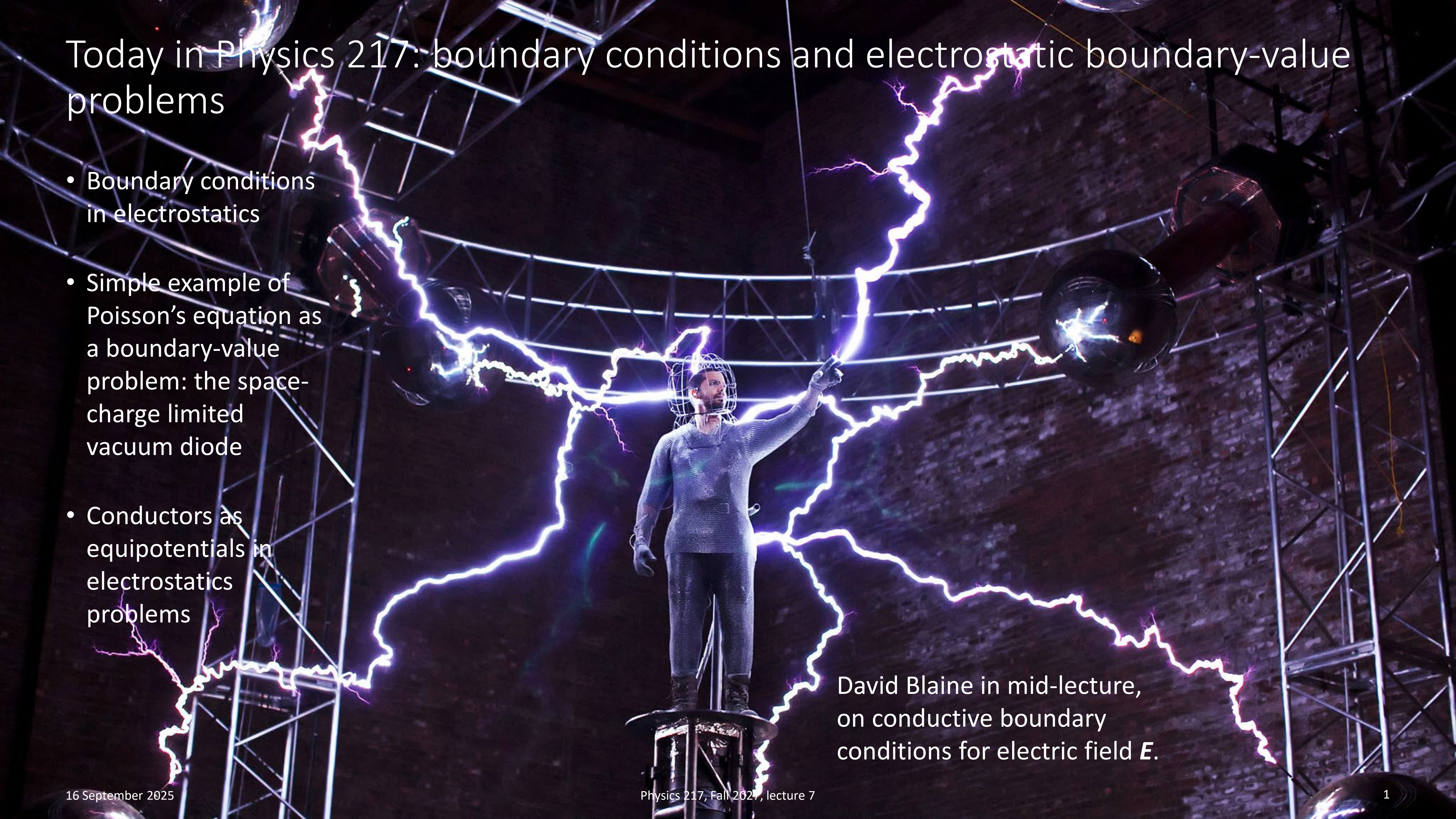
Today in Physics 217: boundary conditions and electrostatic boundary-value problems

- Boundary conditions in electrostatics
- Simple example of Poisson’s equation as a boundary-value problem: the space-charge limited vacuum diode
- Conductors as equipotentials in electrostatics problems



David Blaine in mid-lecture, on conductive boundary conditions for electric field ***E***.

Differential equations for the scalar electric potential

- To get V without referring first to $\boldsymbol{E}$:

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho$$

$$\nabla \cdot (-\nabla V) =$$

$$\boxed{\nabla^2 V = -4\pi\rho}$$

Poisson's equation

- In regions where there are no electric charges, $\boxed{\nabla^2 V = 0}$

Laplace's equation

These equations, plus boundary conditions, provide the boundary-value-problem way to calculate V.

Electrostatic boundary-value problems

- We have encountered several differential equations relating the field, scalar potential, and charge density, which we will in general want to solve for ***E*** or V.

- But the solutions aren't unique unless enough constraints – **boundary values** – are supplied.

  - If the **potential**, or **components of the field**, are specified on enough boundary surfaces or lines (e.g. **equipotentials**), a **unique** solution of the differential equation(s) for V or ***E*** in the rest of the volume or area will be obtained.

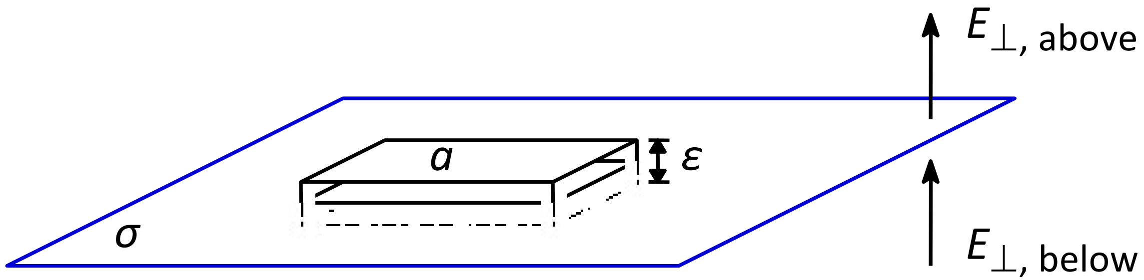
    This will usually be how solutions of Laplace's and Poisson's equations will proceed. They are second order differential equations, and must be provided two boundary conditions for solutions to exist.

