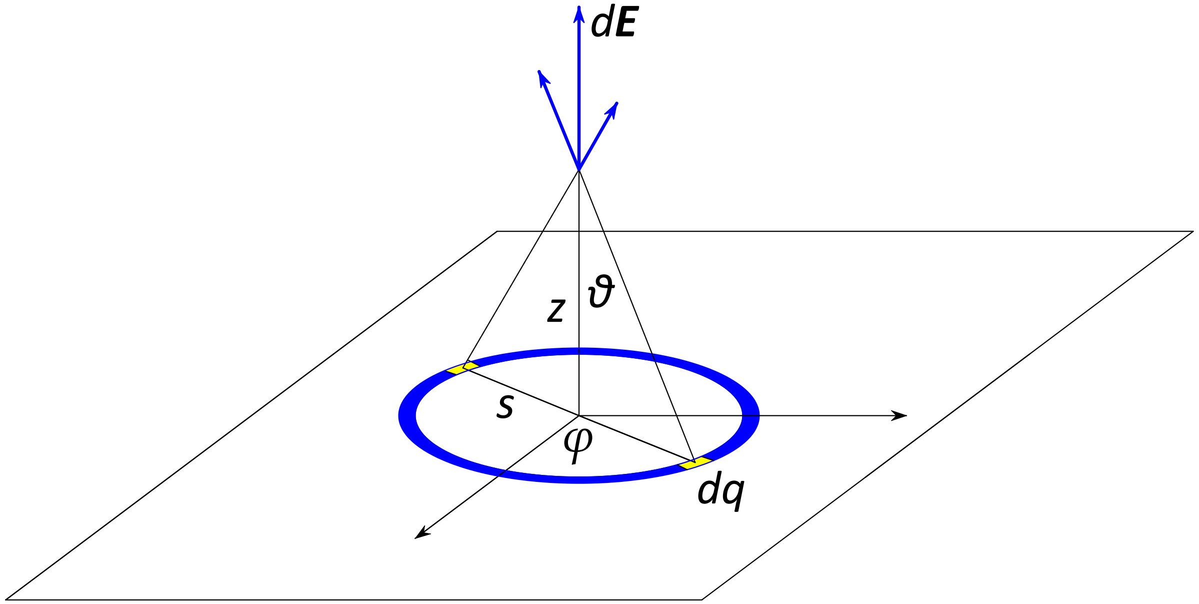
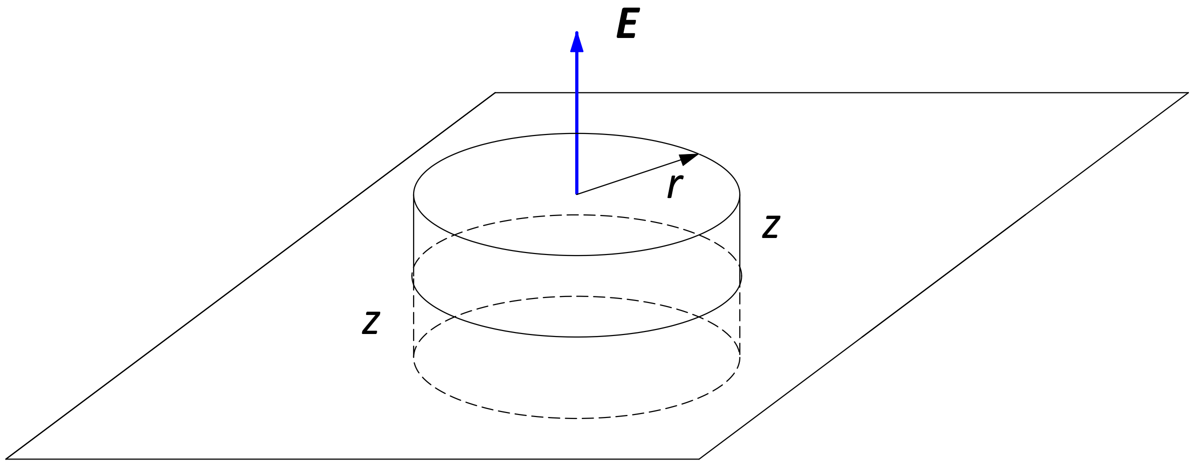
# Today in Physics 217: electrostatic ***E*** as a vector field

- Its divergence, and Gauss's Law
- Its curl, and the electric potential *V*
- Use of integral form of Gauss's Law to calculate ***E***
- Gravity and Gauss's Law



# Divergence of $E$, and Gauss's Law

- $\boldsymbol{E}$ for an arbitrary, static 3-D charge distribution occupying volume $\mathcal{V}$ is given by Coulomb's Law as:

$$\boldsymbol{E}(\boldsymbol{r}) = \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dq = \int_{\mathcal{V}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\boldsymbol{r}')d\tau' \quad \text{, so}$$

$$\nabla \cdot \boldsymbol{E} = \nabla \cdot \int_{\mathcal{V}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\boldsymbol{r}')d\tau' \quad .$$

- The gradient, which only has derivatives with respect to components of $\boldsymbol{r}$ (not $\boldsymbol{r}'$) can be taken inside the integral:

$$\nabla \cdot \boldsymbol{E} = \int_{V} \left( \nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \rho(\boldsymbol{r}')d\tau'$$

