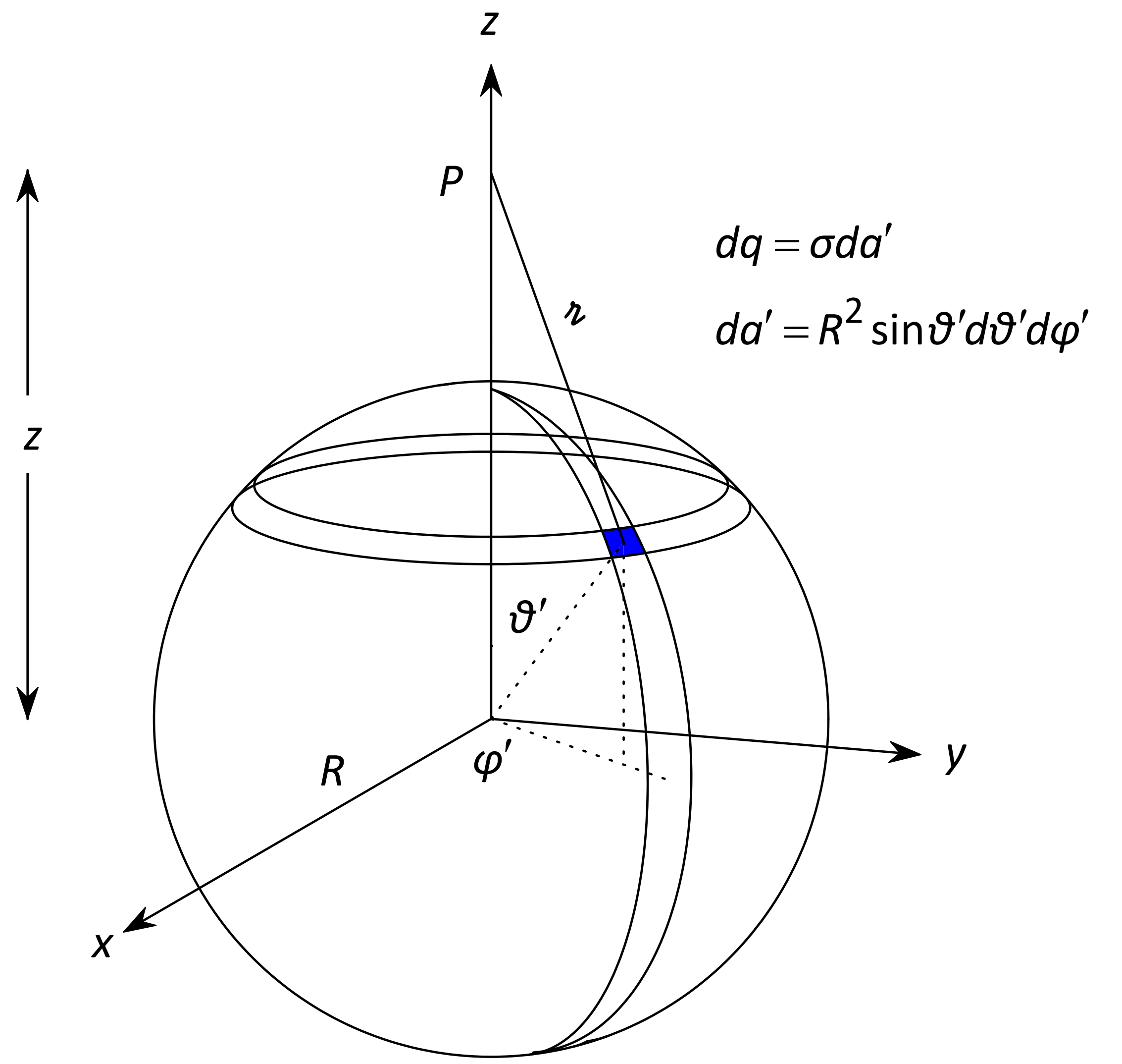
Today in Physics 217: electric potential

- Electric potential *V*
- Examples, including obtaining ***E*** from *V*
- Potential, work, and electrostatic potential energy *W*
- *W* caveats: apparent inconsistency and nonsuperposition



Electric potential

- Because $\nabla \times \boldsymbol{E} = 0$ in electrostatics, we can express $\boldsymbol{E}$ as the gradient of a scalar function:

$$\boldsymbol{E} = -\nabla V$$

where of course V is called the **electric scalar potential**.

- By the gradient theorem, we can write

$$-\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = \int_{\boldsymbol{a}}^{\boldsymbol{b}} (\nabla V) \cdot d\boldsymbol{\ell} = V(\boldsymbol{b}) - V(\boldsymbol{a}) \quad .$$

- Suppose we have agreed upon a standard reference point, $\mathcal{O}$; then

$$-\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = \int_{\mathcal{O}}^{\boldsymbol{b}} (\nabla V) \cdot d\boldsymbol{\ell} + \int_{\boldsymbol{a}}^{\mathcal{O}} (\nabla V) \cdot d\boldsymbol{\ell} \quad .$$

Electric potential (continued)

- This leads us to an integral definition of *V*:

$$V(\mathcal{P}) = -\int_{\mathcal{O}}^{\mathcal{P}} \boldsymbol{E}\cdot d\boldsymbol{\ell} \quad .$$

- Properties of the electric potential:
  - **Arbitrariness**. An arbitrary constant can be added to the potential without changing the field – which, after all, is the fundamental quantity.

    To each constant corresponds a potential reference point. Thus there is always a large selection of appropriate reference points in any electrostatics problem.

Electric potential (continued)

- **Convention**: take $\mathcal{O}$ to lie at infinity, unless the charge distribution itself extends to infinity.

  Just remember that you can actually put the reference point anywhere that doesn't lead to an infinite result for the potential; sometimes you will find reference points not at infinity that will be more convenient for your calculation.

- **Significance**. The magnitude of the electric potential therefore has no physical significance; only **differences** in potential do.

