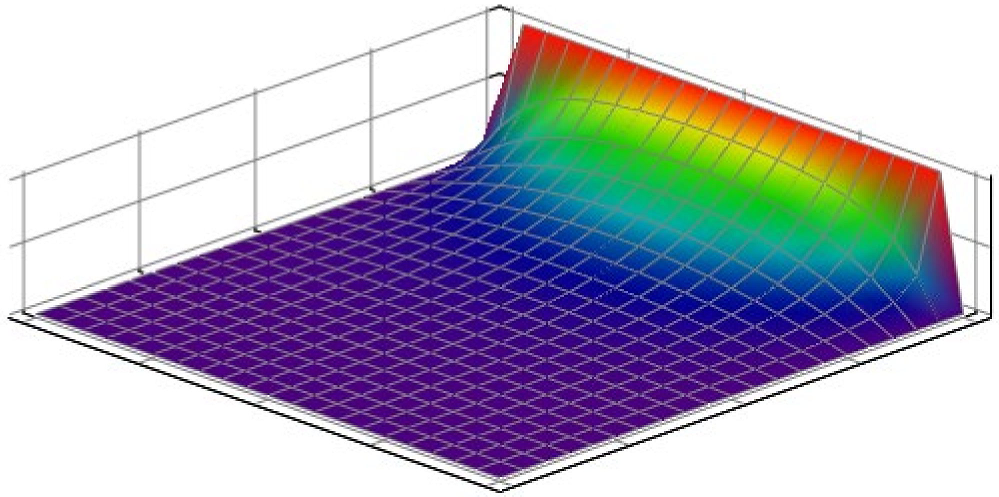
Today in Physics 217: solution of the Laplace equation by separation of variables

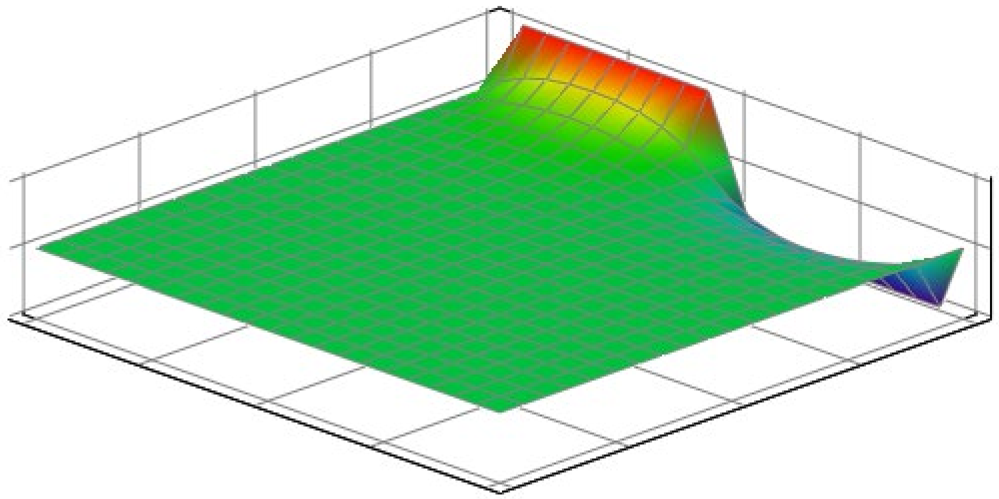


- Introduction to the method, in Cartesian coordinates.

- Example solution of the Laplace equation for the potential in an infinite slot, arbitrary *V* at the bottom, in which we introduce two common features of separation solutions:

  - Completeness and orthogonality of sines.

  - Fourier’s trick.



Introduction to separation of variables

Separation of variables is the easiest direct solution technique. It works best with conducting boundaries for which the surfaces are well behaved – planes, spheres, cylinders, *etc.* – but it's OK with any boundary conditions on V or $\partial V / \partial n$.

Here's how it works, in Cartesian coordinates, in which the Laplace equation is

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

which can't be integrated directly like the 1-D case.

- Consider solutions of the form $\qquad V(x,y,z) = X(x)Y(y)Z(z) \quad .$
- This makes the Laplace equation $\qquad YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0 \quad ,$

  or, dividing through by *XYZ*, $\qquad \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \quad ,$

Introduction to separation of variables (continued)

- This is of the form $f(x)+g(y)+h(z)=0$. The only way for it to be true for all *x,y,z* is for each term to be a **constant**, and for the three constants to add up to zero:

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=A+B-(A+B)=0 \quad .$$

- thus to **separate** the original partial differential equation (**PDE**) into three ordinary ones (**ODEs**):

$$\frac{d^2X}{dx^2}-AX=0 \quad , \quad \frac{d^2Y}{dy^2}-BY=0 \quad , \quad \frac{d^2Z}{dz^2}+(A+B)Z=0 \quad .$$

- If the equation separates into simple ODEs, as it did here, then the PDE's solution is straightforward.

Introduction to separation of variables (continued)

- Nothing guarantees that V will always factor into functions of x, y, and z alone. In fact, there are certainly many solutions to the Laplace equation which are not of this form.

  However,

  - there are in fact lots of electrostatic problems for which the boundary conditions are specified on well-behaved surfaces, and do turn out to have solutions of this form, and