- Change of variables for the gradient: call the Cartesian components of $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$ *X,Y,Z,* as those of $\boldsymbol{r}$ are *x,y,z.* Then

# Divergence of $E$, and Gauss's Law (continued)

$$\boldsymbol{\nabla} = \left( \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right) = \left( \frac{\partial X}{\partial x}\frac{\partial}{\partial X} \quad \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y} \quad \frac{\partial Z}{\partial z}\frac{\partial}{\partial Z} \right) = \left( \frac{\partial}{\partial X} \quad \frac{\partial}{\partial Y} \quad \frac{\partial}{\partial Z} \right) = \boldsymbol{\nabla}_{\mathscr{r}} \quad ,$$

and

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \int_{\mathcal{V}} \left( \boldsymbol{\nabla}_{\mathscr{r}} \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \rho(\boldsymbol{r}') d\tau' \quad ,$$

which as we saw last week is

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \int_{\mathcal{V}} 4\pi\delta(\boldsymbol{\mathscr{r}}) \rho(\boldsymbol{r}') d\tau' = \int_{\mathcal{V}} 4\pi\delta(\boldsymbol{r} - \boldsymbol{r}') \rho(\boldsymbol{r}') d\tau' \quad ;$$

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho(\boldsymbol{r}) \quad .}$$

Gauss's Law

- Integrate this over volume, and use the divergence theorem, for a familiar result:

$$\int_{\mathcal{V}} \nabla \cdot \boldsymbol{E} d\tau = 4\pi \int_{\mathcal{V}} \rho(\boldsymbol{r}) d\tau$$

Surface $\mathcal{S}$ bounds volume $\mathcal{V}$

$$\boxed{\oint_{\mathcal{S}} \boldsymbol{E} \cdot d\boldsymbol{a} = 4\pi Q_{\text{enclosed}}}$$

Gauss's Law, integral form

# Curl of *E*, and electric potential

- Now for the curl of a field given by Coulomb's Law: $\nabla\times\boldsymbol{E}=\nabla\times\int_{\mathcal{V}}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\boldsymbol{r}')d\tau'=\int_{\mathcal{V}}\left(\nabla\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\rho(\boldsymbol{r}')d\tau'$

  Change variables as before: $=\int_{\mathcal{V}}\left(\nabla_{\mathscr{r}}\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\rho(\boldsymbol{r}')d\tau'$ .

- Call the spherical components of $\boldsymbol{\mathscr{r}}$ $\mathscr{r},\vartheta$, and $\varphi$; then, because the latter two components are zero,

$$\nabla_{\mathscr{r}}\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}=\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}\sin\vartheta\underbrace{\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)_{\varphi}}_{=0}-\frac{\partial}{\partial\varphi}\underbrace{\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)_{\vartheta}}_{=0}\right]+\frac{\hat{\boldsymbol{\vartheta}}}{\mathscr{r}}\left[\frac{1}{\sin\vartheta}\underbrace{\frac{\partial}{\partial\varphi}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}}_{=0}-\frac{\partial}{\partial\mathscr{r}}\mathscr{r}\underbrace{\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)_{\varphi}}_{=0}\right]+\frac{\hat{\boldsymbol{\varphi}}}{\mathscr{r}}\left[\frac{\partial}{\partial\mathscr{r}}\mathscr{r}\underbrace{\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)_{\vartheta}}_{=0}-\underbrace{\frac{\partial}{\partial\vartheta}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}}_{=0}\right]$$

$$=0,\text{ so}$$

$$\boxed{\nabla\times\boldsymbol{E}=0.}$$

# Curl of $\boldsymbol{E}$ (continued)

- Thus, as we saw last week, and discussed in this week's homework, Theorem 2 applies to $\boldsymbol{E}$ derived from Coulomb's Law:

$\boldsymbol{E}$ is the gradient of a scalar potential: $\boldsymbol{E} = -\boldsymbol{\nabla} V$. Electric potential

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{\ell}$$ is independent of path, given $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = 0 \quad .$$

# Summary of electrostatics, so far

Expressed in the language of field theory, with all the empirical facts (like Coulomb's Law) built in:

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \qquad = \rho/\varepsilon_0 \text{ in SI}$$

$$\oint_{\mathcal{S}} \boldsymbol{E} \cdot d\boldsymbol{a} = 4\pi Q_{\text{enclosed}} \qquad = Q_{\text{enclosed}}/\varepsilon_0 \text{ in SI}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0$$