- **Superposition**. The electric potential superposes. If $\boldsymbol{E}$ = $\boldsymbol{E}_1 + \boldsymbol{E}_2 + \ldots$, then

$$V = -\int_{\mathcal{O}}^{\mathcal{P}} \boldsymbol{E}_1 \cdot d\boldsymbol{\ell} - \int_{\mathcal{O}}^{\mathcal{P}} \boldsymbol{E}_2 \cdot d\boldsymbol{\ell} - \ldots = V_1 + V_2 + \ldots$$

  This may seem automatic and trivial until we find, next week, that the closely-related electrostatic potential **energy** does not superpose.

Electric potential (continued)

- From point charges and superposition we obtain the potential from a continuous charge distribution:

$$V_i = -\int_{\mathcal{O}}^{\mathcal{P}} \boldsymbol{E}\cdot d\boldsymbol{\ell} \xrightarrow[\mathcal{O}\to\infty]{} -\int_{\infty}^{r} E dr' = \frac{Q_i}{r} \quad ; \qquad V(\mathcal{P}) = \sum_{i=1}^{N} \frac{Q_i}{\mathscr{r}_i} \xrightarrow[N\to\infty]{} \int_{\mathcal{O}}^{\mathcal{P}} \frac{\rho(\boldsymbol{r}')d\tau'}{\mathscr{r}} \quad .$$

- **Units**. In Gaussian CGS, it's the **statvolt**: $\text{statvolt} = \dfrac{\text{dyne cm}}{\text{esu}} = \dfrac{\text{erg}}{\text{esu}}$

  In SI, it's the **volt**: $\text{volt} = \dfrac{\text{Nt m}}{\text{coul}} = \dfrac{\text{joule}}{\text{coul}}$

- **Correspondence**: $1\ \text{statvolt} \leftrightarrow 299.792458\ \text{volts}$

  The two "correspond" because the potential has different **dimensions** in the two systems, as charge does.

Electric potential (continued)

- **What it's good for**. It's often easier to calculate *V*, and take its gradient to find ***E***, than to calculate ***E*** directly. Reasons:
  - *V* is a scalar; no vector addition to get it.
  - Derivatives are always easier to calculate than integrals.
  - Sometimes it allows avoidance of integrals altogether: if you set up an integral to give *V*, intending then to take its gradient to find ***E***: look first to see whether the gradient theorem – or even the scalar version of the fundamental theorem of calculus – will give you ***E*** without even carrying out the integration. (See problem 8 on Homework #3.)
  - There are many situations in nature in which *V* can be regarded as uniform over a region in space near where one would like to know ***E***. The solution of *V* for space between the uniform-*V* ("equipotential") locations and the reference point – the process of which is called a **boundary-value problem** – can be shown to be unique.
  - Finding *V* by boundary-value solution, and then calculating ***E***, is in these cases usually much easier than calculating ***E*** directly.

Electric potential examples

One of these is an impossible electrostatic field. Which one?

$$\text{a. } \boldsymbol{E} = k\left[(xy)\hat{\boldsymbol{x}} + (2yz)\hat{\boldsymbol{y}} + (3xz)\hat{\boldsymbol{z}}\right]$$

$$\text{b. } \boldsymbol{E} = k\left[\left(y^2\right)\hat{\boldsymbol{x}} + \left(2xy + z^2\right)\hat{\boldsymbol{y}} + (2yz)\hat{\boldsymbol{z}}\right]$$

Here, k is a constant with the appropriate units. For the *possible* one, find the potential, using the origin as your reference point. Check your answer by computing $-\nabla V$.

- To find the impossible one, take the curl of each function.

$$\text{a. } \nabla \times \boldsymbol{E} = k\left(\frac{\partial}{\partial y}(3xz) - \frac{\partial}{\partial z}(2yz)\right)\hat{\boldsymbol{x}} + k\left(\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(3xz)\right)\hat{\boldsymbol{y}}$$

$$+k\left(\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(xy)\right)\hat{\boldsymbol{z}} = k\left(-2y\hat{\boldsymbol{x}} - 3z\hat{\boldsymbol{y}} - x\hat{\boldsymbol{z}}\right) \boxed{\neq 0}$$

Can't be an electrostatic field.

Electric potential examples (continued)

b. $\nabla\times\boldsymbol{E} = k\left(\frac{\partial}{\partial y}(2yz)-\frac{\partial}{\partial z}\left(2xy+z^2\right)\right)\hat{\boldsymbol{x}} + k\left(\frac{\partial}{\partial z}\left(y^2\right)-\frac{\partial}{\partial x}(2yz)\right)\hat{\boldsymbol{y}} + k\left(\frac{\partial}{\partial x}\left(2xy+z^2\right)-\frac{\partial}{\partial y}\left(y^2\right)\right)\hat{\boldsymbol{z}} = \boxed{0}$ OK.

- Integrate ***E*** to get *V*: start at origin, choose path for convenience since the result is path-independent.

$$(0,0,0)\rightarrow(x,0,0)\rightarrow(x,y,0)\rightarrow(x,y,z)$$

$$\boldsymbol{E}\cdot d\boldsymbol{\ell} = ky^2dx = 0 \qquad \boldsymbol{E}\cdot d\boldsymbol{\ell} = k\left(2xy+z^2\right)dy = 2kxydy \qquad \boldsymbol{E}\cdot d\boldsymbol{\ell} = 2k(yz)dz$$

$$\int_{(0,0,0)}^{(x,0,0)}\boldsymbol{E}\cdot d\boldsymbol{\ell} = 0 \quad\Rightarrow\quad \int_{(x,0,0)}^{(x,y,0)}\boldsymbol{E}\cdot d\boldsymbol{\ell} = \int_0^y 2kxydy = kxy^2 \quad\Rightarrow\quad \int_{(x,y,0)}^{(x,y,z)}\boldsymbol{E}\cdot d\boldsymbol{\ell} = \int_0^z 2k(yz)dz = kyz^2$$

$$\therefore\quad V(x,y,z) = 0 - kxy^2 - kyz^2 = \boxed{-k\left(xy^2+yz^2\right)}\ .$$

$$-\nabla V = -\frac{\partial V}{\partial x}\hat{\boldsymbol{x}} - \frac{\partial V}{\partial y}\hat{\boldsymbol{y}} - \frac{\partial V}{\partial z}\hat{\boldsymbol{z}} = k\left(y^2\hat{\boldsymbol{x}} + \left(2xy+z^2\right)\hat{\boldsymbol{y}} + (2yz)\hat{\boldsymbol{z}}\right) = \boldsymbol{E} \qquad ✓✓$$