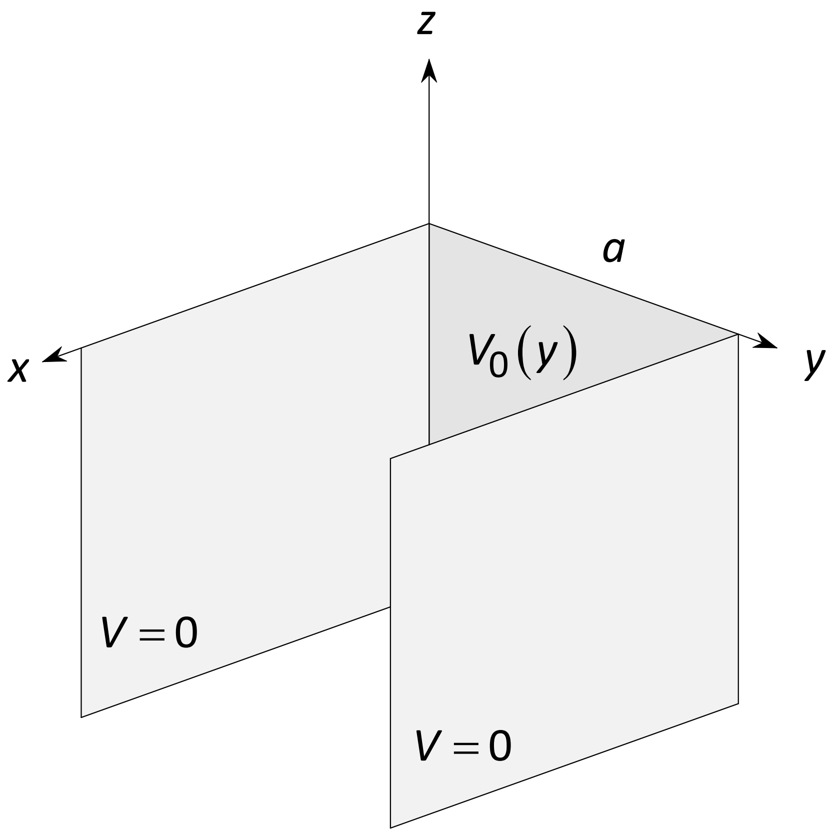
  - the solutions to electrostatics problems are unique, so if separation of variables yields a solution, it's guaranteed to be the correct one.

  - And if the PDE does not separate cleanly like this one did into easily-solved ODEs, then try one of the other techniques we will learn/have learned.

- Separation of variables is also a very useful PDE solution technique in quantum mechanics, where one finds many problems in which the boundary conditions are specified on regular, well-behaved surfaces.

Introductory example: the infinite conducting slot

Griffiths, example 3.3: Two infinite, grounded, metal plates lie parallel to the *x*-*z* plane, one at $y = 0$, the other at $y = a$. The end at $x = 0$ is closed off with an infinite strip insulated from the two plates and maintained at a specified potential $V_0(y)$. Find the potential V inside this slot.



- Only a section of the arrangement appears at right, so we can see inside. The plates stretch to $z \to \pm\infty$ and $x \to \infty$.

- The slot is infinite in both directions along *z*, so the solution can't depend upon *z*; we write the Laplace equation as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 V}{\partial z^2}}}_{0} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad .$$

The infinite slot (continued)

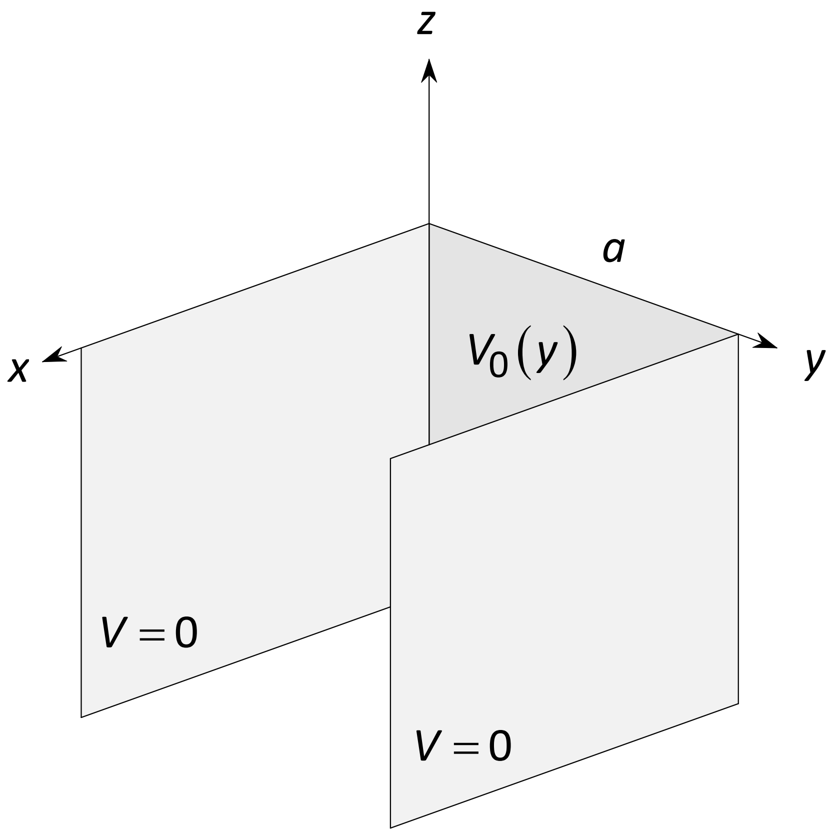
- Suppose $V(x,y) = X(x)Y(y)$; then the PDE separates into

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0 \qquad \Rightarrow \qquad \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 = k^2 - k^2 \quad ,$$

or:

$$\text{I.} \quad \frac{d^2X}{dx^2} - k^2X = 0 \qquad , \qquad \text{II.} \quad \frac{d^2Y}{dy^2} + k^2Y = 0 \quad .$$

- We chose k^2 rather than, say, A, to indicate that this constant is non-negative, and that the other one $(-k^2)$ is non-positive.
  - Also with hindsight, as $-k^2 = (ik)^2$ turns out to be useful.