  - Specification of the charge density results in specification of boundary conditions on field and potential, as follows.

Surface charges and electric-field boundary conditions

Consider a surface with charge per unit area *σ,* not necessarily uniform. **What is the relation of this charge to the electric field on either side of the surface?**

- Zoom in on an area small enough to consider flat, and draw a Gaussian surface with planar symmetry which has sides $\varepsilon\left(\to 0\right)$ perpendicular to the surface.



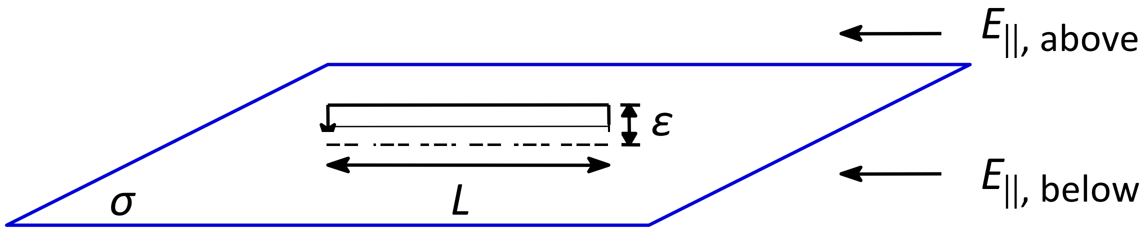
- Then, since the flux through the sides is negligible,

$$\oint \boldsymbol{E}\cdot d\boldsymbol{a} = 4\pi Q_{\text{enclosed}} \quad\Rightarrow\quad \left(E_{\perp,\text{ above}} - E_{\perp,\text{ below}}\right)a = 4\pi\sigma a \quad\Rightarrow\quad \left(E_{\perp,\text{ above}} - E_{\perp,\text{ below}}\right) = 4\pi\sigma \quad .$$

Charge sheets make discontinuities of $4\pi\sigma$ in $E_\perp$.

Surface charges and electric-field boundary conditions (continued)

- Now consider a loop, $\varepsilon\left(\rightarrow 0\right)$ by L, perpendicular to the surface, bisected by the surface, and small enough that the surface is flat over its dimensions.



- In a line integral of field along this loop the contribution of the short sides is negligibly small, so

$$\oint \boldsymbol{E}\cdot d\boldsymbol{\ell}=0 \;\Rightarrow\; \left(E_{||,\text{ above}}-E_{||,\text{ below}}\right)L=0 \;\Rightarrow\; E_{||,\text{ above}}-E_{||,\text{ below}}=0 \quad .$$

The parallel component of ***E*** is continuous across a sheet of charge.

Surface charges and electric-potential boundary conditions

And from these come boundary conditions on *V*:

- For two points ***a*** and ***b***, $\pm\varepsilon/2$ from the surface,

$$V_{\text{above}} - V_{\text{below}} = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E}\cdot d\boldsymbol{\ell} \xrightarrow[\varepsilon\to 0]{} 0$$

  i.e. *V* is continuous across the charged sheet.

- Then consider the gradient of *V*.

$$\left(\nabla V_{\text{above}} - \nabla V_{\text{below}}\right)_\perp = -\left(E_{\perp,\text{ above}} - E_{\perp,\text{ below}}\right) = -4\pi\sigma \quad .$$

  Or, with *n* as the coordinate axis perpendicular to the surface at this point,

$$\left(\nabla V_{\text{above}} - \nabla V_{\text{below}}\right)\cdot\hat{\boldsymbol{n}} = -4\pi\sigma \quad \Rightarrow \quad \boxed{\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -4\pi\sigma} \quad .$$

Poisson equation solution example

Treated as a boundary-value problem, but with none of the solution machinery we're about to introduce.

Griffiths problem ! 2.54: In a vacuum-tube diode, electrons are boiled off a hot cathode, at potential zero, and accelerated across a gap to the anode, which is held at positive potential V_0.

The cloud of moving electrons within the gap (called the **space charge**) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on, a steady current *I* flows between the plates.

Suppose the plates are large relative to the separation *d*, so that edge effects can be neglected.

Then *V*, *ρ*, and *v* (the speed of the electrons) are functions of *x* alone.