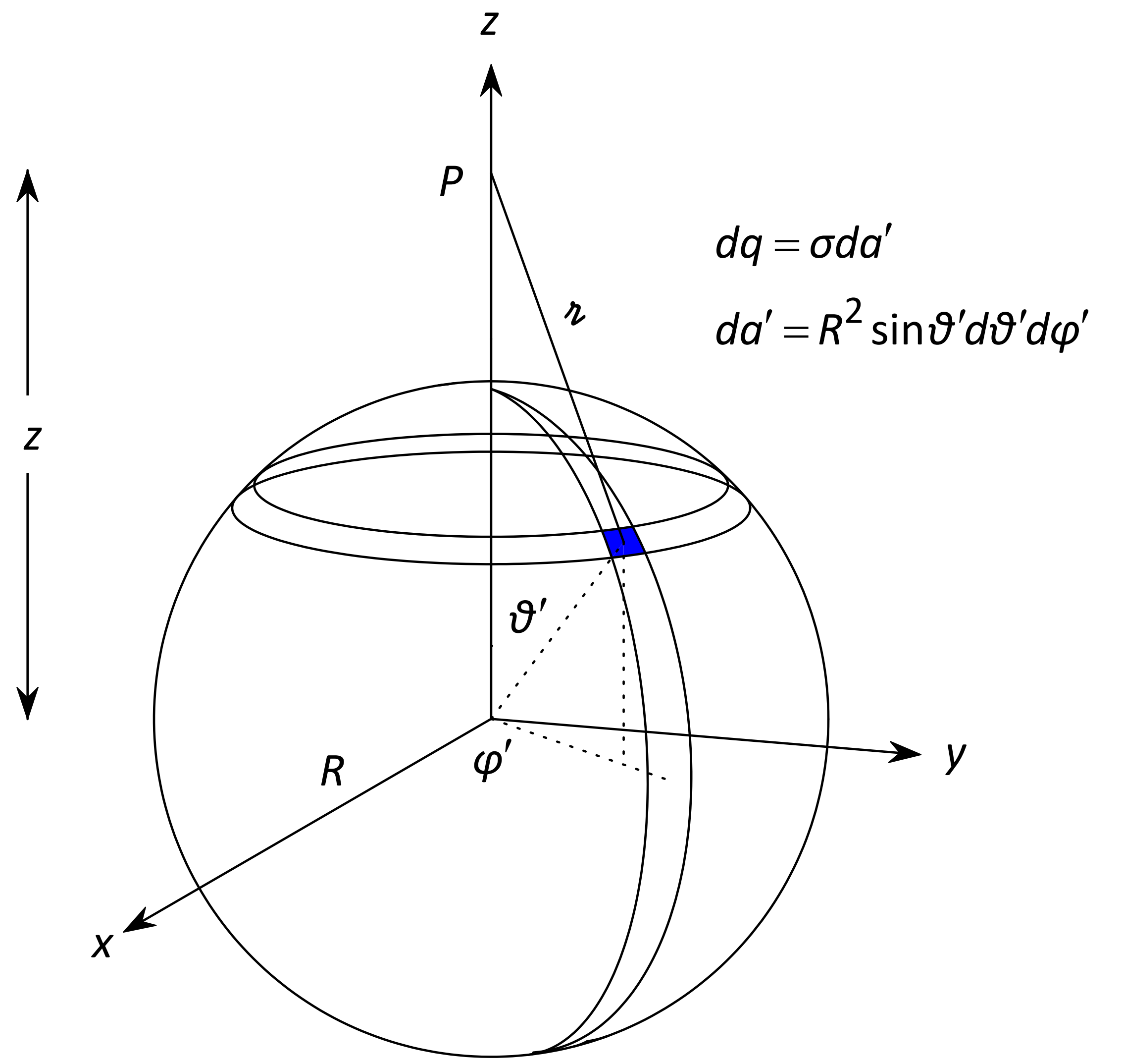
Electric potential examples (continued)

Calculate the electric potential V a distance z away from the center of a spherical shell with radius R and uniform surface charge density σ. Check by calculating $\boldsymbol{E} = -\nabla V$, and comparing with Tuesday's results.

- As before, the distance between dq and P is

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\vartheta' \quad \text{, so}$$

$$V(z) = \int \frac{\sigma da'}{\mathscr{r}} = \sigma \int_0^{2\pi} d\varphi' \int_0^{\pi} \frac{R^2 \sin\vartheta'}{\sqrt{R^2 + z^2 - 2Rz\cos\vartheta'}} d\vartheta' \quad .$$



Electric potential examples (continued)