$$\oint \boldsymbol{E} \cdot d\ell = 0$$

$$\boldsymbol{E} = -\boldsymbol{\nabla} V$$

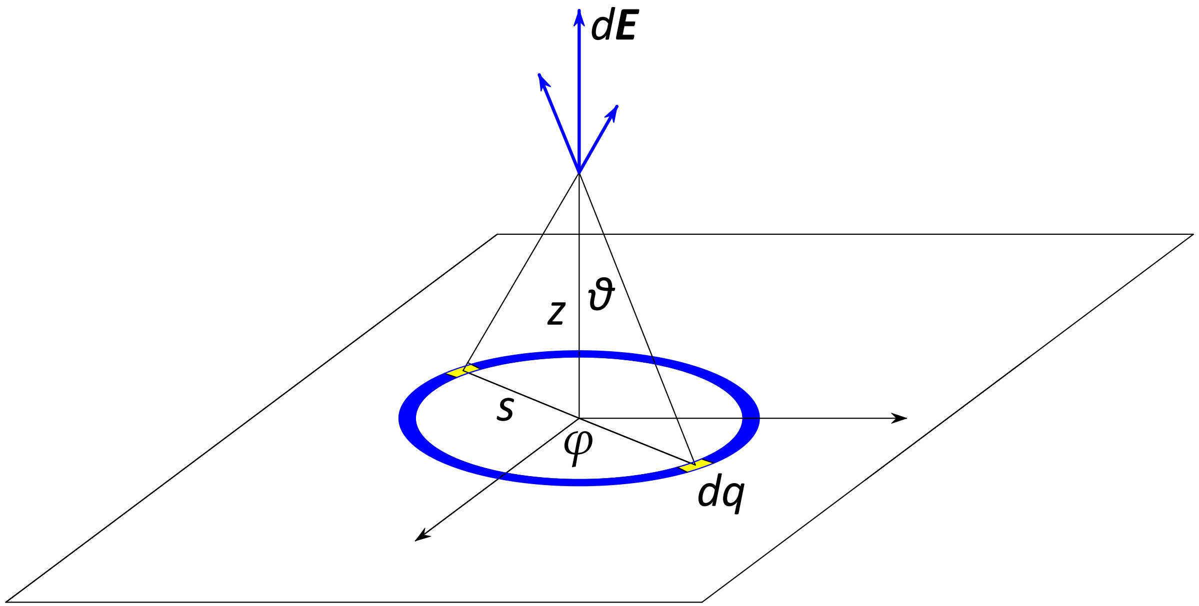
# Use of Gauss's Law in integral form

As you know well: the integral form of Gauss's Law provides a much easier way than Coulomb's Law to calculate ***E*** for symmetrical charge distributions.

Calculate the electric field from an infinite plane, parallel to *x-y,* with uniform charge per unit area *σ*; first, with Coulomb's Law, and second, with Gauss' Law. The answer, as you may remember, is $\boldsymbol{E} = \pm 2\pi\sigma\hat{\boldsymbol{z}}$. (+ above the plane, - below.)

With Coulomb's Law:

- Break the plane into annuli with radius *s* and width *ds*, and break the annuli into segments of width *sdφ*. The charge of each segment is $dq = \sigma s ds d\varphi$ .
- Horizontal components of field from segments at $\varphi$ and $\varphi + \pi$ cancel, and their vertical components add, so above the plane, we have …



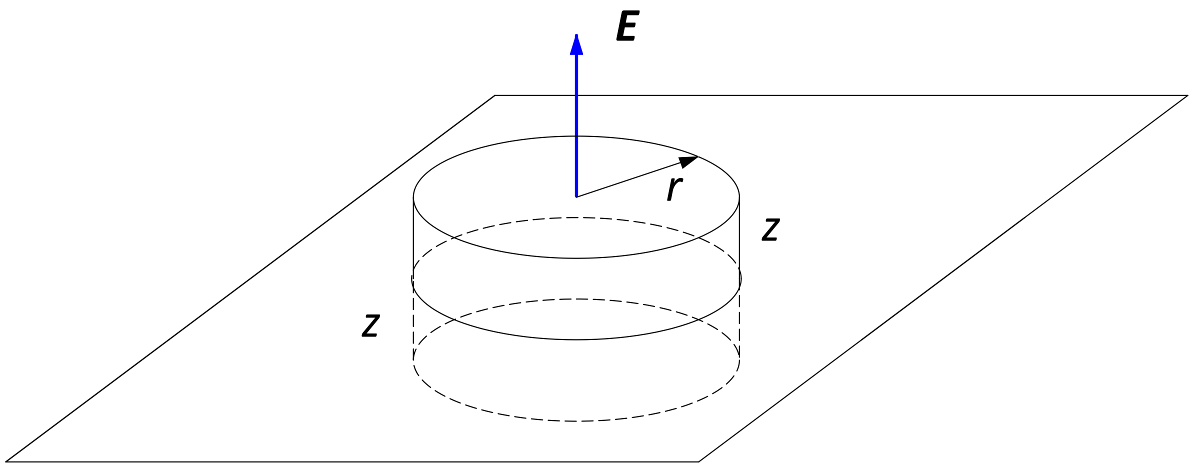
# Use of Gauss's Law in integral form (continued)

$$d\boldsymbol{E} = 2\frac{dq}{\mathscr{r}^2}\cos\vartheta\,\hat{\boldsymbol{z}} = 2\frac{\sigma s}{z^2+s^2}\frac{z}{\sqrt{z^2+s^2}}\,ds\,d\varphi\,\hat{\boldsymbol{z}}$$

$$\boldsymbol{E} = 2\sigma z\hat{\boldsymbol{z}}\int_0^{\pi} d\varphi \int_0^{\infty} s\left(s^2+z^2\right)^{-3/2} ds = \pi\sigma z\hat{\boldsymbol{z}}\int_{z^2}^{\infty} u^{-3/2}du = \pi\sigma z\hat{\boldsymbol{z}}\left.\frac{u^{-1/2}}{-1/2}\right|_{z^2}^{\infty} = 2\pi\sigma\hat{\boldsymbol{z}} \quad \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{z}} \text{ in SI}$$

With Gauss's Law:

- ***E*** must point perpendicular to, and away from, the plane, since the plane is infinite and there's no difference between the view to the right and the view to the left.
- Draw a cylinder, bisected by the plane, and calculate the flux of ***E*** through the cylinder.