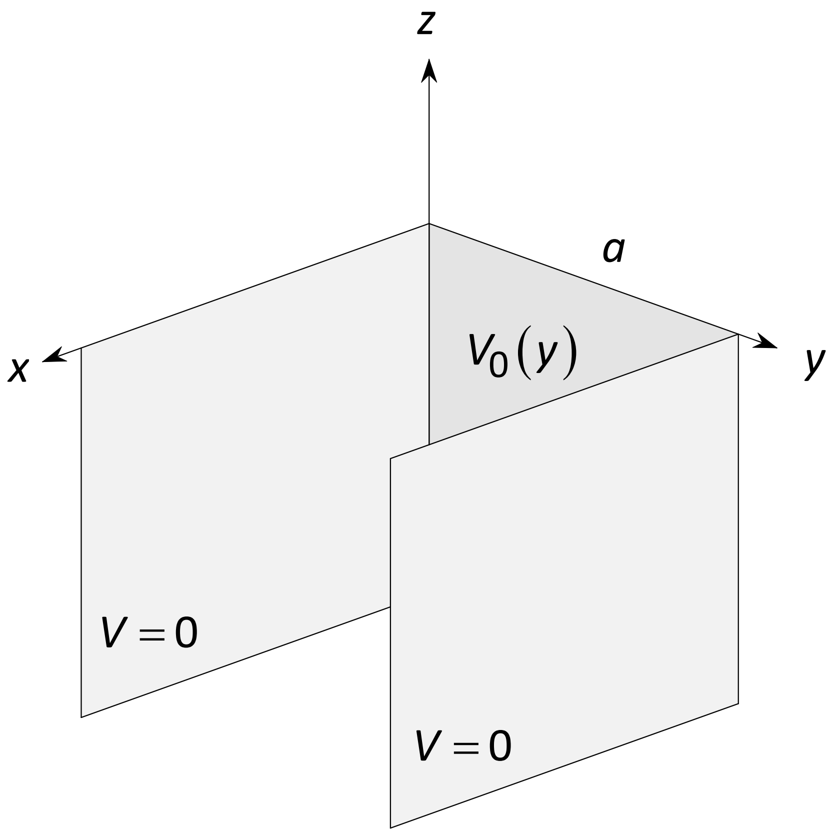


The infinite slot (continued)

- Boundary conditions:

  1. $V \to 0$ as $x \to \infty$
  2. $V = 0$ at $y = 0$
  3. $V = 0$ at $y = a$
  4. $V = V_0(y)$ at $x = 0$

  Reference point an infinite distance away from the non-grounded plane



- Solutions: I’m sure you know equations I and II very well from MATH 165/174 and PHYS 122-123/143-142. But the means by which they’re solved is too useful to forget, so I’ll remind you, first, with I:

$$\frac{d^2X}{dx^2} = k^2X \quad .$$

- Let $v = dX/dx$, and multiply through by $v = dX/dx$: $\quad v\frac{dv}{dx} = k^2X\frac{dX}{dx} \quad .$

The infinite slot (continued)

- Integrate over *x*; invoke the chain rule

$$\int v\frac{dv}{\cancel{dx}}\cancel{dx} = k^2\int X\frac{dX}{\cancel{dx}}\cancel{dx} \quad\Rightarrow\quad \frac{v^2}{2} = k^2\frac{X^2}{2} + S \quad .$$

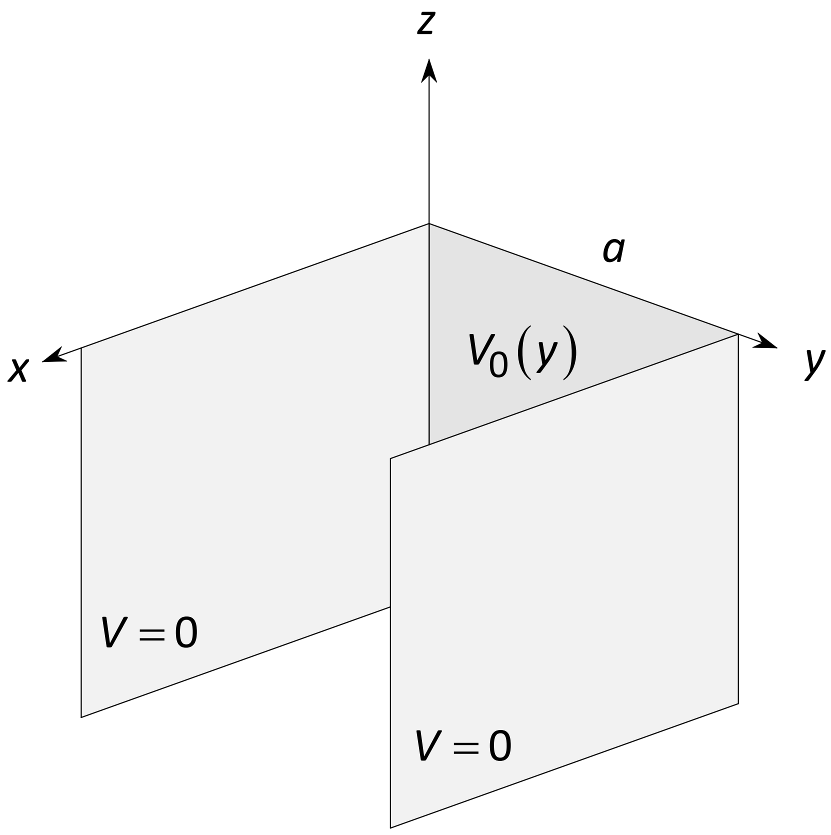
- Both *X* and *dX/dx* must be zero for all *x* at *y* = 0 and *a*, so the integration constant *S* = 0, and

$$v = \frac{dX}{dx} = kX \text{ or } -kX \quad .$$

- Separate and integrate: $\int\frac{dX}{X} = k\int dx \text{ or } -k\int dx$

$$\ln X = kx + T \text{ or } -kx + U$$

$$X = e^{kx+T} = Ae^{kx} \text{ or } e^{-kx+U} = Be^{-kx} ,$$



The infinite slot (continued)

for two **particular** solutions to ODE I. Here A and B are just integration constants.

- Any linear combination of the particular solutions is also a solution to ODE I, and would be more general, so we take