- The first integral is trivial. For the second, substitute

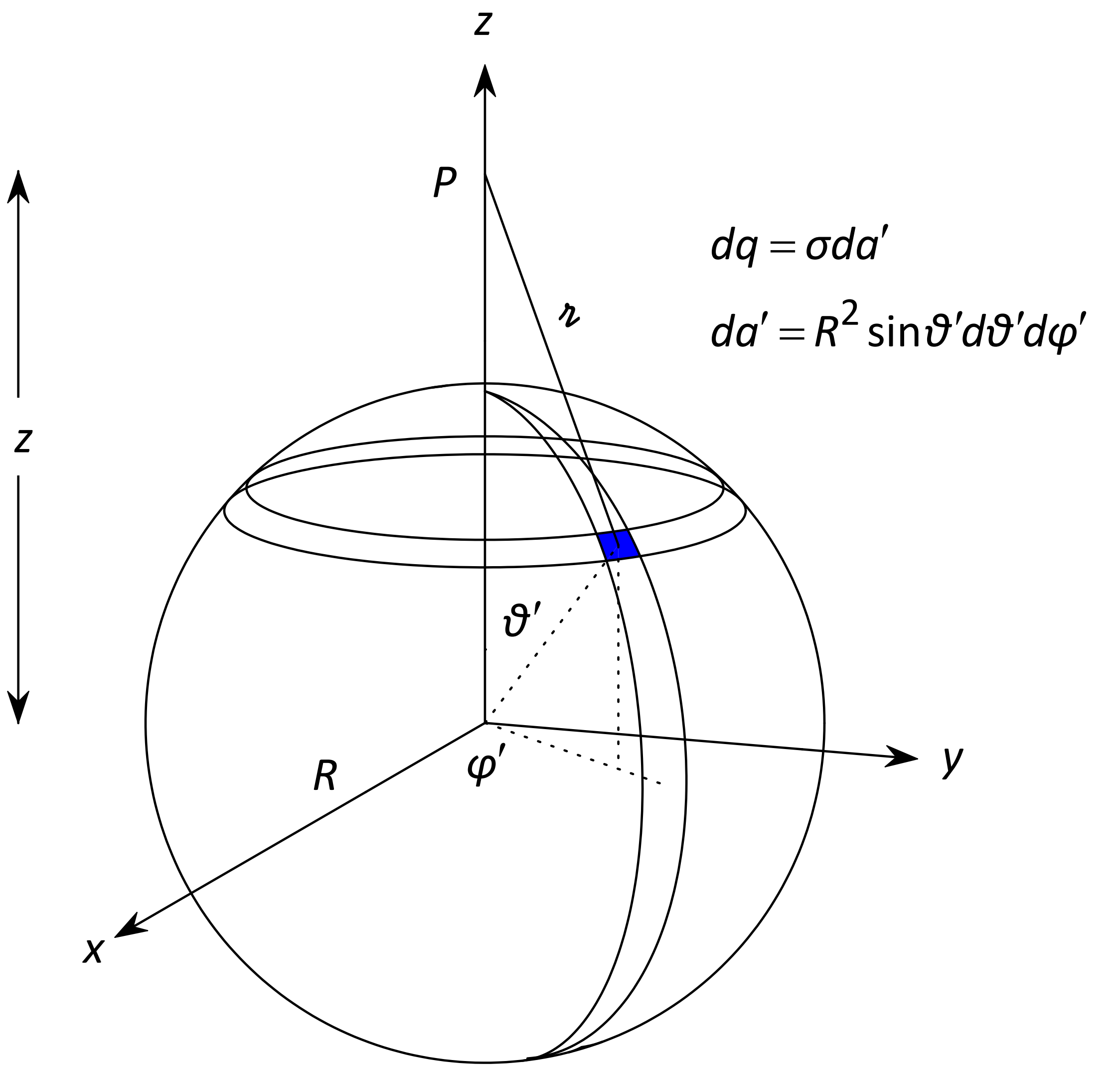
$$w = \cos\vartheta', \quad dw = \not{-}\sin\vartheta' d\vartheta', \quad w = 1 \overset{\leftarrow}{\not\to} -1:$$

$$V(z) = 2\pi\sigma R^2 \int_{-1}^{1} \frac{dw}{\sqrt{R^2 + z^2 - 2Rzw}} \quad .$$

- Then substitute

$$u = R^2 + z^2 - 2Rzw, \quad du = -\not{2}Rzdw,$$

$$u = R^2 + z^2 + 2Rz \overset{\leftarrow}{\not\to} R^2 + z^2 - 2Rz \quad ,$$



Electric potential examples (continued)

to give

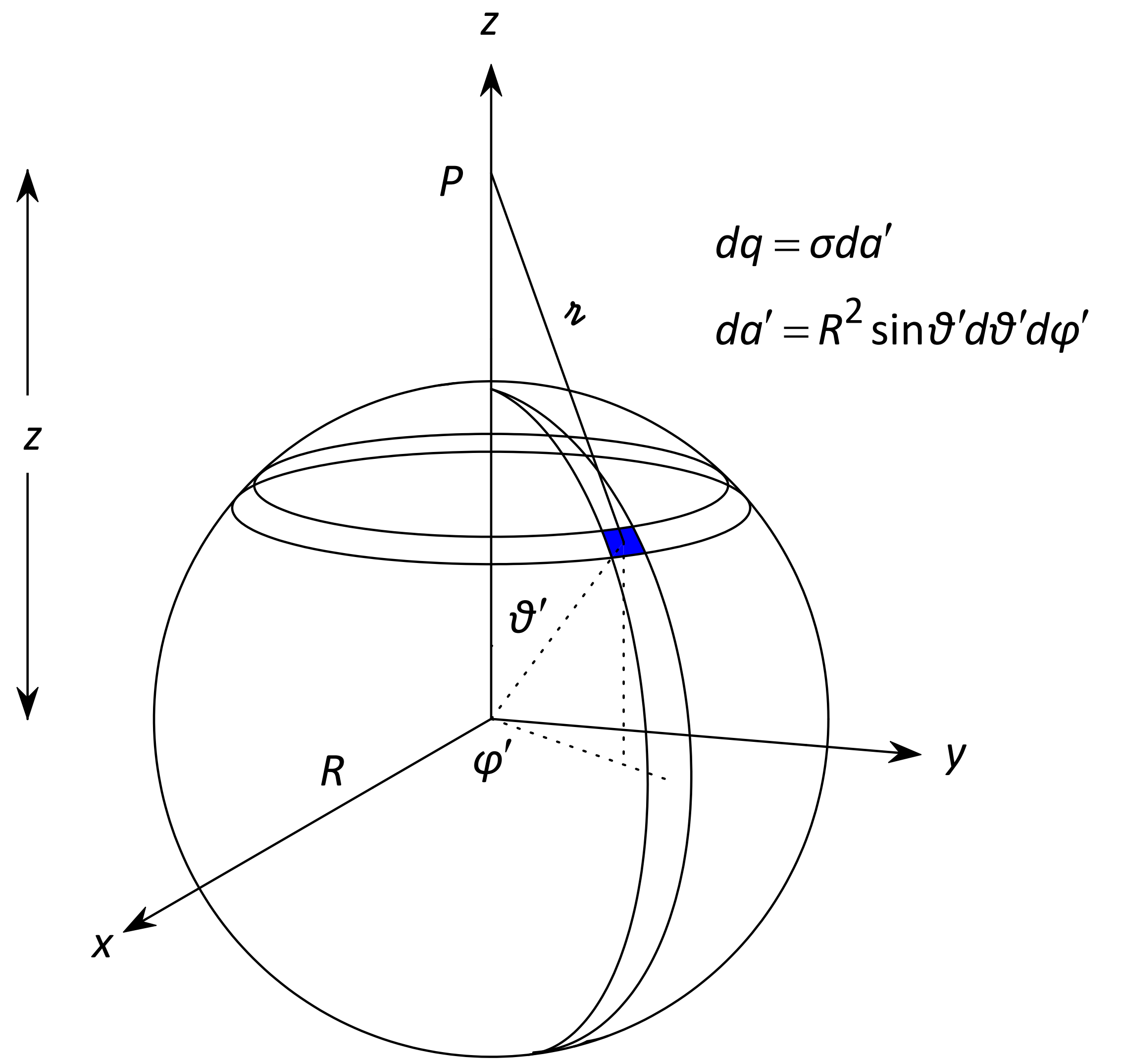
$$V(z) = \frac{\pi\sigma R}{z}\int_{R^2+z^2-2Rz}^{R^2+z^2+2Rz} u^{-1/2}du = \frac{\pi\sigma R}{z}\left[2\sqrt{u}\right]_{R^2+z^2-2Rz}^{R^2+z^2+2Rz}$$

