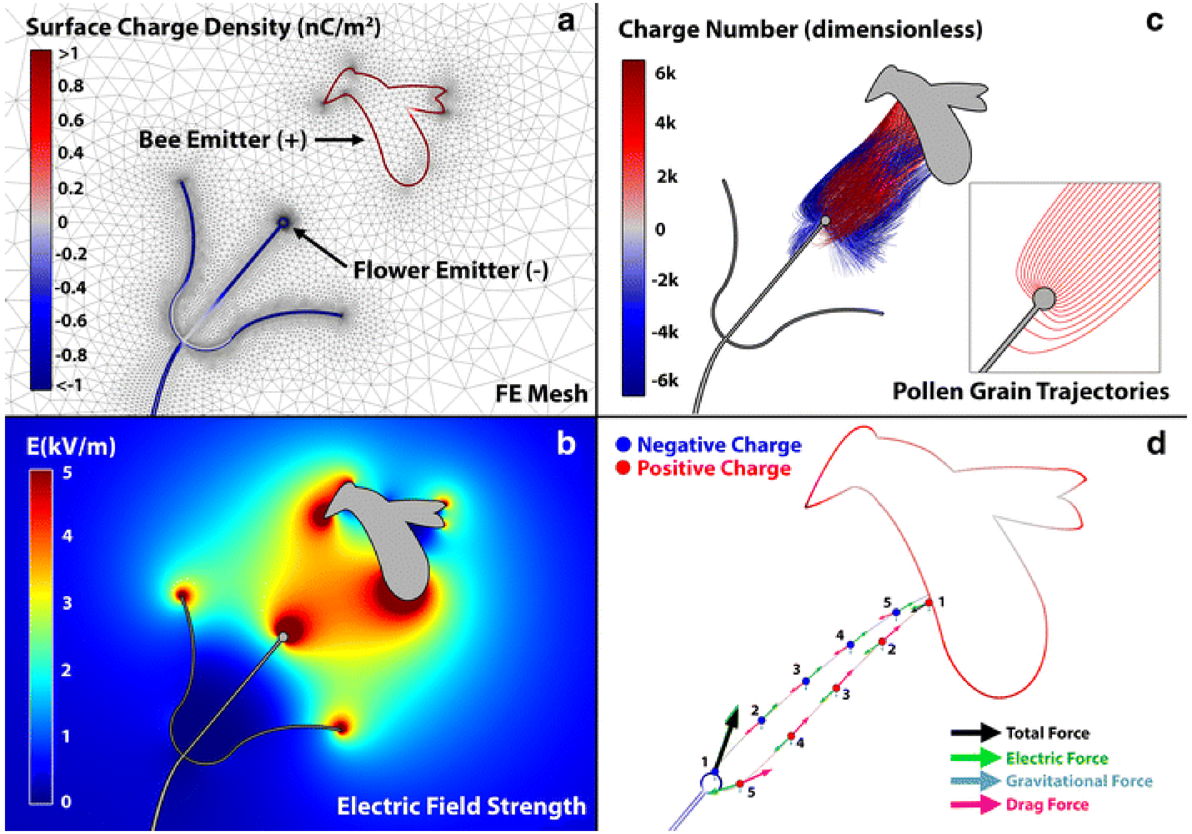
Today in Physics 217: Laplace's Equation

- Conductors, concluded
- Laplace's equation and some simple solutions
- Lemma: the extrema of solutions to Laplace's equation are always on the boundaries, never elsewhere.
- Uniqueness of solutions to Poisson's and Laplace's equations, two of the many such theorems.



Clarke, Morley, & Robert 2017

Field in an empty cavity within a conductor

By empty, we mean there's no charge there.

What is $\boldsymbol{E}$ in an empty cavity within a conductor?