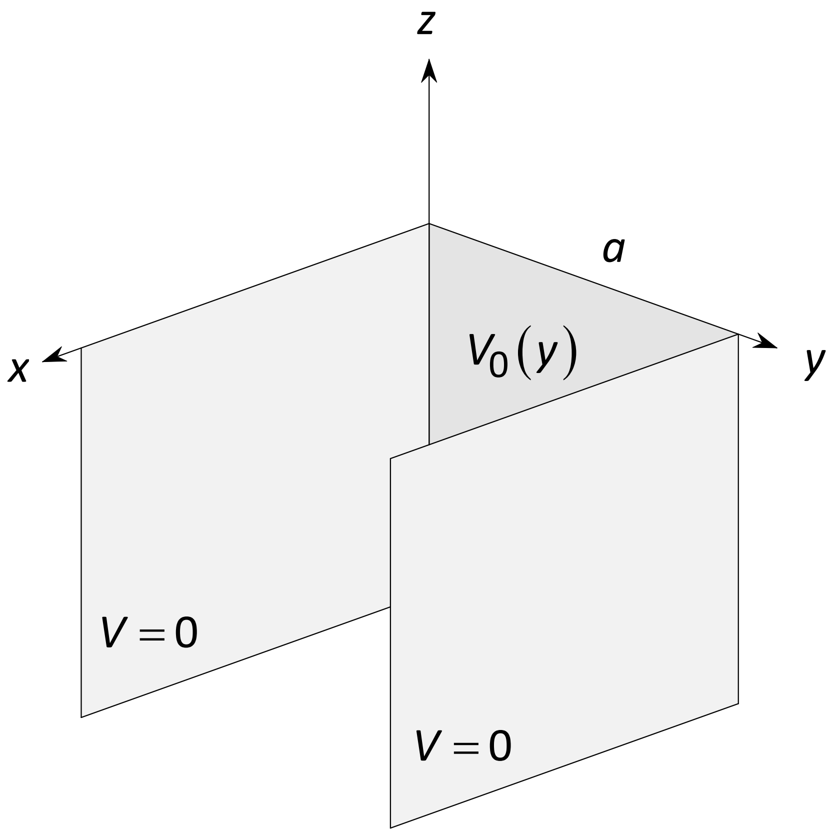
$$X(x) = Ae^{kx} + Be^{-kx} \quad .$$

- Similarly, for ODE II, $d^2Y/dy^2 = -k^2Y$, and a more general solution is

$$Y(y) = C'e^{iky} + D'e^{-iky} \quad .$$

- Or, using Euler's formula, $\sin\alpha = \dfrac{e^{i\alpha} - e^{-i\alpha}}{2i}$ and $\cos\alpha = \dfrac{e^{i\alpha} + e^{-i\alpha}}{2}$, we have

$$Y(y) = C\sin ky + D\cos ky \quad ,$$



The infinite slot (continued)

so $$V(x,y)=\left(Ae^{kx}+Be^{-kx}\right)\left(C\sin ky+D\cos ky\right) \quad .$$

- Apply the boundary conditions:

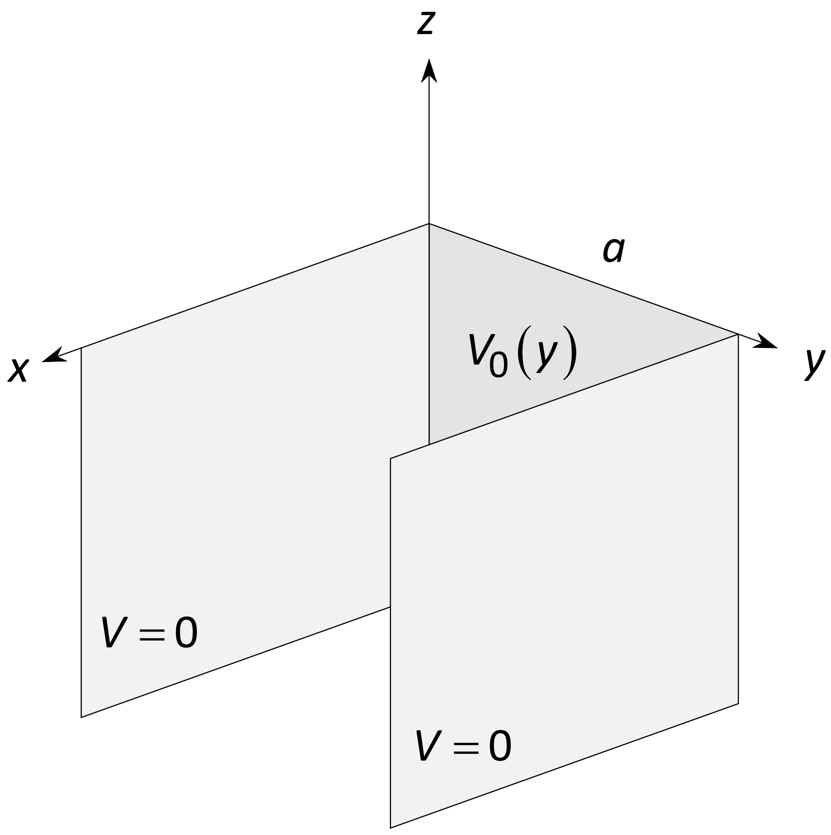
1. $V\to 0$ as $x\to\infty$, but $e^{kx}\to\infty$ as $x\to\infty \quad \Rightarrow \quad A=0 \quad .$
2. $V=0$ at $y=0$, so $0=Be^{-kx}\left(C(0)+D(1)\right) \quad \Rightarrow \quad D=0$

   $$\Rightarrow \quad V(x,y)=BCe^{-kx}\sin ky=Ge^{-kx}\sin ky \quad .$$
3. $V=0$ at $y=a$, so $0=Ge^{-kx}\sin ka \quad \Rightarrow \quad \sin ka=0$

   $$\Rightarrow \quad k=\frac{n\pi}{a} \quad , \quad n=0,1,2,\ldots$$

   $$\Rightarrow \quad V_n(x,y)=G_n e^{-n\pi x/a}\sin\frac{n\pi y}{a} \quad , \quad n=1,2,3\ldots$$