$$= \frac{2\pi\sigma R}{z}\left(\sqrt{R^2+z^2+2Rz} - \sqrt{R^2+z^2-2Rz}\right)$$

$$= \frac{2\pi\sigma R}{z}\left(\sqrt{(z+R)^2} - \sqrt{(z-R)^2}\right)$$

$$= \frac{2\pi\sigma R}{z}\left(|z+R| - |z-R|\right) \quad .$$

$$\frac{\sigma R}{2\varepsilon_0 z}\left(|z+R| - |z-R|\right) \text{ in SI.}$$



Electric potential examples (continued)

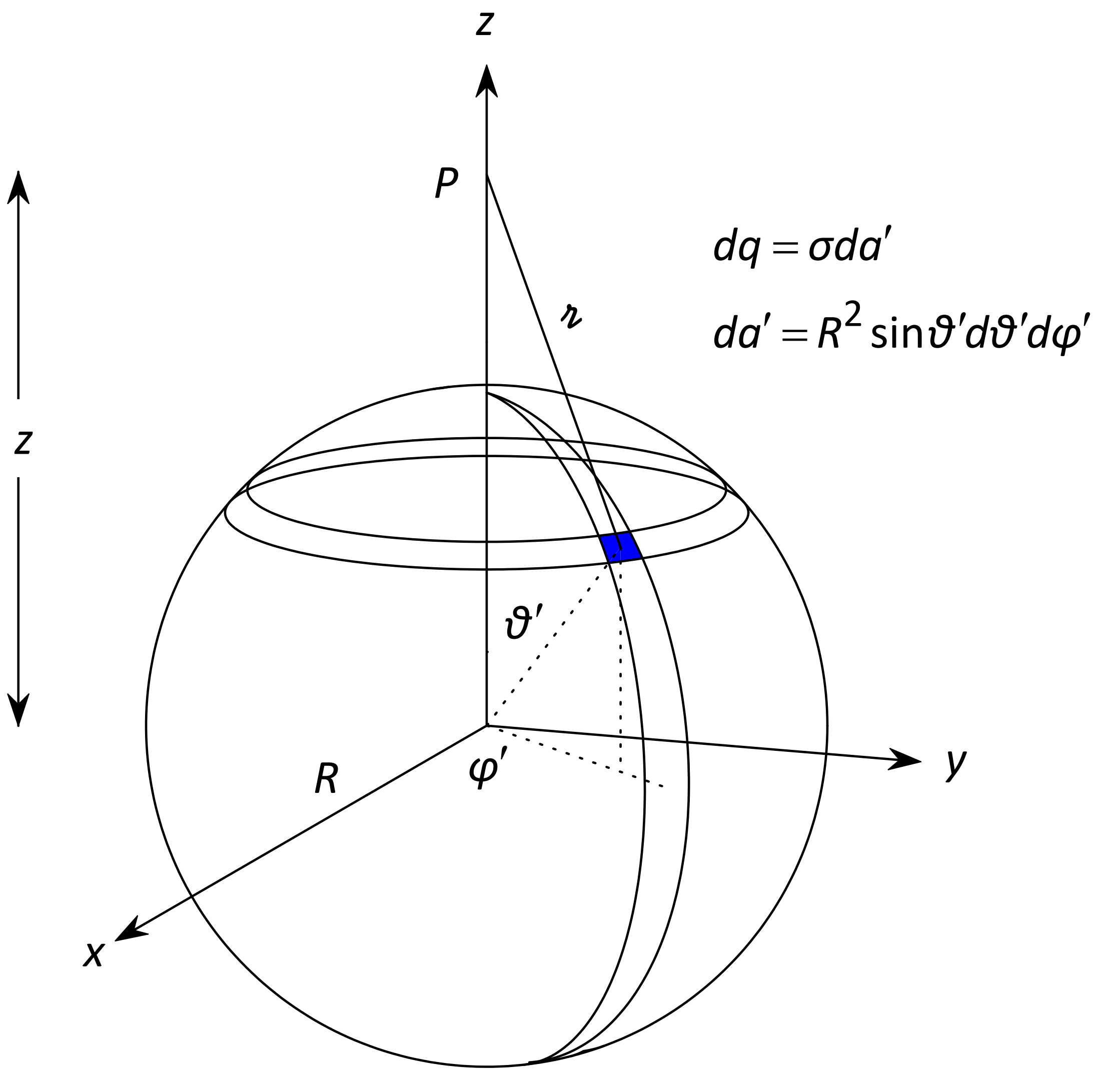
- Two cases, as before: *z* larger than, or smaller than, *R.* (*P* outside, inside)
  - Larger (outside):

$$V(z)=\frac{2\pi\sigma R}{z}(z+R-z+R)=\frac{4\pi R^2\sigma}{z}=\frac{q}{z}\quad .$$