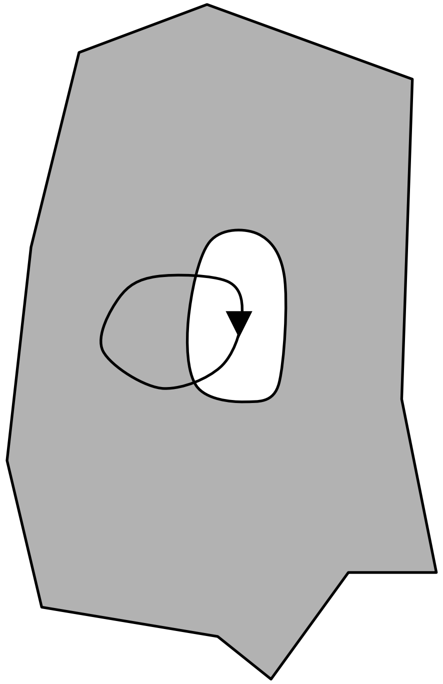
- Consider a loop lying partly within the cavity, and partly within the conductor. $\boldsymbol{E} = 0$ in the conductor, and $\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = 0$, so

$$\int_{\text{path thru cavity}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = 0 \quad .$$

  Since the path is finite, the field within the cavity must be zero.

- This is why you can shield external electrostatic fields with a conducting box -- a Faraday cage – and why you won't get electrocuted if lightning strikes your car while you're inside.

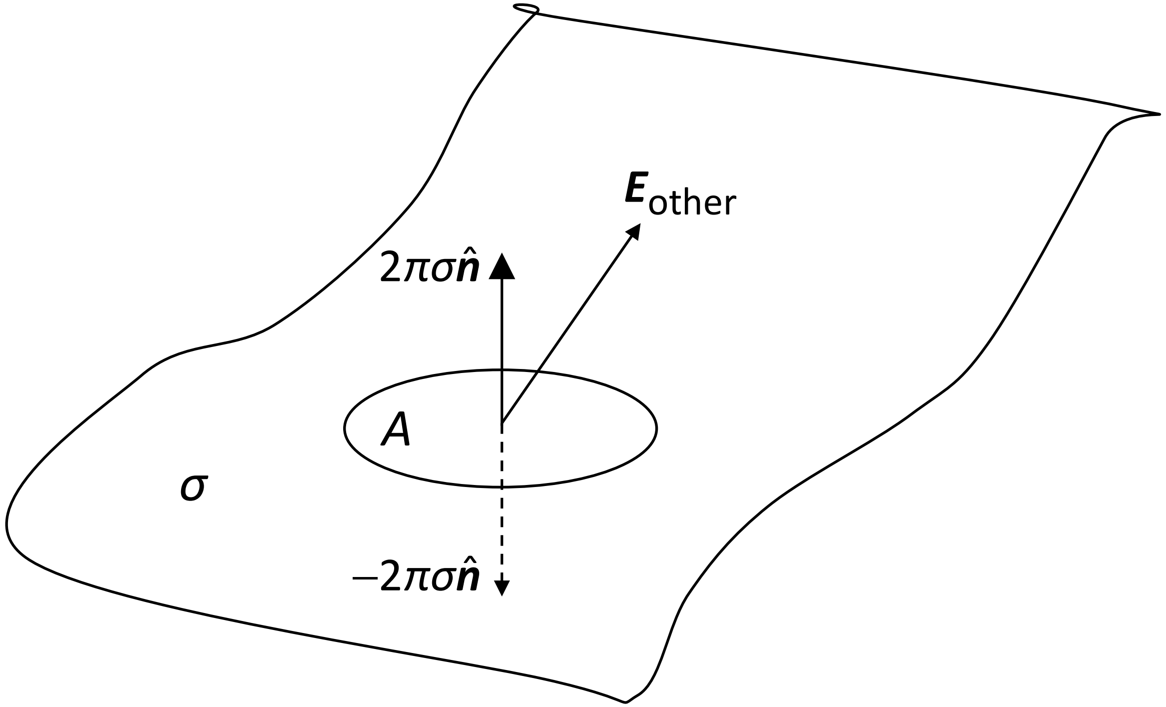
  (Unless the car is not made of a conducting material. Don't try this in a Corvette.)



Forces on conductors

If there's a field difference across a surface charge, as there is on the surface of a charged conductor, then there is a force, too.

