2Q}{S^2}\hat{\boldsymbol{z}} - \frac{2Qz}{S^2H}\hat{\boldsymbol{z}}\left(1+\frac{1}{2}\left(2\frac{H}{z}+\frac{S^2}{z^2}+\frac{H^2}{z^2}\right) - \frac{1}{8}\left(2\frac{H}{z}+\frac{S^2}{z^2}+\frac{H^2}{z^2}\right)^2 - 1 - \frac{1}{2}\frac{S^2}{z^2}+\frac{1}{8}\left(\frac{S^2}{z^2}\right)^2\right)$$

$$= \frac{2Q}{S^2}\hat{\boldsymbol{z}} - \frac{2Qz}{S^2H}\hat{\boldsymbol{z}}\left(1+\frac{1}{2}\left(2\frac{H}{z}+\frac{S^2}{z^2}+\frac{H^2}{z^2}\right) - \frac{1}{8}\left(4\frac{H^2}{z^2}+4\frac{H^3}{z^3}+\frac{S^4}{z^4}+\frac{H^4}{z^4}+2\frac{H^2S^2}{z^4}+4\frac{HS^2}{z^3}\right) - 1 - \frac{1}{2}\frac{S^2}{z^2}+\frac{1}{8}\left(\frac{S^2}{z^2}\right)^2\right)$$

Ignore

$$\cong \frac{2Qz}{S^2H}\frac{HS^2}{2z^3}\hat{\boldsymbol{z}} + \frac{2Qz}{S^2H}\frac{H^3}{2z^3}\hat{\boldsymbol{z}} = \boxed{\frac{Q}{z^2}\left(1+\frac{H^2}{S^2}\right)}\quad .$$

Coulomb's law recovered, far from cylinder ✓✓

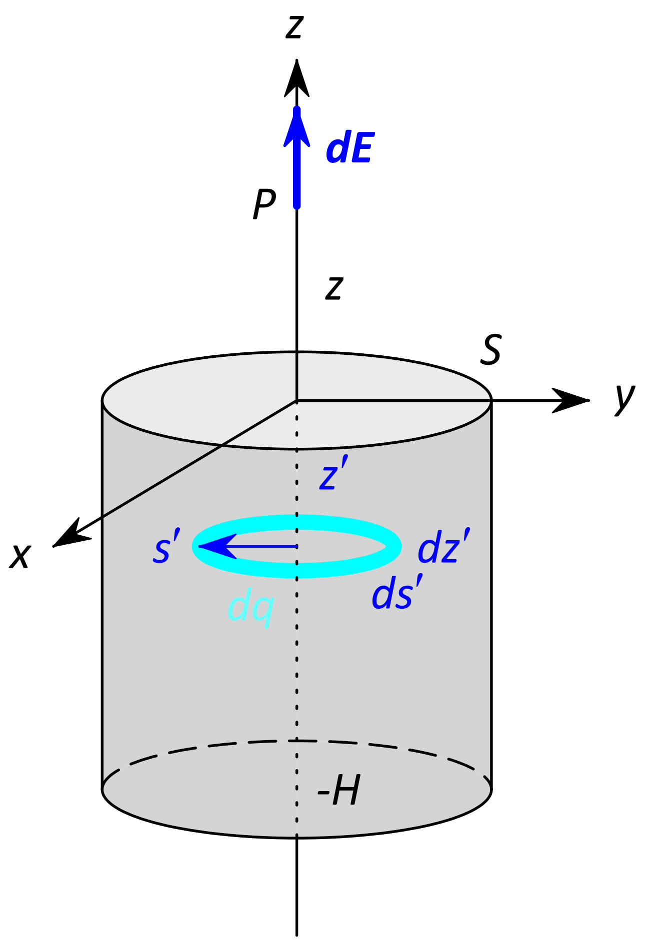
Calculation of $\boldsymbol{E}$ from charge distributions, using Coulomb's law (cont'd)

- Extra check, for comparison to near-future result: what if $S \gg z, H$, as though the cylinder were a very thin, extremely wide wafer with charge per unit area $\sigma = \rho H$? Then

$$\boldsymbol{E} = 2\pi\rho H\hat{\boldsymbol{z}} - 2\pi\rho\hat{\boldsymbol{z}}\left(\sqrt{S^2 + (z+H)^2} - \sqrt{S^2 + z^2}\right)$$

$$\cong 2\pi\rho H\hat{\boldsymbol{z}} = \boxed{2\pi\sigma\hat{\boldsymbol{z}}} \quad .$$

- As we will see next Tuesday, ✓✓.



Calculation of *E* from charge distributions, using Coulomb's law (cont'd)

- Notes, and suggestions for your submitted homework:
  - Each problem starts with a diagram. If you can't draw a good diagram as you start a problem, you should ask your instructors for help.

    A diagram is good if it shows you explicitly how to construct the Coulomb's Law integral, including trigonometric factors, symmetries, and integration bounds.
  - Integrals are carried out in detail.

    On homework and exams, please do the math in complete detail unless you are explicitly allowed to look up a result. You will need to use all you learned in MATH 161-165 or 171-174, and eventually MATH 281. Most of the integrals you will encounter can be looked up, of course, but you will benefit from the math practice.
  - Check your results in appropriate limits to make sure they're sensible, have the right dimensions, etc. You should get in the habit of doing that, for every physics problem you work out.