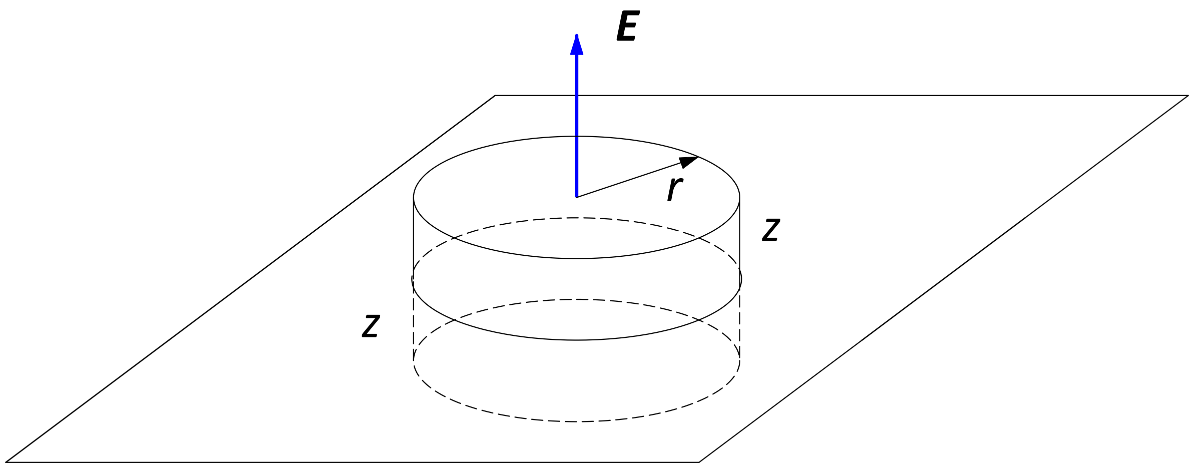
# Use of Gauss's Law in integral form (continued)

- By symmetry, ***E*** is perpendicular to the area element vectors on the cylinder walls, parallel to those on the circular faces, and constant on those faces, so

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = 2E\pi s^2 = 4\pi Q_{\text{enclosed}} = 4\pi^2 s^2 \sigma, \text{ or}$$

$$\boldsymbol{E} = \pm 2\pi\sigma\hat{\boldsymbol{z}}.$$

Conceptually harder setup – finding and exploiting symmetry – but easier math.