   Ignore the trivial solution $G_0=0\Rightarrow V(x,y)=0.$

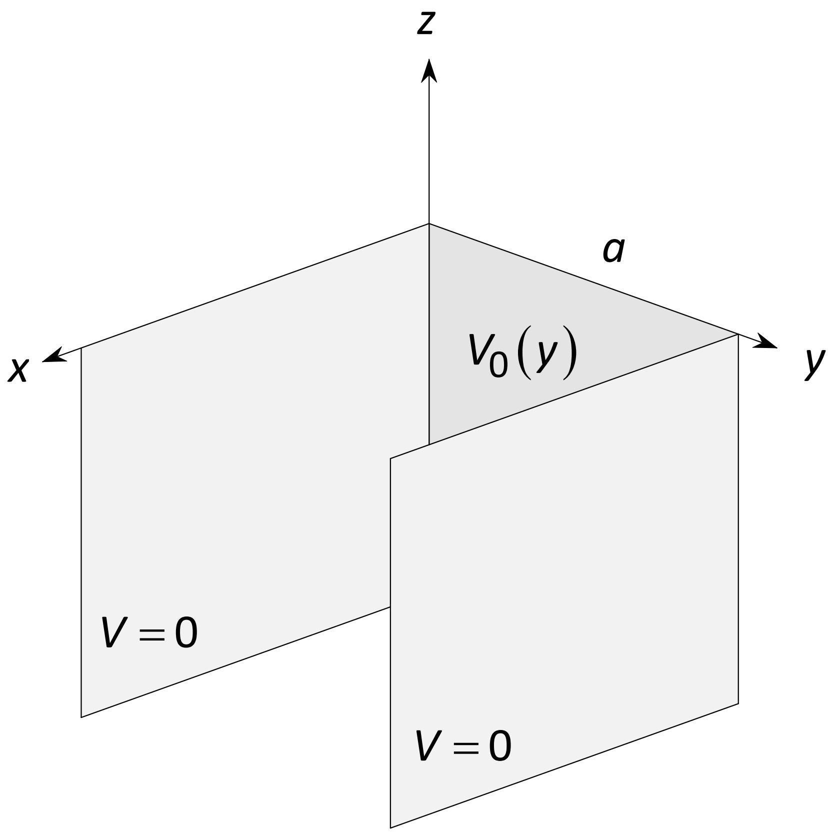


The infinite slot (continued)

- The remaining boundary condition (4) needs to be used to determine the G_n.
  - Note first that this solution **won't work unless V_0 itself is sinusoidal.**
  - BUT, if the $V_n(x,y)$ are all solutions, then a linear combination of them is a solution too:

$$V(x,y) = \sum_{n=1}^{\infty} V_n(x,y)$$

because $\nabla^2 V = \nabla^2 \sum_{n=1}^{\infty} V_n(x,y) = \sum_{n=1}^{\infty} \nabla^2 V_n(x,y) = 0$



The infinite slot (continued)

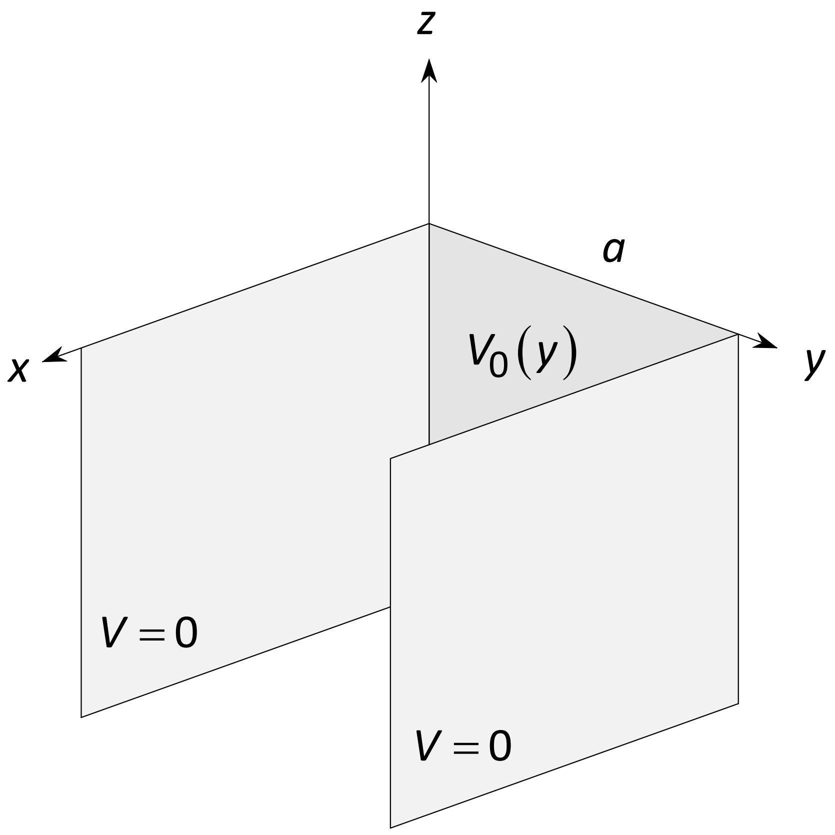
- And an über-general solution can match arbitrary functions $V_0(y)$ at $x = 0$ (boundary condition 4), because the linear combination of all of them,

$$V_0(y) = \sum_{n=1}^{\infty} G_n e^{-n\pi(0)/a} \sin\frac{n\pi y}{a} = \sum_{n=1}^{\infty} G_n \sin\frac{n\pi y}{a} \quad .$$

  is a **Fourier sine series** representation of $V_0(y)$.

- In MATH 281 you have learned, or will learn, that the sines form a **complete** set of functions, for which any arbitrary function of of y can be expressed as a series like this.

- Thus **this** solution works for **any** specified $V_0(y)$, and all we have left to do is to determine the **Fourier coefficients** G_n, via boundary condition 4.



The infinite slot (continued)

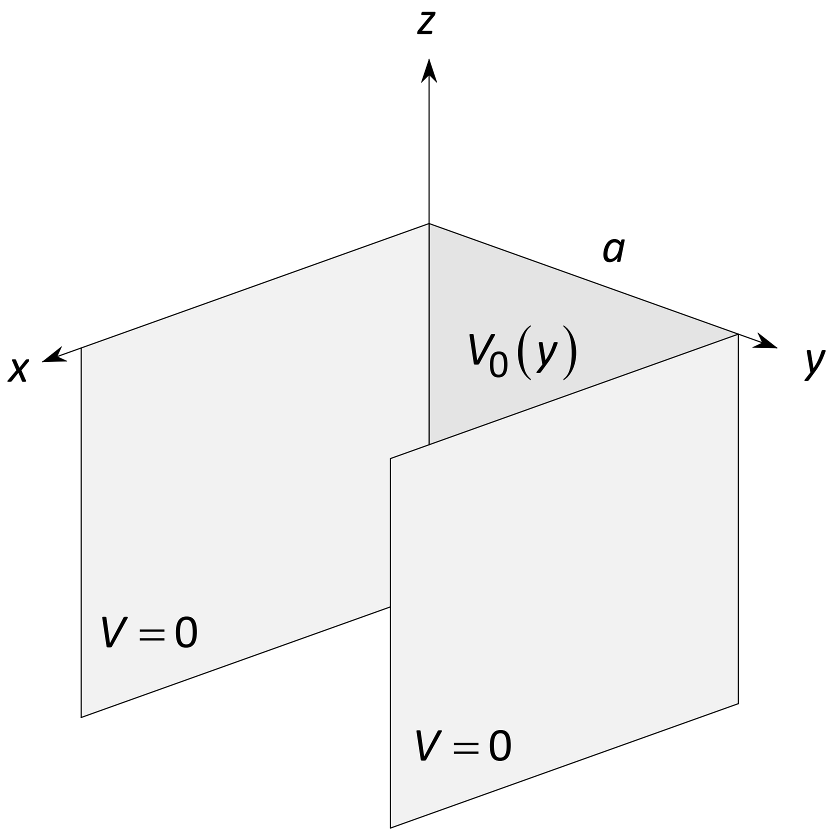
- We do this with **Fourier's trick**: multiply both sides of the equation by $\sin m\pi y/a$, $m = 1,2,3,\ldots$ (note the different index *m*):