  - Smaller (inside): this means $|z-R|=R-z$, so

$$V(z)=\frac{2\pi\sigma R}{z}(z+R-R+z)=4\pi R\sigma=\frac{q}{R}\quad .$$

Uniform, nonzero *V* inside shell.



Electric potential examples (continued)

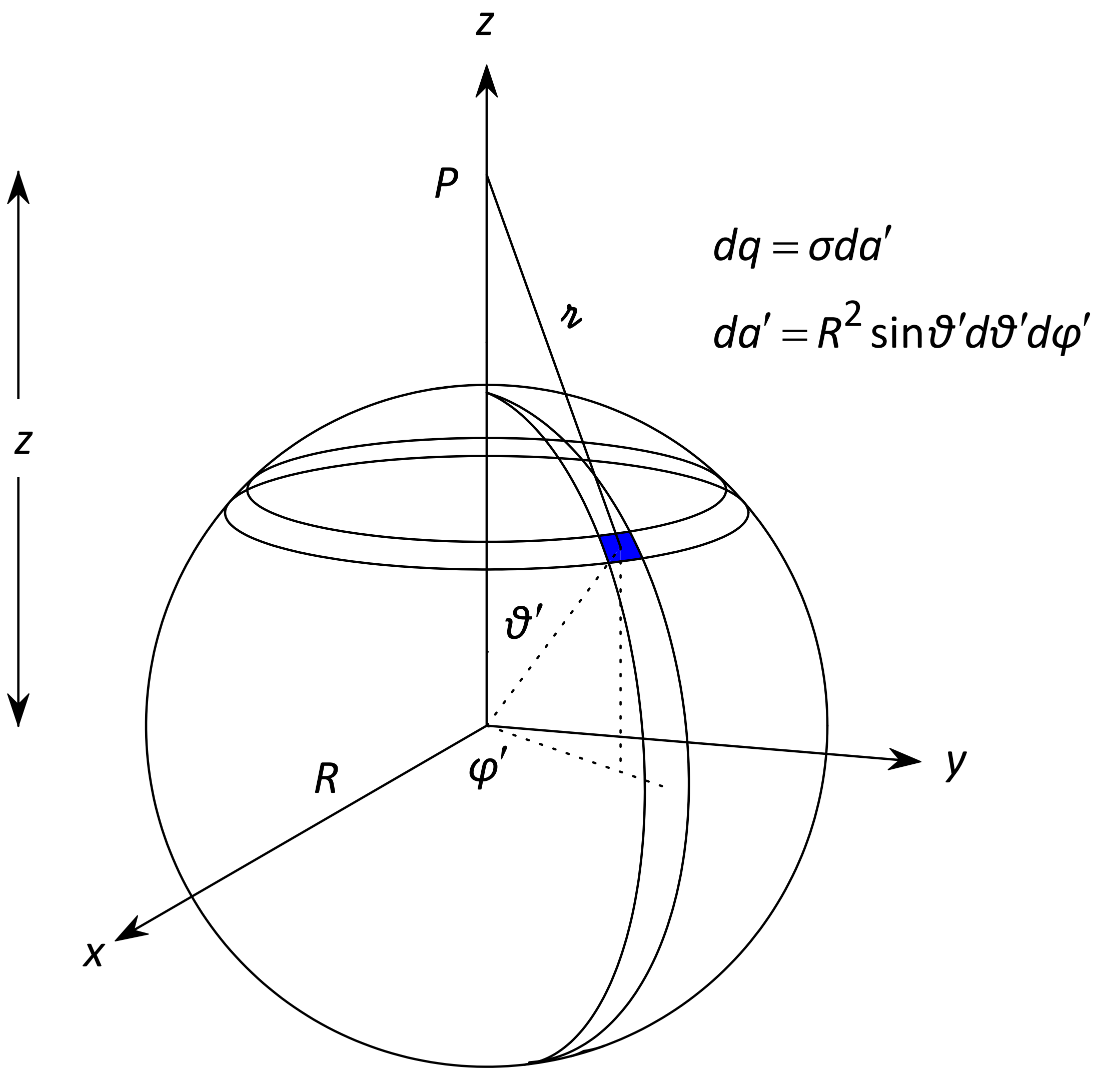
- Check with $\boldsymbol{E}$:

$$V(z)=\frac{2\pi\sigma R}{z}\left(\left|z+R\right|-\left|z-R\right|\right)=\begin{cases}\dfrac{q}{z} & , z\geq R\\[2ex] \dfrac{q}{R} & , z\geq R\end{cases}$$

$$\boldsymbol{E}(z)=-\boldsymbol{\nabla}V=\begin{cases}-\hat{\boldsymbol{z}}\dfrac{\partial}{\partial z}\dfrac{q}{z}=\dfrac{q}{z^2}\hat{\boldsymbol{z}} & , z\geq R \quad ;\\[2ex] -\hat{\boldsymbol{z}}\dfrac{\partial}{\partial z}\dfrac{q}{R}=0 & , z\geq R \quad .\end{cases}$$

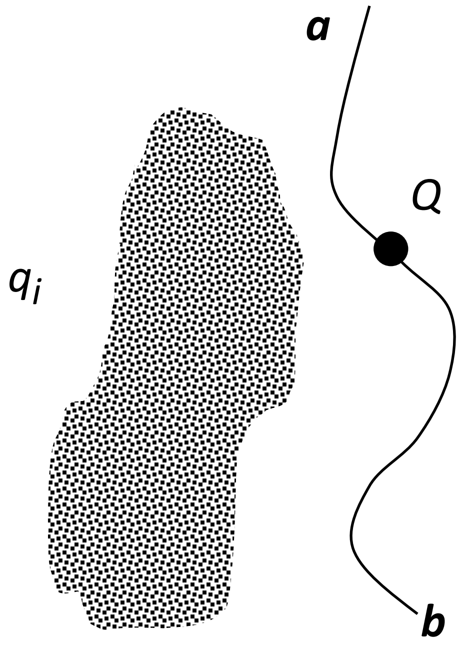
✓✓

Same result as Tuesday, of course.