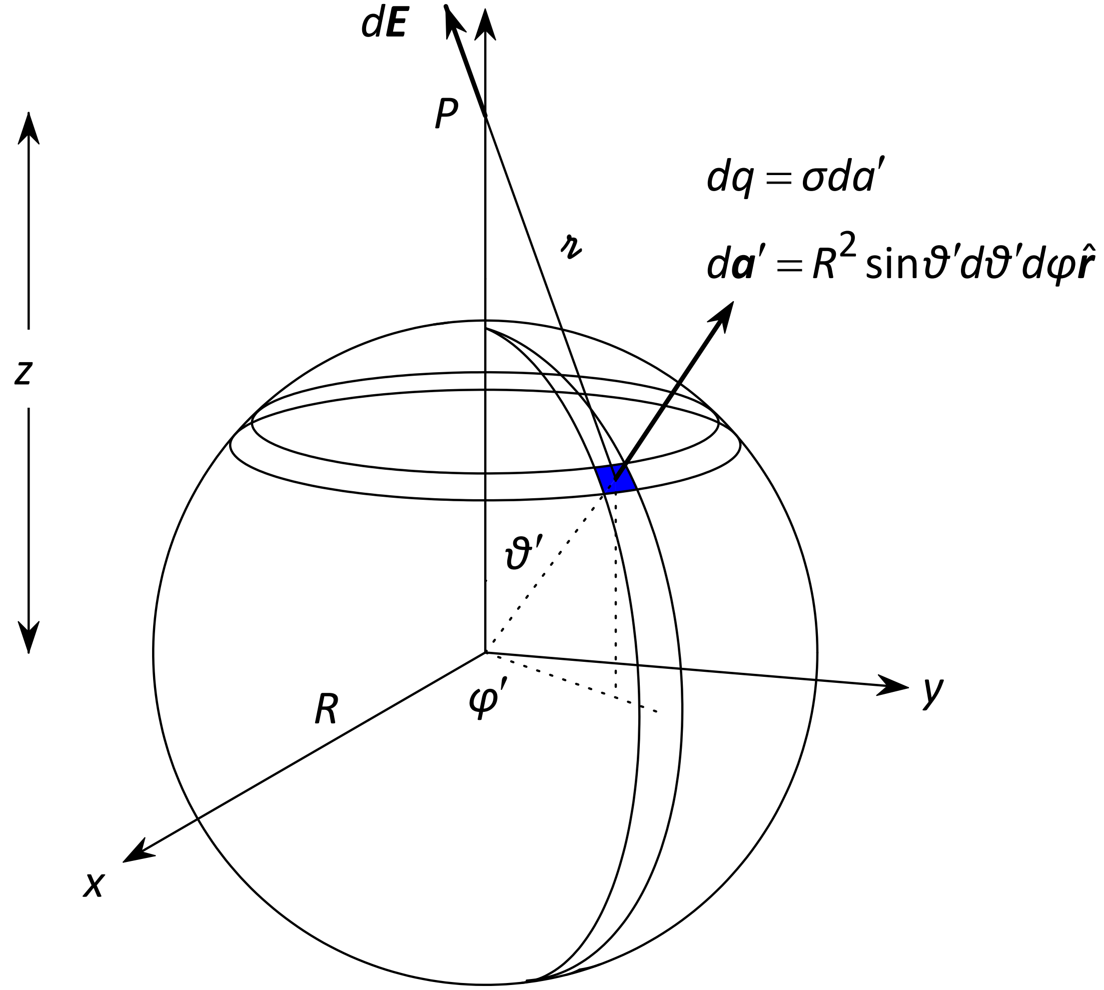


# Use of Gauss's Law in integral form (continued)

Show that the electric field ***E* outside** a uniformly-charged spherical shell – radius *R*, density *σ* – is the same as that from a point charge of the same magnitude, the same distance away as the sphere's center, and that ***E* inside** a uniformly-charged spherical shell is zero. Also show that the same result is obtained using Coulomb's Law or Gauss's Law.

With Coulomb's Law:

- We use the spherical-coordinate infinitesimal area element introduced last Tuesday, $d\mathbf{a}' = R^2 \sin\vartheta' d\vartheta' d\varphi' \hat{\mathbf{r}}$, to construct the charge element $dq = \sigma da'$.

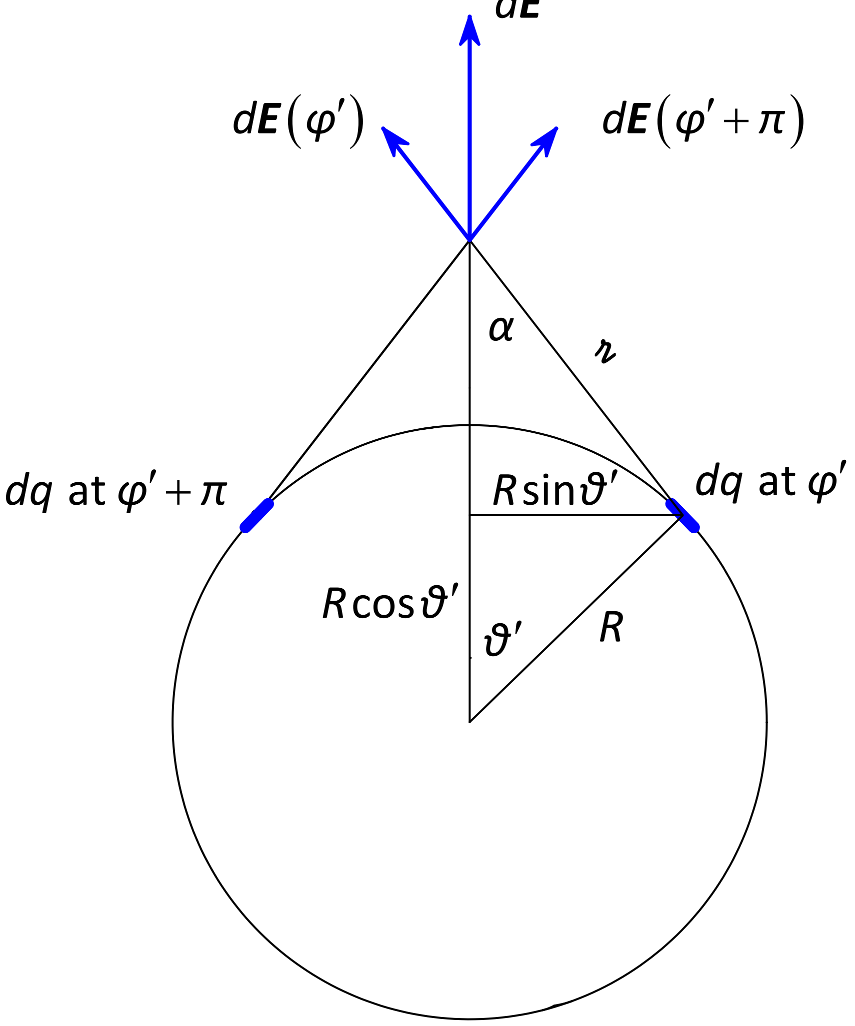


# Use of Gauss's Law in integral form (continued)

- A view to the plane at azimuth $\varphi'$ shows more easily that

$$
\begin{aligned}
\mathscr{r}^2 &= R^2 \sin^2\vartheta' + \left(z - R\cos\vartheta'\right)^2 \\
&= R^2 \sin^2\vartheta' + z^2 + R^2\cos^2\vartheta' - 2Rz\cos\vartheta' \\
&= R^2 + z^2 - 2Rz\cos\vartheta' \quad .
\end{aligned}
$$

- Consider two area elements at azimuth $\varphi'$ and $\varphi' + \pi$: as before, the horizontal components of their contribution to ***E*** cancel, and the vertical components add.