- Consider a small patch with area *A* on a charged sheet.
- The total field near the center of the patch is the sum of the patch's field and the field from the rest of the charge distribution ("other"). Only the "other" $\boldsymbol{E}$ exerts a force on the patch. What is $\boldsymbol{E}_{\text{other}}$?
- Just above and just below the center of the patch, we have



$$\boldsymbol{E}_{\text{above}} = \boldsymbol{E}_{\text{other}} + 2\pi\sigma\hat{\boldsymbol{n}}$$
$$\underline{\boldsymbol{E}_{\text{below}} = \boldsymbol{E}_{\text{other}} - 2\pi\sigma\hat{\boldsymbol{n}}}$$
$$\boldsymbol{E}_{\text{above}} + \boldsymbol{E}_{\text{below}} = 2\boldsymbol{E}_{\text{other}} \quad \Rightarrow \quad \boldsymbol{E}_{\text{other}} = \left(\boldsymbol{E}_{\text{above}} + \boldsymbol{E}_{\text{below}}\right)/2 \quad .$$

Forces on conductors (continued)

- We can also recall the boundary condition derived on Tuesday,

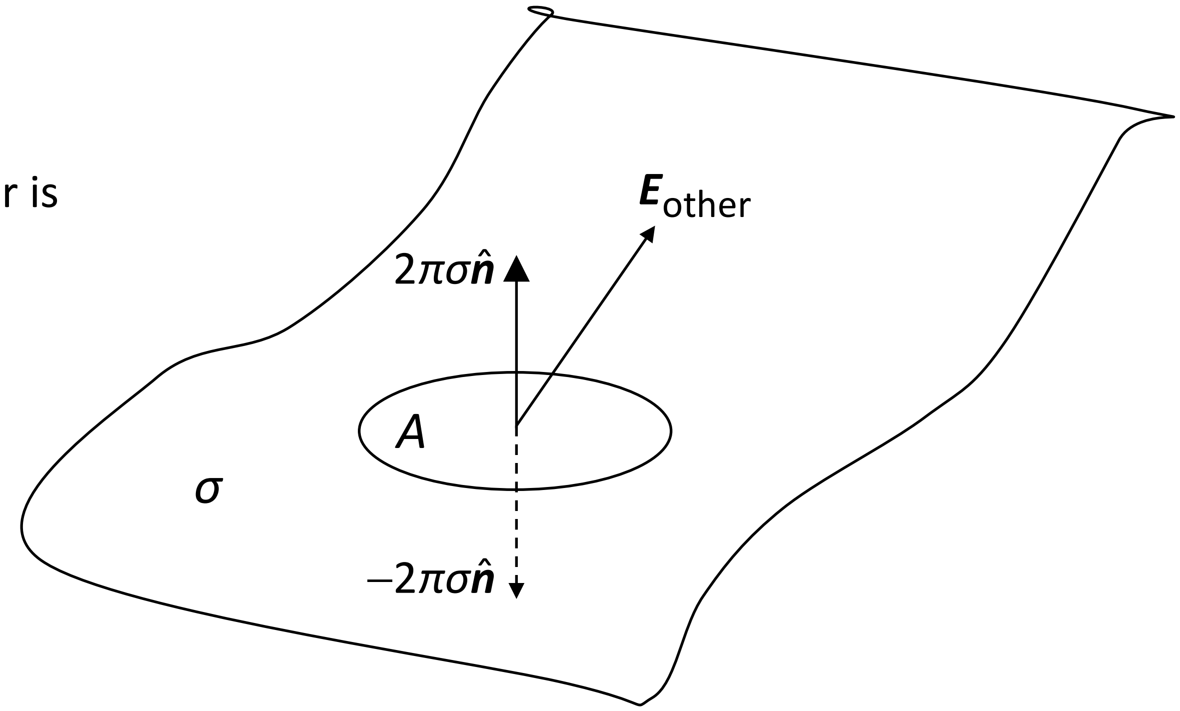
$$\boldsymbol{E}_{\text{above}} - \boldsymbol{E}_{\text{below}} = 4\pi\sigma\hat{\boldsymbol{n}} \quad .$$