$$\sin\frac{m\pi y}{a}V_0(y) = \sum_{n=1}^{\infty} G_n \sin\frac{n\pi y}{a}\sin\frac{m\pi y}{a}$$

- Then integrate both sides over *y*, from zero to *a*:

$$\int_0^a V_0(y)\sin\frac{m\pi y}{a}dy = \sum_{n=1}^{\infty} G_n \int_0^a \sin\frac{n\pi y}{a}\sin\frac{m\pi y}{a}dy$$

- This is a **Fourier transform** of both sides. Now focus on the integral within the sum.



The infinite slot (continued)

- Integrate by parts twice:

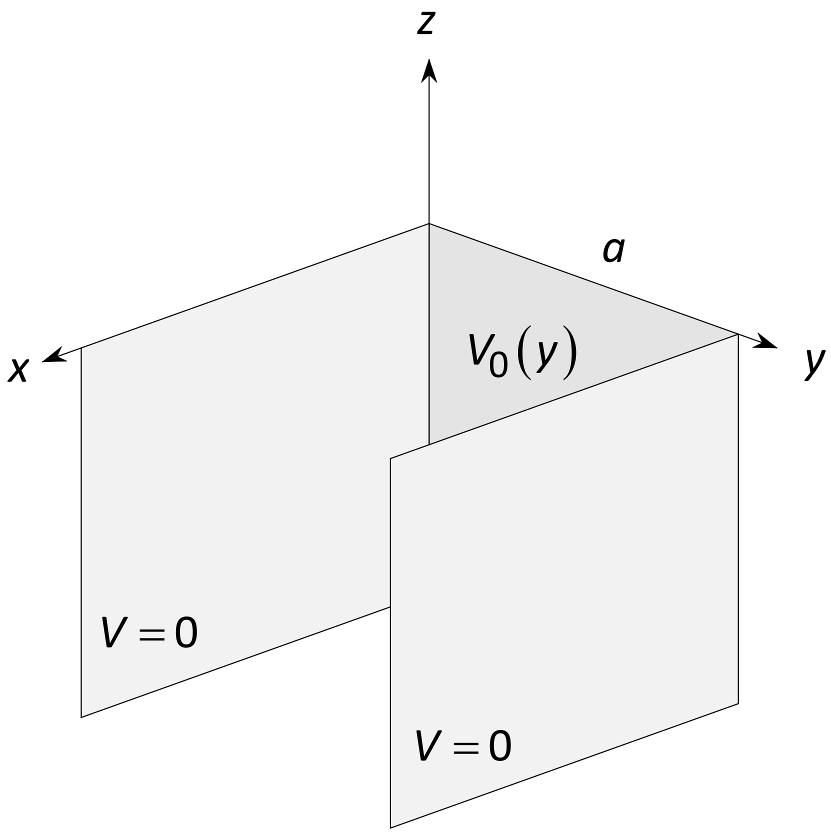
$$\int_0^a \sin\frac{m\pi y}{a}\sin\frac{n\pi y}{a}dy = -\frac{a}{n\pi}\overset{0}{\cancel{\sin\frac{m\pi y}{a}}}\cos\frac{n\pi y}{a}\Bigg|_0^a + \frac{m}{n}\int_0^a \cos\frac{m\pi y}{a}\cos\frac{n\pi y}{a}dy$$

$$= \frac{m}{n}\left[\frac{a}{n\pi}\cos\frac{m\pi y}{a}\overset{0}{\cancel{\sin\frac{n\pi y}{a}}}\Bigg|_0^a + \frac{m}{n}\int_0^a \sin\frac{m\pi y}{a}\sin\frac{n\pi y}{a}dy\right]$$

to obtain
$$\left(1-\frac{m^2}{n^2}\right)\int_0^a \sin\frac{m\pi y}{a}\sin\frac{n\pi y}{a}dy = 0 \quad .$$

- There are two possibilities; either $m = n$ or $m \neq n$. If the latter is the case, then

$$\int_0^a \sin\frac{m\pi y}{a}\sin\frac{n\pi y}{a}dy = 0 \quad (m \neq n).$$



The infinite slot (continued)

- If on the other hand $m = n$, then

$$\int_0^a \sin^2\frac{n\pi y}{a}dy = -\frac{a}{n\pi}\sin\overset{0}{\cancel{\frac{n\pi y}{a}}}\cos\frac{n\pi y}{a}\bigg|_0^a + \int_0^a \cos^2\frac{n\pi y}{a}dy = \int_0^a\left(1-\sin^2\frac{n\pi y}{a}\right)dy \quad ,$$