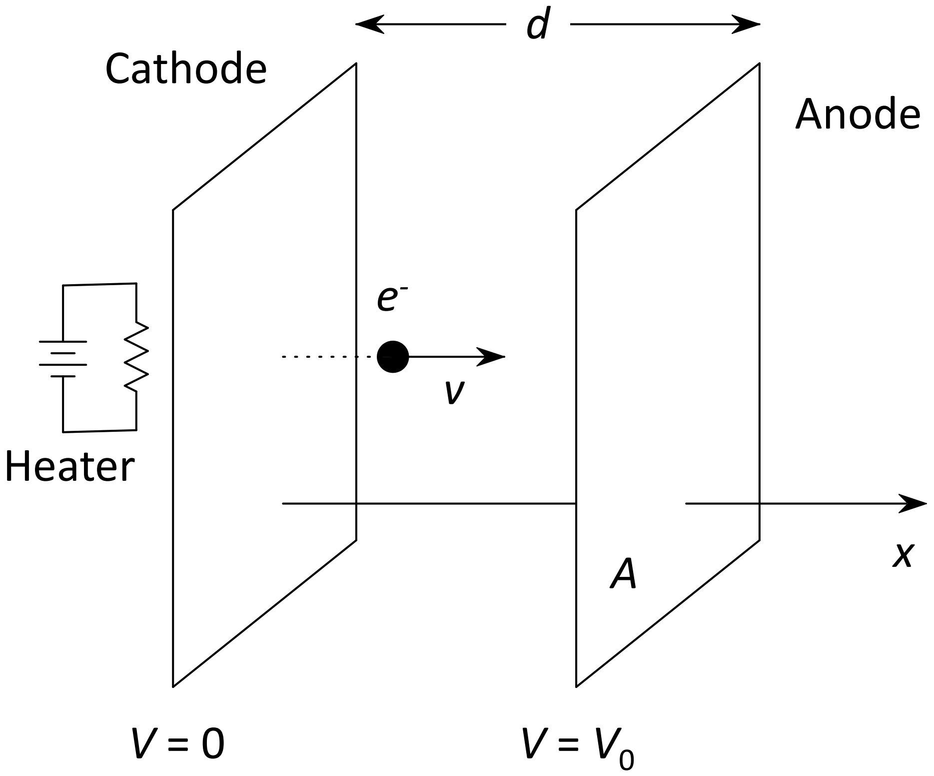
(a) Solve Poisson's equation for the potential between the anode and cathode, as a function of *x*, V_0, and *d*.

(b) Show that $I = KV_0^{3/2}$, and find the constant *K*. This expression is called **Child's Law**.

Example: the space-charge limited vacuum diode (continued)

- As soon as electrons boil off in large numbers, their presence slows down further electron ejection because they're so slow at first.
  - This is what is meant by **space-charge limited**.
- The space charge is denser near the cathode.
- Although individual electrons move, the space charge distribution is constant after a while, **so this *is* an electrostatics problem.**
- So: Poisson's equation, 1-D: $\nabla^2 V = \frac{d^2V}{dx^2} = -4\pi\rho(x)$.

  Boundary conditions: $V(0) = 0, \quad V(d) = V_0, \quad \boldsymbol{E}(0) = 0$.



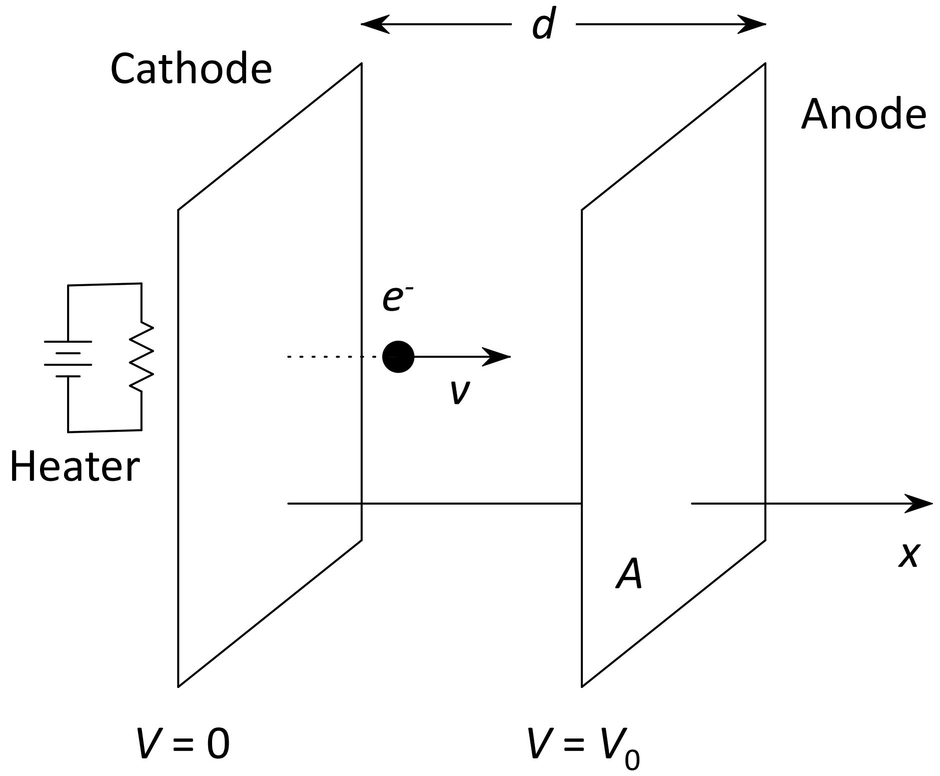
Example: the space-charge limited vacuum diode (continued)

- Solution: first, consider a charge q that makes it to where the potential is V. The field has done work $W = qV$, so

$$W = qV = \Delta\left(KE\right) = \frac{1}{2}mv^2 \quad \Rightarrow \quad v = \sqrt{2qV/m} \quad .$$

- Also consider a slice of the space charge, with thickness dx:

$$dq = \rho A dx \quad \Rightarrow \quad \frac{dq}{dt} = \rho A \frac{dx}{dt} = \rho A v$$



Example: the space-charge limited vacuum diode (continued)

- But $dq/dt = I$, which is constant if the charge distribution has reached a steady state, so