# Use of Gauss's Law in integral form (continued)

- So $$d\boldsymbol{E} = \hat{\boldsymbol{z}}2\frac{dq}{\mathscr{r}^2}\cos\alpha = \hat{\boldsymbol{z}}2\frac{\sigma R^2\sin\vartheta' d\vartheta' d\varphi'}{R^2+z^2-2Rz\cos\vartheta'}\frac{z-R\cos\vartheta'}{\sqrt{R^2+z^2-2Rz\cos\vartheta'}} = \hat{\boldsymbol{z}}2\sigma R^2\frac{\sin\vartheta'\left(z-R\cos\vartheta'\right)}{\left(R^2+z^2-2Rz\cos\vartheta'\right)^{3/2}}d\vartheta' d\varphi' \quad ,$$

$$\boldsymbol{E} = \hat{\boldsymbol{z}}2\sigma R^2\int_0^{\pi} d\varphi' \int_0^{\pi}\frac{\sin\vartheta'\left(z-R\cos\vartheta'\right)}{\left(R^2+z^2-2Rz\cos\vartheta'\right)^{3/2}}d\vartheta' \quad .$$

The first integral is trivial: it just comes out to $\pi$.

- For the second, substitute $w = \cos\vartheta'$, $dw = \not{-}\sin\vartheta' d\vartheta'$, $w = 1 \overset{\leftarrow}{\not\rightarrow} -1$:

$$\boldsymbol{E} = \hat{\boldsymbol{z}}2\pi\sigma R^2\int_{-1}^{1}\frac{\left(z-Rw\right)}{\left(R^2+z^2-2Rzw\right)^{3/2}}dw$$

# Use of Gauss's Law in integral form (continued)

- In this integral's first term, substitute $u = R^2 + z^2 - 2Rzw,\ du = \cancel{-}2Rz\,dw,\ u = R^2 + z^2 + 2Rz \overset{\leftarrow}{\cancel{\rightarrow}} R^2 + z^2 - 2Rz$:

$$z\int_{-1}^{1} \frac{1}{\left(R^2 + z^2 - 2Rzw\right)^{3/2}}dw = \frac{1}{2R}\int_{R^2+z^2-2Rz}^{R^2+z^2+2Rz} u^{-3/2}du = \frac{1}{2R}\left[-2u^{-1/2}\right]_{R^2+z^2-2Rz}^{R^2+z^2+2Rz} = \frac{1}{R}\left[\frac{1}{\sqrt{R^2+z^2-2Rz}} - \frac{1}{\sqrt{R^2+z^2+2Rz}}\right]$$

- The second term needs to be integrated by parts, to get rid of the factor of *w* in the integrand's numerator. Take

$$u = w \qquad dv = \frac{-R\,dw}{\left(R^2+z^2-2Rzw\right)^{3/2}}$$

$$du = dw \qquad v = \frac{1}{z}\frac{1}{\sqrt{R^2+z^2-2Rzw}} \quad .$$

# Use of Gauss's Law in integral form (continued)

- Then stuff these into the usual formula for integration by parts, $\int\limits_C u\,dv = uv\big|_C - \int\limits_C v\,du$:

$$\int_{-1}^{1} \frac{-Rw}{\left(R^2+z^2-2Rzw\right)^{3/2}}\,dw = \frac{1}{z}\frac{w}{\sqrt{R^2+z^2-2Rzw}}\Bigg|_{-1}^{1} + \frac{1}{z}\int_{-1}^{1}\frac{dw}{\sqrt{R^2+z^2-2Rzw}} \quad .$$

- In the second term, use (again) $u = R^2+z^2-2Rzw$, $du = \cancel{-}2Rz\,dw$, $u = R^2+z^2+2Rz \overset{\leftarrow}{\not\to} R^2+z^2-2Rz$, and it turns into

$$\frac{1}{z}\int_{-1}^{1}\frac{dw}{\sqrt{R^2+z^2-2Rzw}} = \frac{1}{2Rz^2}\int_{R^2+z^2-2Rz}^{R^2+z^2+2Rz} u^{-1/2}\,du = \frac{1}{2Rz^2}\Big[2\sqrt{u}\,\Big]_{R^2+z^2-2Rz}^{R^2+z^2+2Rz}$$

$$= \frac{1}{Rz^2}\left(\sqrt{R^2+z^2+2Rz} - \sqrt{R^2+z^2-2Rz}\right) \quad .$$

# Use of Gauss's Law in integral form (continued)

- So, putting all these terms together, and factoring out $1/z^2$ as we do, we get

$$\boldsymbol{E} = \hat{\boldsymbol{z}}\frac{2\pi\sigma R^2}{z^2}\left[\frac{z^2}{R}\left(\frac{1}{\sqrt{R^2+z^2-2Rz}}-\frac{1}{\sqrt{R^2+z^2+2Rz}}\right)+z\left(\frac{1}{\sqrt{R^2+z^2-2Rz}}-\frac{1}{\sqrt{R^2+z^2+2Rz}}\right)\right.$$

$$\left.-\frac{1}{R}\left(\sqrt{R^2+z^2+2Rz}-\sqrt{R^2+z^2-2Rz}\right)\right] \quad .$$

- It will save writing, and possibly be a little clearer, if we express the terms under the square roots as

$$\sqrt{R^2+z^2+2Rz} = \sqrt{(z+R)^2} = |z+R| \quad ,$$

$$\sqrt{R^2+z^2-2Rz} = \sqrt{(z-R)^2} = |z-R| \quad .$$

We need the **positive** roots, since they represent the length of $\mathscr{r}$, which is always positive.