- So, for instance, the field just above a charged conductor is $\boldsymbol{E} = 4\pi\sigma\hat{\boldsymbol{n}}$, the field below the surface is zero, and

$$\boldsymbol{F} = q\boldsymbol{E}_{\text{other}} = \sigma A \frac{1}{2}\left(4\pi\sigma\hat{\boldsymbol{n}} + 0\right) = 2\pi\sigma^2 A\hat{\boldsymbol{n}} \quad .$$

- Or, express this as a pressure (force per unit area) in terms of the field just outside the conductor:

$$P = \frac{F}{A} = 2\pi\sigma^2 = \frac{1}{8\pi}E^2_{\text{above}} \quad .$$



Solving the Laplace equation

So far: In electrostatics, we're usually looking for ***E***.

- If there isn't symmetry appropriate for the use of Gauss' Law, we use the generalized Coulomb's Law:

$$\boldsymbol{E} = \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \rho(r') d\tau' \quad .$$

- Or, if that's too hard, use

$$V = \int \frac{\rho(r') d\tau'}{\mathscr{r}} \quad , \quad \boldsymbol{E} = -\boldsymbol{\nabla} V \quad .$$

- Even that's too hard sometimes, so we wind up trying the Poisson equation:

$$\nabla^2 V = -4\pi\rho \quad ; \quad \boldsymbol{E} = -\boldsymbol{\nabla} V \quad .$$

- But very often the charge density is zero **where we want to compute *V* and *E***, so this becomes the Laplace equation:

$$\nabla^2 V = 0 \quad ; \quad \boldsymbol{E} = -\boldsymbol{\nabla} V \quad .$$

Solving the Laplace equation (continued)

- So a large fraction of our work from now through the midterm will be on solutions to the Laplace equation.

- General solution in 1-D: just integrate indefinitely twice, and accept the two integration constants:

$$\frac{d^2V}{dx^2}=0 \quad \Rightarrow \quad \frac{dV}{dx}=a \quad \Rightarrow \quad V(x)=ax+b \quad .$$

- To find out what the integration constants are, consult boundary conditions. Suppose for example we were told that V = 4 at x = 1, and 0 at x = 5. This gives two equations in two unknowns, easily solved:

$$\left.\begin{aligned}4&=a+b\\0&=5a+b\end{aligned}\right\}\Rightarrow a=-1,\, b=5 \quad \Rightarrow \quad V(x)=-x+5 \quad .$$

Solving the Laplace equation (continued)

- General solution in spherical symmetry, in which things only depend upon *r*, and not on the other two spherical coordinates:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \quad \Rightarrow \quad r^2\frac{\partial V}{\partial r} = a \quad \Rightarrow \quad \frac{\partial V}{\partial r} = \frac{a}{r^2} \quad \Rightarrow \quad V = a\int\frac{dr}{r^2} = -\frac{a}{r} + b \quad .$$

- Boundary conditions: suppose that $V = 0$ at $r \to \infty, V = V_0$ at $r = R$. Then the first of these implies $b = 0$, the second implies that $a = V_0 R$.

- General solution in cylindrical coordinates, but independent of φ and z:

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0 \quad \Rightarrow \quad s\frac{\partial V}{\partial s} = a \quad \Rightarrow \quad V = a\int\frac{ds}{s} = a\ln s + b' = a\ln\left(\frac{s}{b}\right) \quad .$$

You get the idea.

Solving the Laplace equation (continued)

- A key feature of solutions to the Laplace equation is that their **extrema** (maxima or minima) occur **only at the boundaries**, never elsewhere. Which figures:

  - In 1-D, maxima and minima of a continuous function $f(x)$ are normally identified by the signs of their second derivatives d^2f/dx^2 at locations x where their first derivatives df/dx vanish.

  - Similarly in 2-D and 3-D, for scalar functions $f(\mathbf{r})$, first derivatives $\boldsymbol{\nabla} f$, and second derivatives $\nabla^2 f$.