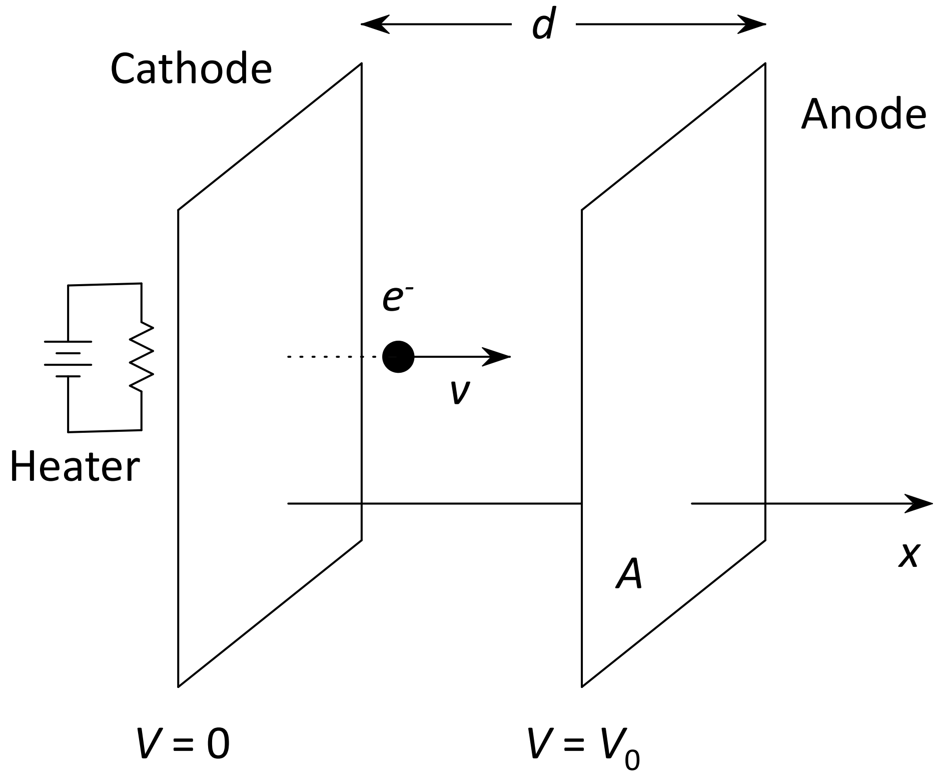
$$\frac{d^2V}{dx^2} = -4\pi\rho = -4\pi\frac{I}{Av} = -4\pi\frac{I}{A}\sqrt{\frac{m}{2qV}} = \beta V^{-1/2} \quad ,$$

where $\beta = -4\pi\frac{I}{A}\sqrt{\frac{m}{2q}}$.

- To solve, substitute $dV/dx = V'$, and multiply through by V':

$$V'\frac{d^2V}{dx^2} = \beta V^{-1/2}V'$$

$$V'\frac{dV'}{dx} = \beta V^{-1/2}\frac{dV}{dx} \quad .$$



Example: the space-charge limited vacuum diode (continued)

- Integrate with respect to *x*: $$\int V'\frac{dV'}{dx}dx=\beta\int V^{-1/2}\frac{dV}{dx}dx$$

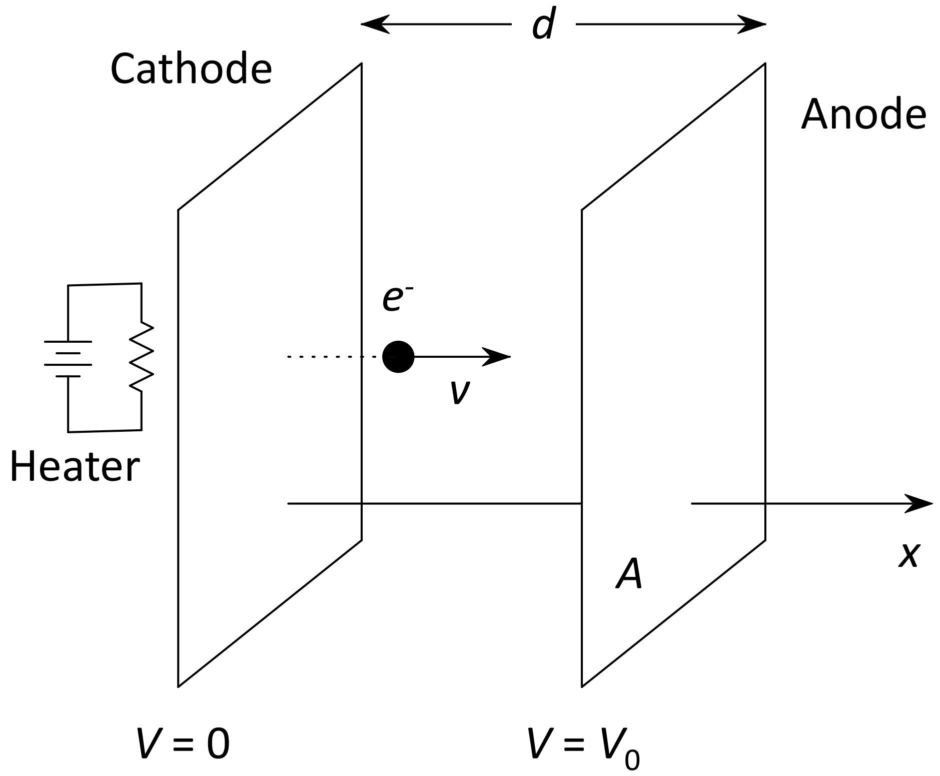
$$\frac{1}{2}V'^2=2\beta V^{1/2}+C \quad .$$

But $V(0) = 0$, and $E(0)=0=-\frac{dV}{dx}(0)=-V'(0)$, so $C=0$, and

$$V'=\frac{dV}{dx}=2\sqrt{\beta}V^{1/4}$$

$$\int V^{-1/4}dV=2\sqrt{\beta}\int dx$$

$$\frac{4}{3}V^{3/4}=2\sqrt{\beta}x+D \quad .$$



Example: the space-charge limited vacuum diode (continued)