# Use of Gauss's Law in integral form (continued)

- This gives us

$$\boldsymbol{E} = \hat{\boldsymbol{z}}\frac{2\pi\sigma R^2}{z^2}\left[\frac{z^2}{R}\left(\frac{1}{|z-R|}-\frac{1}{|z+R|}\right)+z\left(\frac{1}{|z-R|}-\frac{1}{|z+R|}\right)-\frac{1}{R}\left(|z+R|-|z-R|\right)\right]$$

$$= \hat{\boldsymbol{z}}\frac{2\pi\sigma R^2}{z^2}\left[\frac{z^2}{R}\left(\frac{1}{|z-R|}-\frac{1}{|z+R|}\right)+z\left(\frac{1}{|z-R|}-\frac{1}{|z+R|}\right)-\frac{1}{R}\left(\frac{|z^2-R^2|}{|z-R|}-\frac{|z^2-R^2|}{|z+R|}\right)\right]$$

$$= \hat{\boldsymbol{z}}\frac{2\pi\sigma R^2}{z^2}\left[\frac{\cancel{\frac{z^2}{R}}+z-\cancel{\frac{z^2}{R}}-R}{|z-R|}-\frac{\cancel{\frac{z^2}{R}}+z+\cancel{\frac{z^2}{R}}+R}{|z+R|}\right]$$

$$\boxed{= \hat{\boldsymbol{z}}\frac{2\pi\sigma R^2}{z^2}\left[\frac{z-R}{|z-R|}+\frac{z+R}{|z+R|}\right]}\ .$$

# Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

- The two cases to consider: $z$ larger than, or smaller than, $R$. ($P$ outside, inside)

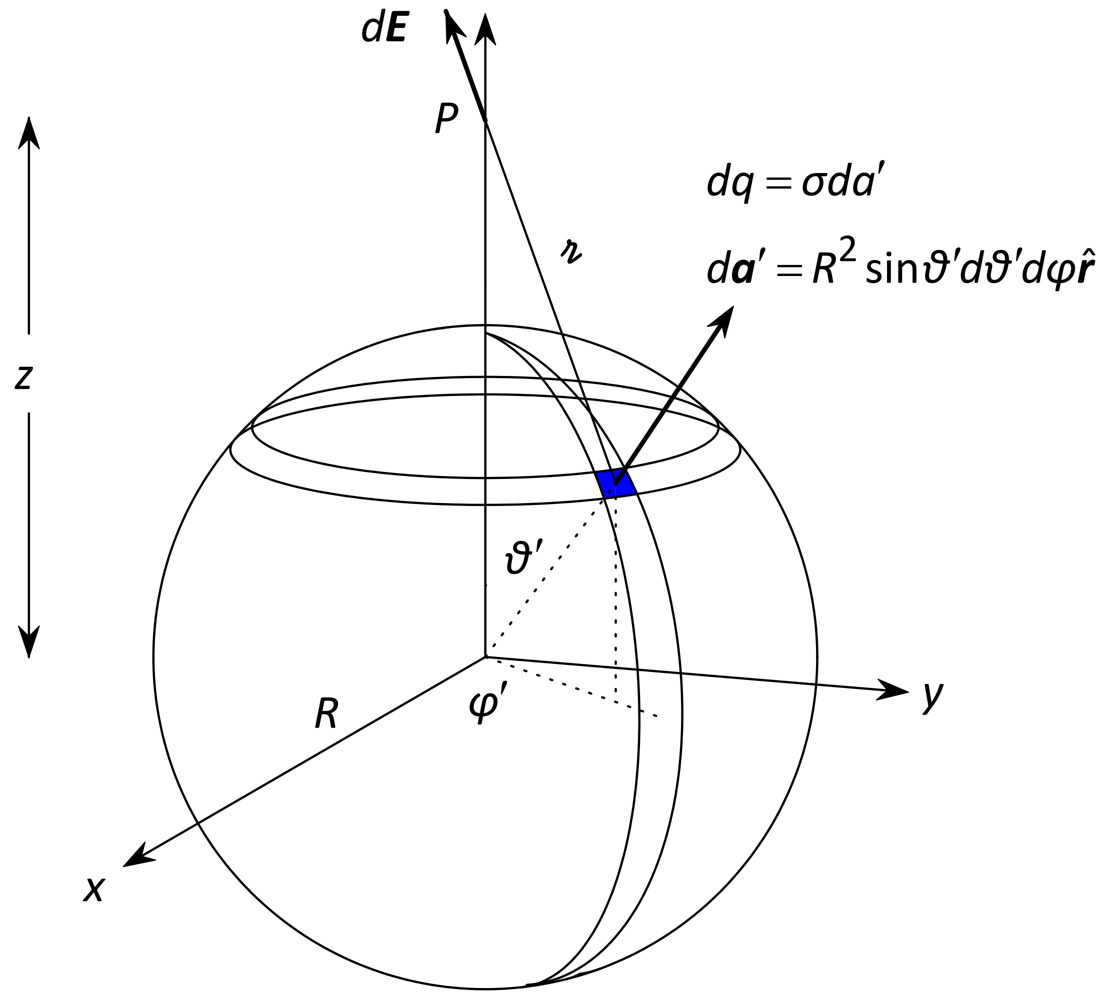
  - $z > R$ (outside):