- One good way to express this is to note that Laplace-equation solutions are such that their values at a given point are equal to the average values in their neighborhood, independent of the neighborhood's size. For example, in 1-D:

$$V(x) = ax + b = \frac{1}{2}a(x+d+x-d) + b = \frac{1}{2}\left[a(x+d) + b + a(x-d) + b\right]$$

$$= \frac{1}{2}\left[V(x+d) + V(x-d)\right] = \langle V \rangle \quad \text{, independent of } d.$$

Let's prove this important result in 3-D next.

Extrema of solutions to the Laplace Equation

Uniqueness proofs go better if we prove this first.

Lemma: If $V(\boldsymbol{r})$ is a solution to the Laplace equation, its average over any spherical surface is equal to the value at the center of the sphere.

- For a sphere of radius R, centered for our convenience at the coordinate origin, this average is

$$\langle V(R)\rangle = \frac{1}{4\pi R^2}\int V(\boldsymbol{r})\,da = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} V(\boldsymbol{r})\sin\vartheta\, d\vartheta\, d\varphi \quad .$$

- Take its derivative with respect to R:

$$\frac{d}{dR}\langle V(R)\rangle = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\partial}{\partial R}V(\boldsymbol{r})\sin\vartheta\, d\vartheta\, d\varphi \quad .$$

- The sphere's radius is perpendicular to its surface. Along one of these radii, say the one pointing in the $\hat{\boldsymbol{r}}$ direction, the integrand is the component of the gradient of V in that direction: $\partial V/\partial R = (\boldsymbol{\nabla} V)\cdot\hat{\boldsymbol{r}}$.

Extrema of solutions to the Laplace Equation (continued)

- Substitute this into the integral, and then notice that doing so has removed any explicit dependence on *R*, so we can restore the original factors:

$$\frac{d}{dR}\langle V(R)\rangle = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}(\boldsymbol{\nabla} V)\cdot\hat{\boldsymbol{r}}\sin\vartheta d\vartheta d\varphi = \frac{1}{4\pi R^2}\int_0^{2\pi}\int_0^{\pi}(\boldsymbol{\nabla} V)\cdot\hat{\boldsymbol{r}}R^2\sin\vartheta d\vartheta d\varphi$$

$$= \frac{1}{4\pi R^2}\oint_{\text{sphere}}(\boldsymbol{\nabla} V)\cdot d\boldsymbol{a} = \frac{1}{4\pi R^2}\int_{\text{sphere}}\nabla^2 V d\tau = 0 \quad ,$$

  where we used the divergence theorem in the second to last step, and the Laplace equation itself in the last.

- That this derivative is zero means that $\langle V(R)\rangle$ is independent of *R.* In particular it means that all spheres give the same average value, even vanishingly small ones:

$$\langle V(R)\rangle = \lim_{R\to 0}\langle V(R)\rangle = V(0) \quad \text{, q.e.d.}$$

  This would not be true, were there local extrema between the boundaries.

Uniqueness of solutions of the Poisson and Laplace equations

There are two basic sorts of theorems of concern to us as we work with the Poisson and Laplace equations: existence of solutions, and uniqueness of solutions.

- Existence is a much harder proof, best left to mathematicians. The existence of solutions to these two second-order PDEs has been proven, but the exposition of the proof takes many lectures and deep math skills, which is why one doesn't see these proofs in E&M textbooks, or even in mathematical physics textbooks.
- Also after one's first few solutions most are willing just to read the proof in a math textbook like that by L.C. Evans: Partial Differential Equations, second edition.
- On the contrary, physicists lean hard on uniqueness proofs and need to understand them thoroughly.
- Fortunately they're simpler. For Poisson/Laplace + boundary conditions, they mostly follow a pattern like this
  - Suppose that there are two solutions, V_1 and V_2, and by superposition $U = V_1 - V_2$.
  - Show that U is everywhere zero, so that V_1 and V_2 are identical and therefore unique.

Uniqueness of solutions of the Poisson and Laplace equations (continued)

This one spans all the different kinds of boundary conditions we will use, all at once, so Griffiths's first uniqueness theorem is a sub-case of it.