- $V(0) = 0$ once again mandates that $D = 0$. Also $V(d) = V_0$, so

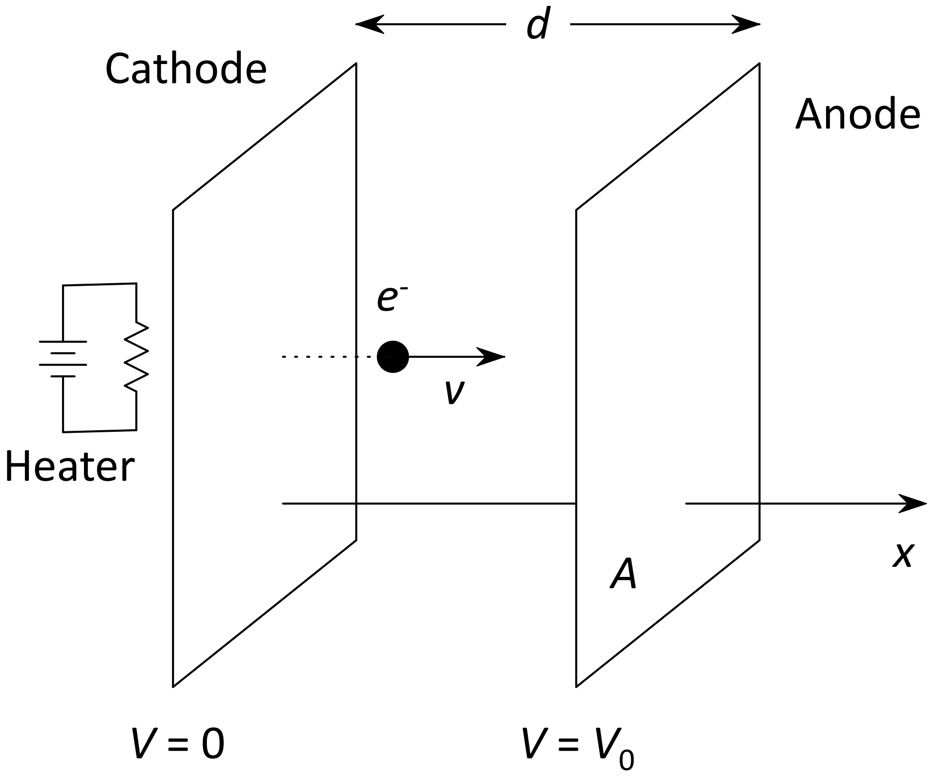
$$\frac{4}{3}V_0^{3/4} = 2\sqrt{\beta}d \quad \Rightarrow \quad \beta = \left(\frac{2}{3d}\right)^2 V_0^{3/2}$$

$$\Rightarrow \quad V(x) = \left(\frac{3}{2}\sqrt{\beta}x\right)^{4/3} = \left(\frac{3}{2}\left(\frac{2}{3d}\right)V_0^{3/4}x\right)^{4/3} = V_0\left(\frac{x}{d}\right)^{4/3} \qquad \text{(a)}$$

- To work out the current (part b), start by getting the charge density and speed:

$$\rho = -\frac{1}{4\pi}\frac{d^2V}{dx^2} = -\frac{1}{4\pi}\frac{V_0}{d^{4/3}}\frac{4}{3}\frac{1}{3}x^{-2/3} = -\frac{V_0}{9\pi\left(d^2x\right)^{2/3}}$$

$$v = \sqrt{\frac{2qV}{m}} = \sqrt{\frac{2qV_0}{m}}\left(\frac{x}{d}\right)^{2/3}$$



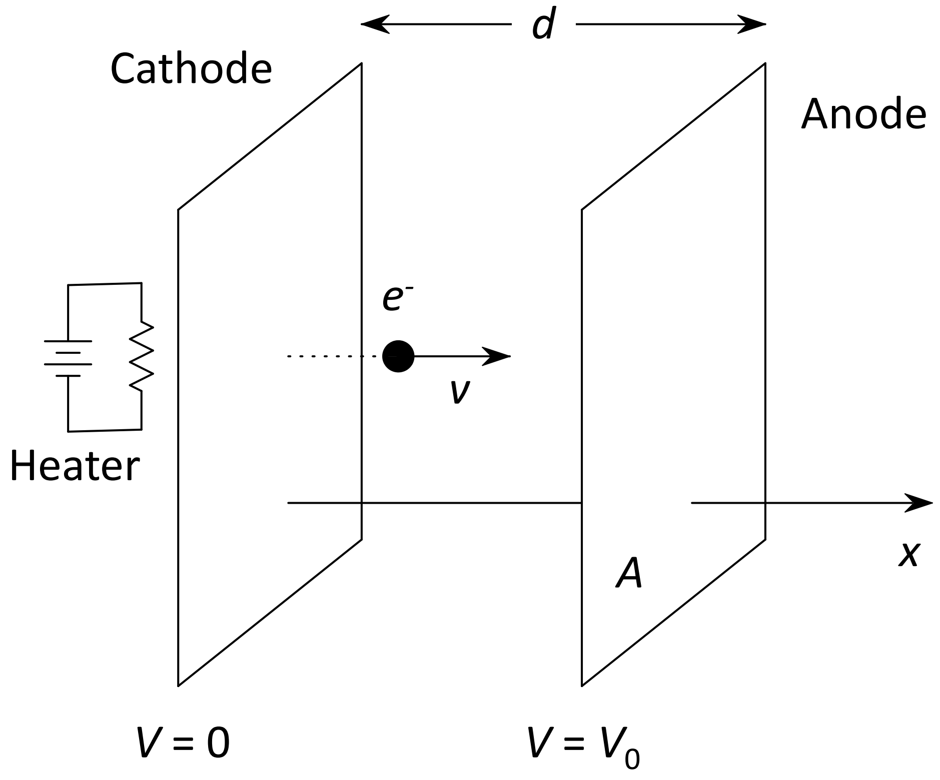
Example: the space-charge limited vacuum diode (continued)