Work done in motion of a test charge

Move a test charge Q around in vicinity of other charges. How much work is done moving it from $\boldsymbol{a}$ to $\boldsymbol{b}$?



- The force exerted on Q by those charges' fields is $\boldsymbol{F} = Q\boldsymbol{E}$; the force **we** need to exert, **by Newton's third law**, is - $Q\boldsymbol{E}$. So the work **we** do is

$$W = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{F}\cdot d\boldsymbol{\ell} = -Q\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E}\cdot d\boldsymbol{\ell} = Q\left[V(\boldsymbol{b}) - V(\boldsymbol{a})\right] \quad ,$$

  independent of path because the path integral of electrostatic $\boldsymbol{E}$ is path-independent.

- Corollary: the work required **for us** to bring charge Q to point $\mathcal{P}$ from infinity is

$$W = Q\left[V(\mathcal{P}) - V(\infty)\right] = QV(\mathcal{P}) \quad .$$

Electrostatic potential energy

To obtain the **potential energy** of an assembly of charges, bring them from infinity in one by one, and calculate the work you do. That work ***W*** is the potential energy.

Consider assembling the charge distribution above: a bunch of point charges, q_i.

- Bring in the first one: $$W_1 = 0$$
- Bring in the second one: $$W_2 = q_2\left(\frac{q_1}{\mathscr{r}_{12}}\right)$$
- And the third one: $$W_3 = q_3\left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right)$$
- And the fourth: $$W_4 = q_4\left(\frac{q_1}{\mathscr{r}_{14}} + \frac{q_2}{\mathscr{r}_{24}} + \frac{q_3}{\mathscr{r}_{34}}\right)$$

Electrostatic potential energy (continued)

- So far, for the first four charges, the total work is

$$W = q_2\left(\frac{q_1}{\mathscr{r}_{12}}\right) + q_3\left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right) + q_4\left(\frac{q_1}{\mathscr{r}_{14}} + \frac{q_2}{\mathscr{r}_{24}} + \frac{q_3}{\mathscr{r}_{34}}\right)$$

$$= \frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}}$$

- Evidently, for N charges, we'd get $W = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{2} \sum_{i=1}^{N} q_i \sum_{\substack{j=1, \\ i \neq j}}^{N} \frac{q_j}{\mathscr{r}_{ij}} = \frac{1}{2} \sum_{i=1}^{N} q_i V\left(\mathcal{P}_i\right)$

Count each pair of charges just once

Electrostatic potential energy (continued)

- If the collection of charges is finite and continuous $(N \to \infty)$, this becomes

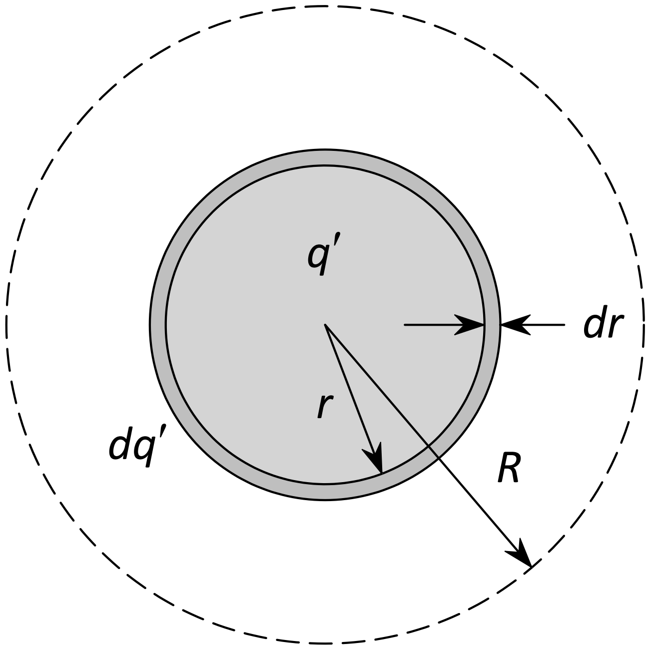
$$W = \frac{1}{2}\int_{\mathcal{V}} dqV = \frac{1}{2}\int_{\mathcal{V}} \rho V d\tau \quad .$$

- As usual, we'd also have $W = \frac{1}{2}\int_{\mathcal{S}} \sigma V da$, and $W = \frac{1}{2}\int_{\mathcal{C}} \lambda V d\ell$,

  for surface- and line-charge distributions.

Electrostatic potential energy (continued)

Find the electrostatic potential energy for a sphere with radius R and uniform charge density ρ: total charge $q = 4\pi\rho R^3 / 3$.



- Envision building the sphere one infinitesimal shell at a time, and observe the process in the middle at first, when a charge $q' = 4\pi\rho r^3 / 3$ is already there.