The Poisson equation, when subjected to boundary conditions on the solution, its first derivative, or combinations of these, yields unique solutions.

- Let V be continuous, and its derivatives continuous through second order, in $\mathcal{V}$ bounded by $\mathcal{S}$. Suppose

$$\nabla^2 V = f \ , \quad \text{functions of } x,y,z \text{ in } \mathcal{V} \ ; \quad p\frac{dV}{dn} + hV = g, \quad \text{functions of } x,y,z \text{ on } \mathcal{S}.$$

  where p and $h \geq 0$, but are never zero simultaneously; and where d/dn is the component of the gradient normal to $\mathcal{S}$.

- If V_1 and V_2 two solutions, then $U = V_1 - V_2$ is a solution to the Laplace equation:

$$\nabla^2 U(x,y,z) = \nabla^2 V_1(x,y,z) - \nabla^2 V_2(x,y,z) = 0 \ ; \quad p\frac{dU}{dn} + hU = 0 \quad .$$

Uniqueness of solutions of the Poisson and Laplace equations (continued)

- And thus by the Lemma, ***U* has can have no extrema in $\mathcal{V}$.** It is smooth or flat between the boundaries.

- Furthermore, by $\nabla^2 U = 0$, the divergence theorem, and Product Rule #5, we can write

$$\int_{\mathcal{S}} \left(U\boldsymbol{\nabla}\right) \cdot d\boldsymbol{a} = \int_{\mathcal{S}} U \frac{dU}{dn} da = \int_{\mathcal{V}} U^2 d\tau \quad .$$

- So suppose U = 0 everywhere on $\mathcal{S}$. Then $\int_{\mathcal{S}} U \frac{dU}{dn} da = 0$, and since U^2 can't be negative, then U = 0 everywhere in $\mathcal{V}$.

- And by the same token, if $dU/dn = 0$ everywhere on $\mathcal{S}$, then U = uniform everywhere in $\mathcal{V}$.

- And if $\frac{dU}{dn} + hU = 0$, then $\int_{\mathcal{S}} U \frac{dU}{dn} da = -h \int_{\mathcal{S}} U^2 da \leq 0$, while $\int_{\mathcal{V}} U^2 d\tau$ is still nonnegative, so again U = uniform everywhere in $\mathcal{V}$.

Uniqueness of solutions of the Poisson and Laplace equations (continued)

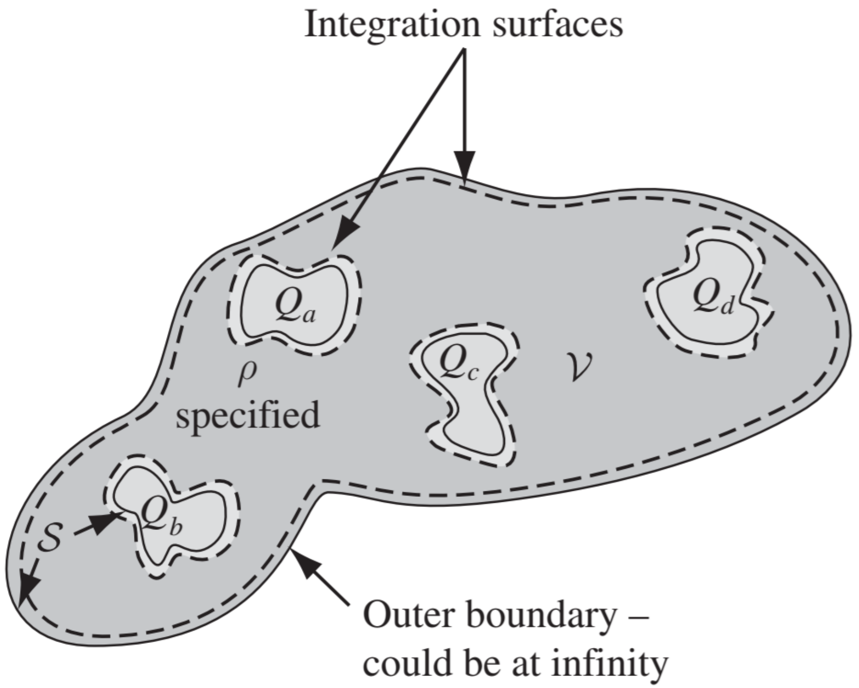
- Thus the cases for $p\frac{dU}{dn}+hU=0$:
  - p = 0. Then U = 0 everywhere on $\mathcal{S}$; thus ***U* = 0** everywhere within $\mathcal{V}$ as well, and V_1 and V_2 are identical: **the solution is unique**, q.e.d. (This case is Griffiths's **First** uniqueness theorem.)
  - h = 0. Then $dU/dn=0$ everywhere on $\mathcal{S}$, and everywhere within $\mathcal{V}$; that is, ***U* is uniform within $\mathcal{S}$ + $\mathcal{V}$**. The two solutions can be said to be the same, **unique apart from this additive constant**, q.e.d. And thus, like the additive constant ambiguity of the electric potential, V_1 and V_2 would give the same fields and forces.
  - Mixed: p and h both nonnegative, both functions of position. If U = 0 **anywhere** on $\mathcal{S}$, then ***U* = 0 everywhere within $\mathcal{V}$** as well, as the additive constant can have no other value besides 0, and the solutions are unique, q.e.d. If not, then ***U* is uniform within $\mathcal{S}$ + $\mathcal{V}$**, and the solutions are unique apart from an additive constant, q.e.d.