- Thus $$I = \rho A v = -\frac{V_0 A}{9\pi\left(d^2 x\right)^{2/3}}\sqrt{\frac{2qV_0}{m}}\left(\frac{x}{d}\right)^{2/3}$$

$$= -\frac{A}{9\pi d^2}\sqrt{\frac{2q}{m}}V_0^{3/2} = KV_0^{3/2},$$

where $$K = -\frac{A}{9\pi d^2}\sqrt{\frac{2q}{m}}. \quad \text{(b)}$$ Child's Law

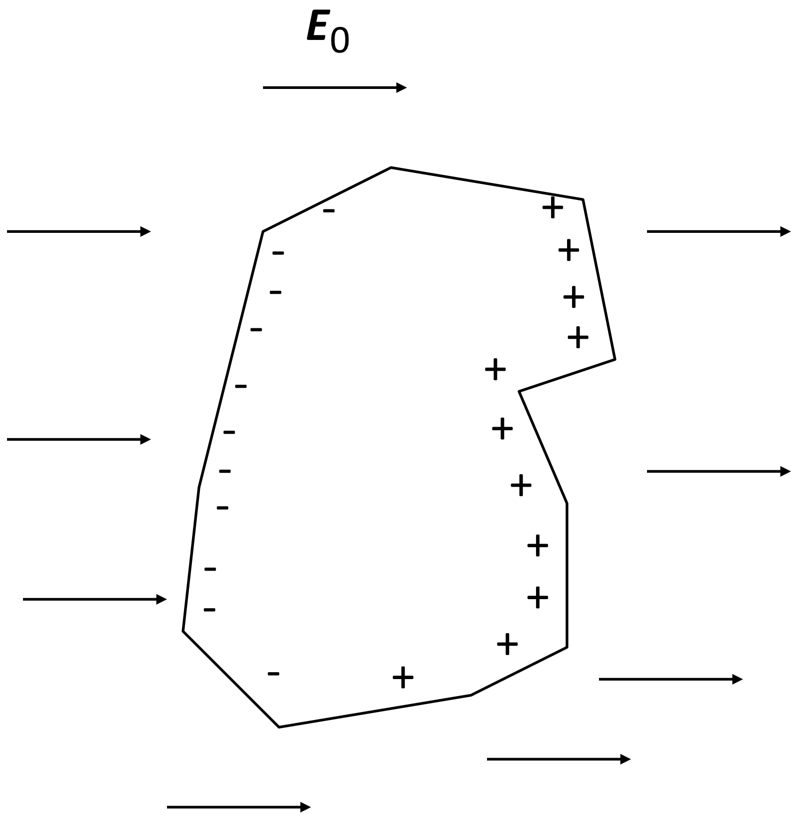
- Note that the current and voltage are not directly proportional, unlike Ohm's law.



Conductors, and the boundary conditions they imply

Conductors are materials that contain charges that can move about freely (within the material) in response to an applied electric field.

- Metals are good examples; for *most* metals, the mobile carriers are electrons.
- Better conductors have higher densities of free (mobile) charges.



A metal in an externally-applied electric field. Electrons have moved to the “upstream” surface, leaving fixed positive charges behind.

Conductors (continued)

Implications of the free, mobile charges in conductors **in electrostatics:**

- ***E* = 0 inside a conductor.** If it weren't, charges would move; the free charges move whenever the field is nonzero, and stop only when ***E*** = 0 inside.

- ***ρ* = 0 inside a conductor.** This is because, for any surface completely within the conductor,

$$Q_{\text{enclosed}} = \frac{1}{4\pi}\oint \boldsymbol{E}\cdot d\boldsymbol{a} = 0$$

  because ***E*** = 0 there.

- **Any net charge lies on the surface**. That's as far as it can travel under the influence of the external electric field; also, there'd be nonvanishing ***E*** and *ρ* if this weren't true.

Conductors (continued)

- ***V* is uniform throughout a conductor**: because, for any two points ***a*** and ***b*** inside,

$$V(\mathbf{b})-V(\mathbf{a})=-\int_{\mathbf{a}}^{\mathbf{b}}\mathbf{E}\cdot d\ell=0$$

  Thus **the surfaces of conductors are equipotentials**.

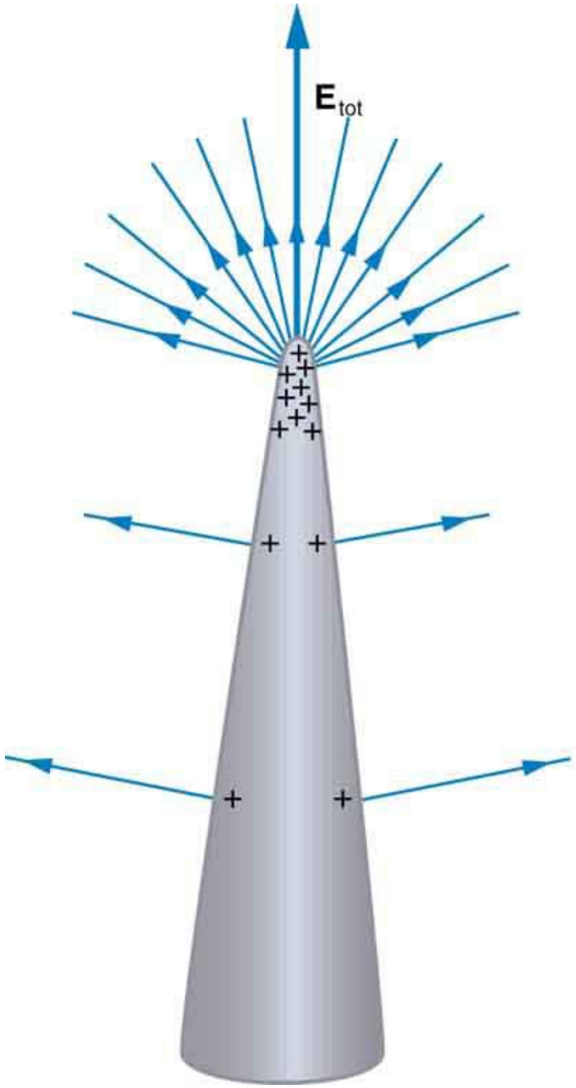
- ***E* is perpendicular to the surface of conductors.** Otherwise, free charges would flow along the surface until any parallel component were zero.