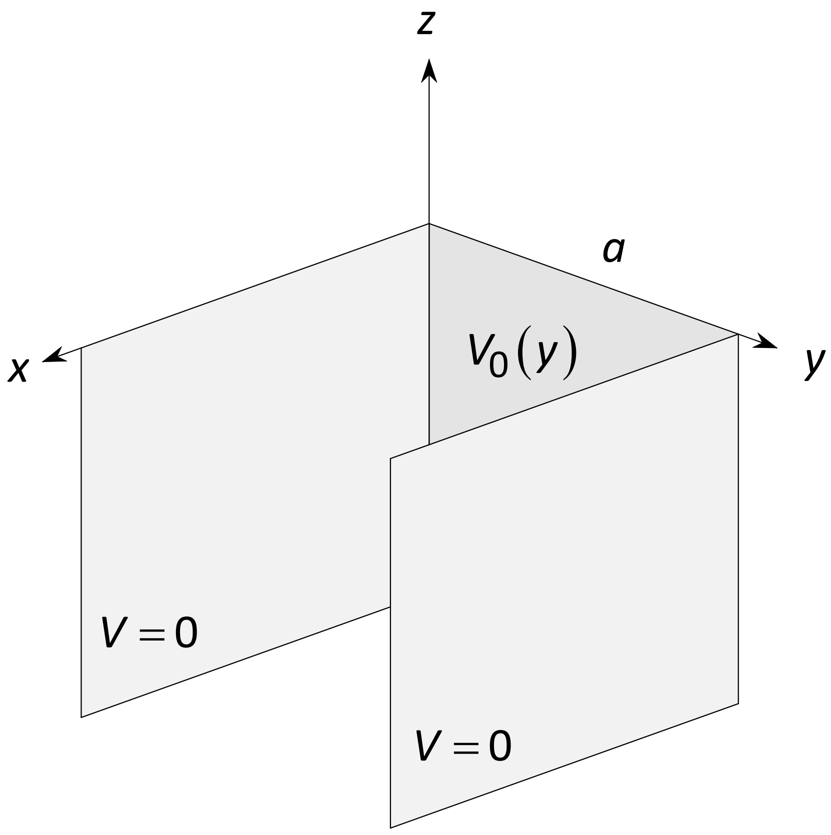
or

$$= 2\int_0^a \sin^2\frac{n\pi y}{a}dy = \int_0^a dy = a$$

- So

$$\int_0^a \sin\frac{m\pi y}{a}\sin\frac{n\pi y}{a}dy = \begin{Bmatrix} \frac{a}{2}, \ m=n \\ 0, \ m\neq n \end{Bmatrix} = \frac{a}{2}\delta_{mn} \quad ;$$

the functions $\sin(n\pi y/a), n = 1,2,3,\ldots$ are **orthogonal** in the range $y = 0 \to a$.



The infinite slot (continued)

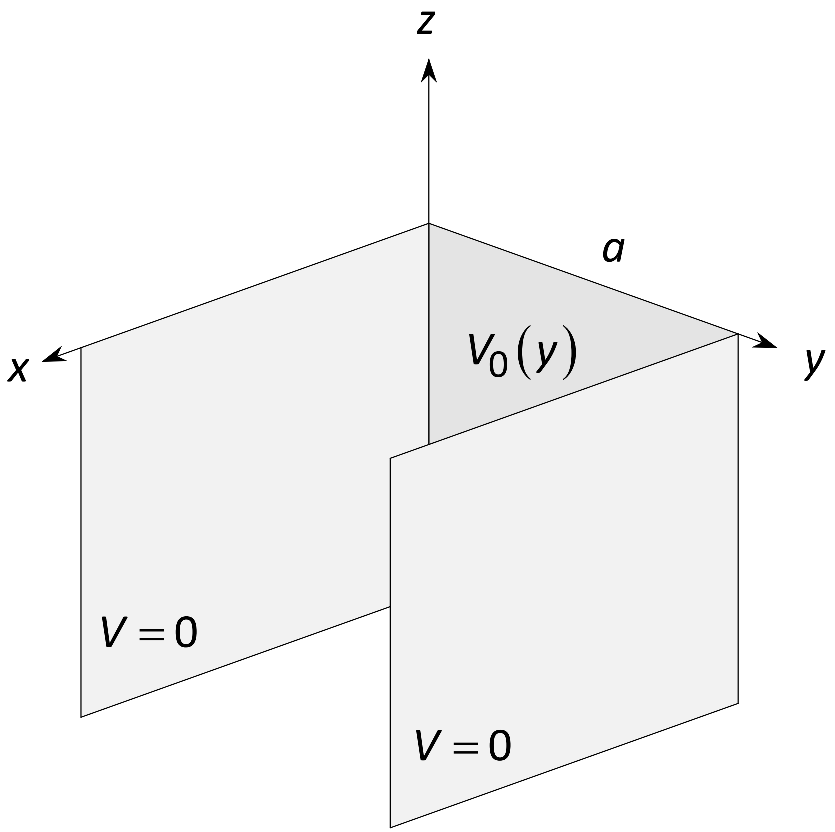
- Now back to boundary condition 4 to finish evaluating the Fourier coefficients G_n:

$$\int_0^a V_0(y)\sin\frac{m\pi y}{a}dy = \sum_{n=1}^{\infty} G_n \int_0^a \sin\frac{n\pi y}{a}\sin\frac{m\pi y}{a}dy$$

$$= \sum_{n=1}^{\infty} G_n \delta_{mn}\frac{a}{2} = G_m\frac{a}{2} \quad .$$

- And we're done: the complete solution is

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} G_n e^{-n\pi x/a}\sin\frac{n\pi y}{a} \quad , \quad \text{where} \qquad G_n = \frac{2}{a}\int_0^a V_0(y)\sin\frac{n\pi y}{a}dy \quad .}$$



That the solution in Cartesian coordinates comes out as a sine series is why mathematicians often refer to Laplace-equation solutions as "harmonic."