Uniqueness of solutions of the Poisson and Laplace equations (continued)

The **Second** uniqueness theorem covers boundaries applied by charged conductors. Sketch of the proof here; see Griffiths section 3.1.6 for interesting details.

In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the **total charge** on **each** conductor is given.

- ***E*** = 0 inside conductors. Any distributed charge would move *until* that were true.
- There would be only one final resting place for each of the charges.
- Thus $V = \int (\rho/\mathscr{r})\,d\tau$ integrates to one and only one answer at each point in space.



Griffiths figure 3.6

Uniqueness of solutions of the Poisson and Laplace equations (continued)

- As for **total charge** on **each** conductor, keep in mind the Purcell example Griffiths offers:
  - Four conductors, each with ±*q*: obviously plausible solution to the Poisson equation.
  - Connect in pairs as shown: now there are two, each with zero total charge.
  - Although Figure 3.8 looks as plausible a solution as 3.7 in its matchup of +*q* and –*q*, so does Figure 3.9.
  - The second uniqueness theorem forbids (at least) one of them, in this case Figure 3.8. As one would also conclude from the conductors in Figure 3.8 not being equipotentials, at least for the instant after hooking up the wire.

Fig. 3.7

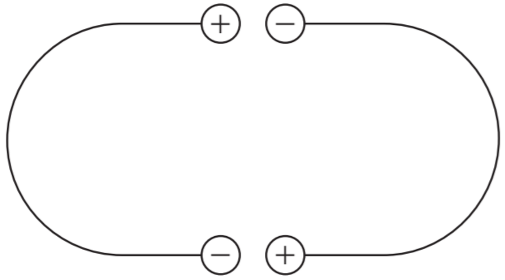


Fig. 3.8

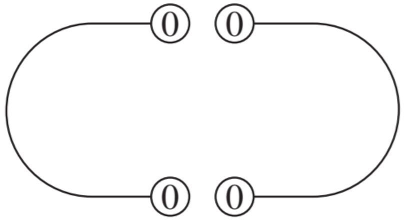


Fig. 3.9

Utility of uniqueness of solutions to the Laplace equation

- One can try any means at one's disposal to derive or guess a solution to the Laplace equation. If one can find a solution by any means that fits the boundary conditions, or gives the specified sum of charges on conducting boundaries, then one has found the **unique** correct answer.

- So some of the approaches we will use to solve the Laplace equation, though they seem like dirty tricks:

  Separation of variables
  Method of images
  Multipole expansion
  Method of relaxation (finite-element analysis)

  do offer systematic paths to solutions to the full boundary value problems, and as the solutions are unique, we can excuse the funny looks to the approaches.