$$\frac{z-R}{|z-R|}=1=\frac{z+R}{|z+R|} \quad \Rightarrow \quad \boxed{\mathbf{E}=\hat{\mathbf{z}}\frac{4\pi\sigma R^2}{z^2}=\hat{\mathbf{z}}\frac{Q}{z^2}}$$

  so **the spherical shell behaves to the outside world as though its charge is concentrated at the sphere's center**.

  - $z < R$ (inside) means $|z-R| = R - z$, so

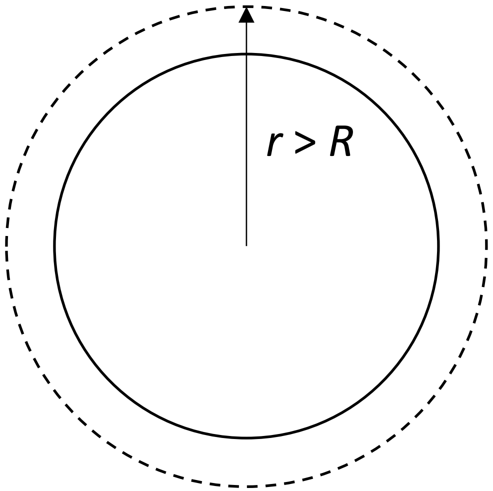
$$\frac{z-R}{R-z}+\frac{z+R}{z+R}=-1+1=0 \quad \Rightarrow \quad \boxed{\mathbf{E}=0}$$

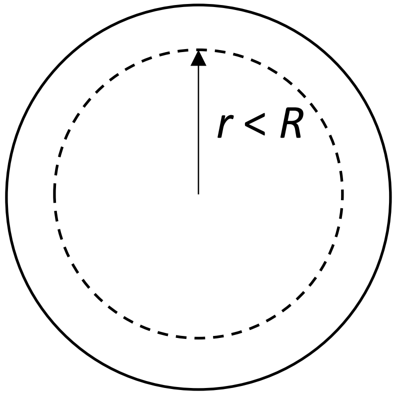


# Use of Gauss's Law in integral form (continued)

And now with Gauss's Law, as you did in PHYS 122 or 142:

- First note that the field must be spherically symmetric because the charges are, and it must point radially outward or inward – that is, ***E*** is perpendicular to all sphere's centered at the same point as the charged sphere. So draw two Gaussian spheres, one inside and one outside the charged shell:




$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = 4\pi Q_{\text{enclosed}}$$

$$r > R:$$

$$(E)\left(4\pi r^2\right) = 4\pi\left(4\pi R^2\sigma\right) = 4\pi Q \quad \Rightarrow \quad \boldsymbol{E} = \hat{\boldsymbol{r}}\frac{Q}{r^2}$$

$$r < R:$$

$$(E)\left(4\pi r^2\right) = 0 \quad \Rightarrow \quad \boldsymbol{E} = 0$$

# Gauss' Law for gravity

Newton was the first to realize these results, in the context of the other $1/r^2$ force, gravity. He convinced himself by use of a proof similar to our Coulomb's law demonstration, Gauss still not having been born by then. We could have saved Newton a lot of trouble by pointing out the following.

- The force of gravity on a mass *M* from a mass *m* is $\boldsymbol{F} = \frac{GmM}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$ .

- Gravitational forces superpose: the force on *M* from *N* charges is

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{Gm_1M}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{Gm_2M}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \ldots = M\sum_{i=1}^{N} G\frac{m_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i \equiv M\boldsymbol{g}(\boldsymbol{r}) \quad .$$

- For a continuous distribution of mass (density $\rho(\boldsymbol{r})$), the gravitational field $\boldsymbol{g}$ is obtained by letting $N \to \infty$:

$$\boldsymbol{g}(\boldsymbol{r}) = G\int_V \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\boldsymbol{r}')d\tau' \quad .$$

# Gauss' Law for gravity (continued)

- Take the divergence of both sides, and carry out the resulting integral on the RHS, as we did on pages 2-3, and we get

$$\boxed{\nabla\cdot\boldsymbol{g}=4\pi G\rho(\boldsymbol{r})}$$

Gauss's Law for gravity

- Now integrate this result over volume, and use the divergence theorem, as we also did on pages 2-3:

$$\boxed{\oint\boldsymbol{g}\cdot d\boldsymbol{a}=4\pi GM_{\text{enclosed}}}$$