- To bring in the next shell, with charge $dq' = 4\pi\rho r^2 dr$, we have to work against the electric field of the inner spherical charge:

$$\mathbf{E}'(\mathbf{r}') = \frac{4\pi\rho r^3}{3r'^2}\hat{\mathbf{r}}$$

$$= dq' V(r)$$

$$dW' = \int_{\mathbf{a}}^{\mathbf{b}} d\mathbf{F}' \cdot d\boldsymbol{\ell} = \boxed{-dq' \int_{\infty}^{r} \mathbf{E}'(\mathbf{r}') \cdot (\hat{\mathbf{r}} dr')} = -4\pi\rho r^2 dr \frac{4\pi\rho r^3}{3} \int_{\infty}^{r} \frac{dr'}{r'^2} = -\frac{16\pi^2\rho^2 r^5}{3} dr \left[-\frac{1}{r'}\right]_{\infty}^{r} = \frac{16\pi^2\rho^2 r^4}{3}$$

$$W = \frac{16\pi^2\rho^2}{3} \int_0^R r^4 dr = \frac{16\pi^2\rho^2}{3} \frac{R^5}{5} = \frac{3}{5R}\left(\frac{4\pi\rho R^3}{3}\right)^2 = \frac{3q^2}{5R} \quad .$$

The energy density of the electrostatic field in free space

- Now back to $W = \frac{1}{2}\int_{\mathcal{V}} dqV = \frac{1}{2}\int_{\mathcal{V}} \rho V d\tau$. Eliminate density and potential from these expressions, in favor of the electric field, by using $\rho = \frac{1}{4\pi}\nabla\cdot\boldsymbol{E}$, $\nabla V = -\boldsymbol{E}$, and $\nabla\cdot(f\boldsymbol{A}) = (\nabla\cdot\boldsymbol{A})f + \boldsymbol{A}\cdot(\nabla f)$:

$$W = \frac{1}{8\pi}\int_{\mathcal{V}}(\nabla\cdot\boldsymbol{E})Vd\tau = \frac{1}{8\pi}\int_{\mathcal{V}}\left[\nabla\cdot(V\boldsymbol{E}) - \boldsymbol{E}\cdot\nabla V\right]d\tau = \frac{1}{8\pi}\int_{\mathcal{V}}\left[\nabla\cdot(V\boldsymbol{E}) + E^2\right]d\tau = \frac{1}{8\pi}\left[\oint_{\mathcal{S}} V\boldsymbol{E}\cdot d\boldsymbol{a} + \int_{\mathcal{V}} E^2 d\tau\right] \; .$$

Divergence theorem

- If we extend the integration region $\mathcal{V}$ to include all of space, and the charge distribution is finite in extent, then $\boldsymbol{E}$ and V approach zero at the surface $\mathcal{S}$ (which "surrounds infinity"). Note that

$$\lim_{r\to\infty} E \propto \frac{1}{r^2} \quad , \quad \lim_{r\to\infty} V \propto \frac{1}{r} \quad , \quad \lim_{r\to\infty} A \propto r^2 \quad \Rightarrow \quad \lim_{r\to\infty}\oint_{\mathcal{S}} V\boldsymbol{E}\cdot d\boldsymbol{a} \propto \frac{1}{r} = 0$$

The energy density of the electrostatic field in free space

- Doing so, we get

$$W = \frac{1}{8\pi} \int_{\substack{\text{all}\\ \text{space}}} E^2 d\tau \qquad = \frac{\varepsilon_0}{2} \int_{\substack{\text{all}\\ \text{space}}} E^2 d\tau \quad \text{in SI.}$$

- So ***E* fields themselves store energy**. One often sees this expressed as an energy density, *u*:

$$u_E = \frac{E^2}{8\pi} \quad . \qquad = \frac{\varepsilon_0 E^2}{2} \quad \text{in SI.}$$

Caveats regarding $W = \frac{1}{8\pi}\int E^2 d\tau$

1. Consider a point charge, for which the charge density is $\rho = q\delta^3(\boldsymbol{r})$. How much work is involved in assembly of this charge distribution? On the one hand,

$$W = \frac{1}{2}\int \rho V d\tau = \frac{q}{2}\int \delta^3(\boldsymbol{r})\frac{q}{r}r^2 \sin\vartheta dr d\vartheta d\varphi = 2\pi q^2 \int_0^\infty \delta(r) r dr = 0$$

but on the other,

$$W = \frac{1}{8\pi}\int E^2 d\tau = \frac{q^2}{2}\int_0^\infty \frac{1}{r^4}r^2 dr = -\frac{q^2}{6}\frac{1}{r^3}\bigg|_0^\infty \to \infty$$

The reason for the difference is related to the troublesome divergence of $\hat{\boldsymbol{r}}/r^2$, as seen in Griffiths problems 1.16 and 1.39. Inconsistency is avoided by restoring the surface integral to the expression of potential energy in terms of field:

Caveats regarding $W = \frac{1}{8\pi}\int E^2 d\tau$ (continued)

For our point charge,

$$\nabla\cdot(V\mathbf{E}) + E^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{q}{r}\frac{q}{r^2}\right) + \frac{q^2}{r^4} = \frac{q^2}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \frac{q^2}{r^4} = -\frac{q^2}{r^4} + \frac{q^2}{r^4} = 0 \quad \Rightarrow \quad W = \int\left[\nabla\cdot(V\mathbf{E}) + E^2\right]d\tau = 0$$

Not, therefore, inconsistent. But use $W = \frac{1}{2}\int E^2 d\tau$ with care, keeping that surface integral in mind.

2. As you know, forces, electric fields, and electric potentials obey the principle of superposition. **Potential energy does not.** Consider:

$$W = \frac{1}{8\pi}\int E^2 d\tau = \frac{1}{8\pi}\int(\mathbf{E}_1 + \mathbf{E}_2 + \ldots)\cdot(\mathbf{E}_1 + \mathbf{E}_2 + \ldots)d\tau$$

$$\neq \frac{1}{8\pi}\int\left(E_1^2 + E_2^2 + \ldots\right)d\tau = W_1 + W_2 + \ldots \quad ,$$

because cross terms such as $\frac{1}{8\pi}\int 2\mathbf{E}_1\cdot\mathbf{E}_2 d\tau$ and $\frac{1}{8\pi}\int 2\mathbf{E}_1\cdot\mathbf{E}_3 d\tau$ are not necessarily zero.