  - Or think of the boundary conditions on ***E*** derived above: here, ***E*** = 0 inside, and thus

$$E_{\|,\text{ inside}}-E_{\|,\text{ outside}}=0 \quad ;$$

all ***E*** is $\perp$ to the surface.

  - This is why sharp-pointed conductors make good lightning rods: they force ***E*** to be strongest at the point.



What can we do with conductors?

Or, rather, what kinds of homework problems can I assign about conductors?

- Given conductors held at specified potentials, find the fields in between.
- Given conductors held at specified potentials, find the surface charge density.
- Given conductors immersed in specified electric fields, find the density of charge induced on the surface.
- Or vice-versa, for any of these.

Example: concentric conducting spheres

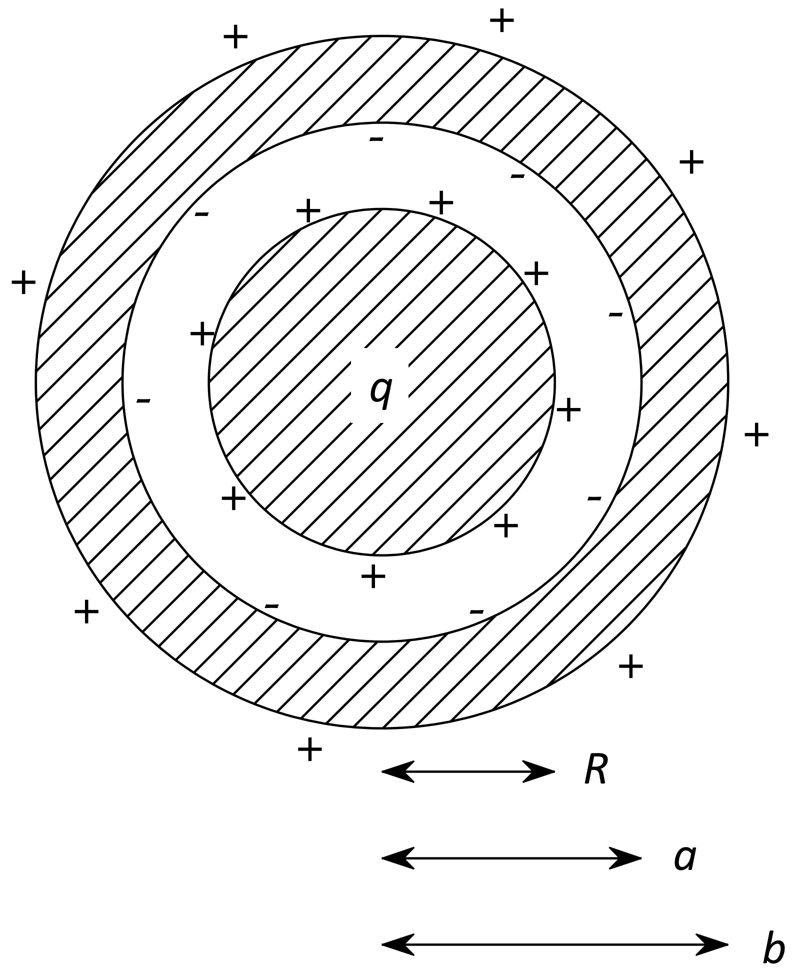
Not solved as a boundary-value problem:

Griffiths, problem 2.35: A metal sphere of radius *R*, carrying charge *q*, is surrounded by a thick concentric metal shell (inner radius *a*, outer radius *b*). The shell carries no net charge.

(a) Find the surface charge density, σ, on each surface.

(b) Find the potential at the center, using infinity as the reference point.

(c) Now the outer surface is **grounded**; that is, it is connected by a conducting wire to zero potential (same potential as at infinity). How do the answers to (a) and (b) change?



Example: concentric conducting spheres (continued)

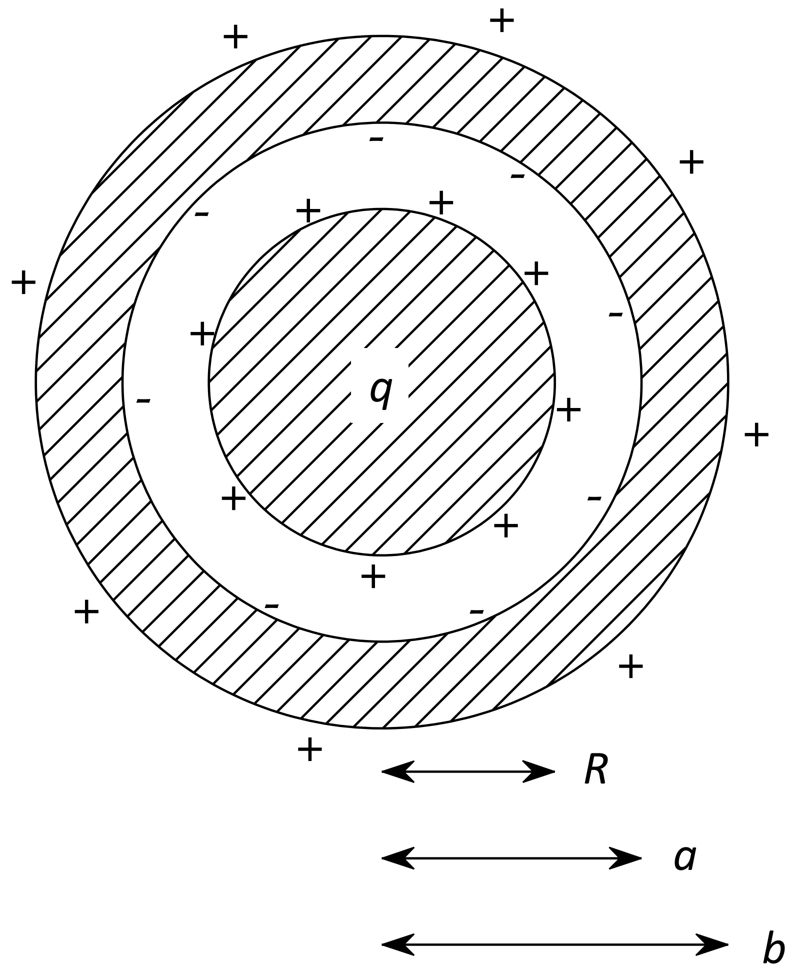
- First, the inner sphere. A conductor carries its charge on its surface, so the density there is the total charge divided by total area:

$$\sigma_R = q \big/ 4\pi R^2 \quad . \qquad \text{(a)}$$

- Next, the inner surface of the shell. A charge is induced on this surface so as to generate an electric field equal and opposite to that from the sphere's charge, within the shell, so that ***E*** = 0 there.

- Spherical charged surfaces generate fields outside themselves that appear as if their charge is concentrated at the center, so the charge on the inner surface is $-q$:

$$\sigma_a = -q \big/ 4\pi a^2 \quad . \qquad \text{(a)}$$



Example: concentric conducting spheres (continued)

- Finally, the outer surface of the shell. The shell has no net charge, and $-q$ was induced on its inner surface, so there must be $+q$ there:

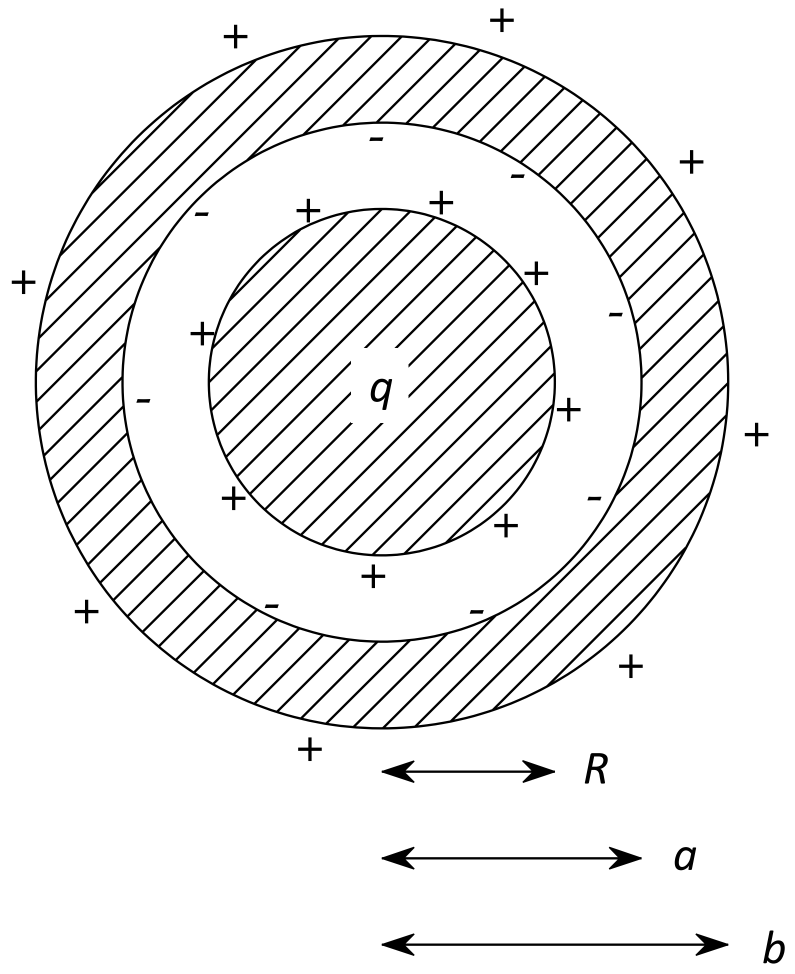
$$\sigma_b = q/4\pi b^2 \quad . \qquad \text{(a)}$$

- The field outside the shell is $\boldsymbol{E} = \hat{\boldsymbol{r}}\, q/r^2$, so the potential of its outer surface is $V = q/b$. This is also the potential at the inner surface, since conductors are equipotentials.

- The field between the inner surface of the shell and the sphere is $\boldsymbol{E} = \hat{\boldsymbol{r}}\, q/r^2$, so

$$V(R) = V(a) - \int_a^R \frac{q}{r^2} dr = \frac{q}{b} + \left.\frac{q}{r}\right|_a^R = \frac{q}{b} + \frac{q}{R} - \frac{q}{a} \quad . \qquad \text{(b)}$$

Since conductors are equipotentials, this applies at the center of the sphere, too.



Example: concentric conducting spheres (continued)

- Now the shell is grounded. The field from the sphere's charge still has to be cancelled within the shell, so the surface charge densities on the sphere and on the inner surface of the shell are the same as before.

- But now the potential on the outer surface of the shell is zero, as at infinity, so the field must be zero outside the shell:

$$V(b)-V(\infty)=-\int_{\infty}^{b}\boldsymbol{E}\cdot d\ell=0 \quad\Rightarrow\quad \boldsymbol{E}=0 \quad .$$

- Thus there's no charge on this surface: $\sigma_b=0$. (c)

- And since the potential is zero on the inner surface too,

$$V(R)=V(a)-\int_{a}^{R}\frac{q}{r^2}dr=\left.\frac{q}{r}\right|_a^R=\frac{q}{R}-\frac{q}{a} \quad . \quad \text{(c)}$$