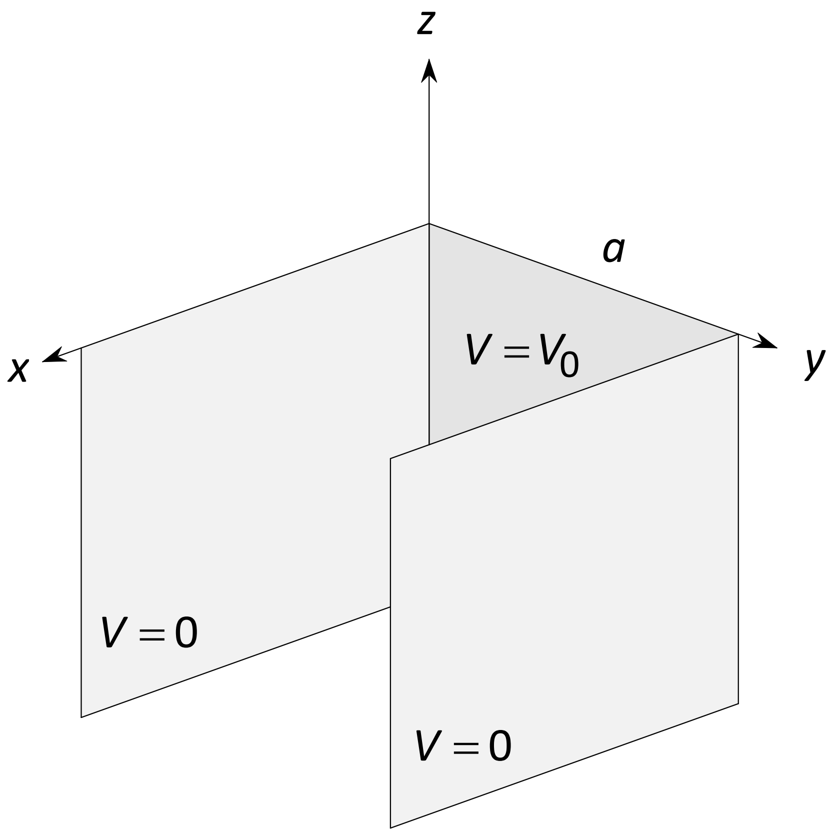
Two concrete examples for the infinite slot

The strip at $x = 0$ is held at a **uniform** potential $V = V_0$. Calculate $V(x,y)$ inside the slot.

- First the Fourier coefficients:

$$G_n = \frac{2V_0}{a}\int_0^a \sin\frac{n\pi y}{a}dy = \frac{2V_0}{n\pi}\int_0^{n\pi} \sin u\,du = -\frac{2V_0}{n\pi}(\cos n\pi - 1)$$

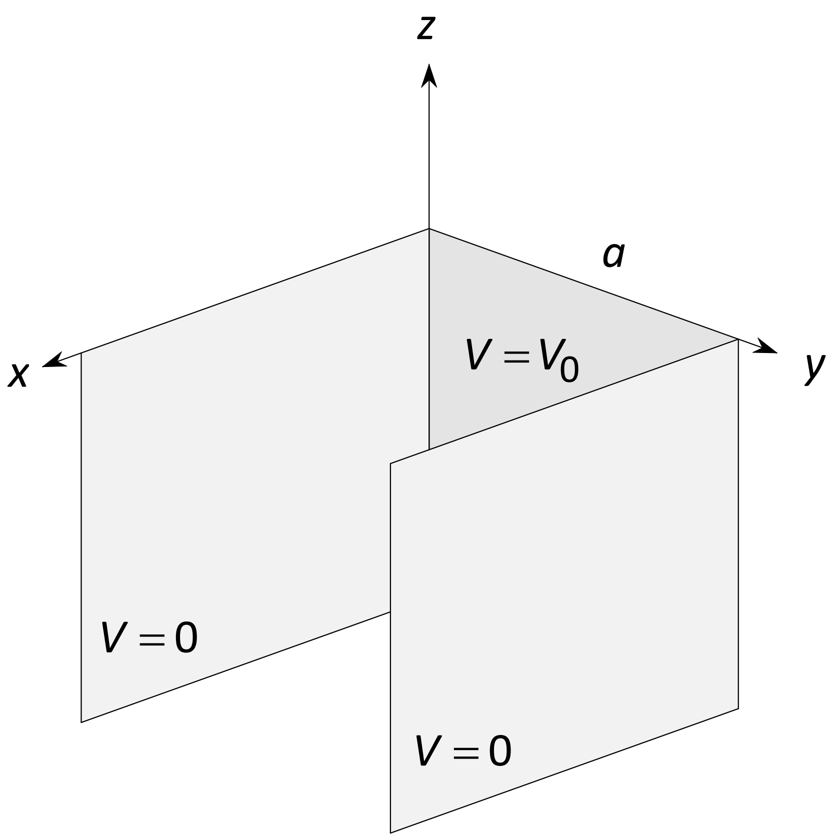
$$= \begin{cases} \dfrac{4V_0}{n\pi}, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even} \end{cases} = \frac{4V_0}{(2m+1)\pi} \quad , \quad m = 0,1,2,...$$



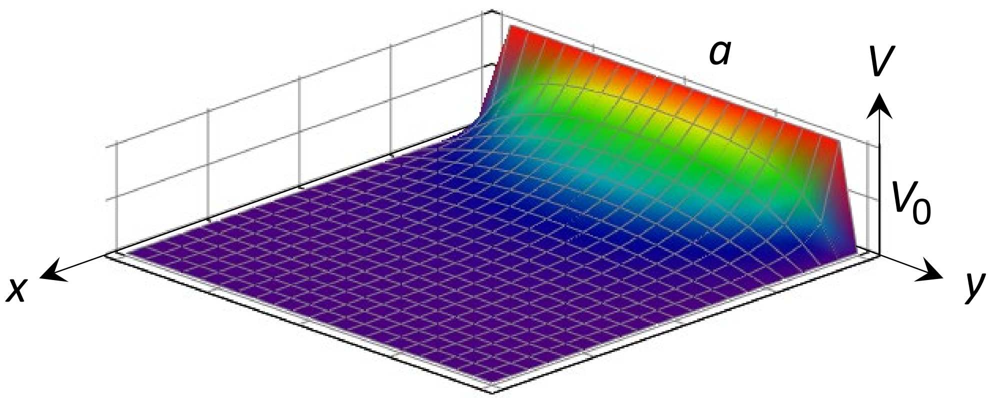
Two concrete examples for the infinite slot (continued)

- Plug in: $V(x,y) = \sum_{n=1}^{\infty} G_n e^{-n\pi x/a} \sin\frac{n\pi y}{a}$

$$\boxed{= \frac{4V_0}{\pi} \sum_{m=0}^{\infty} \frac{e^{-(2m+1)\pi x/a}}{2m+1} \sin\frac{(2m+1)\pi y}{a}} \; .$$



- A filled contour plot of $V(x,y)$, with the $x - y$ plane in the same orientation as the diagram, and V along z.



Two concrete examples of the infinite slot (continued)

Now suppose that the nonzero-potential strip has uniform potential V_0 from y = 0 to $y = a/2$, and $-V_0$ from $y = a/2$ to $y = a$. Once again, calculate V in the slot.

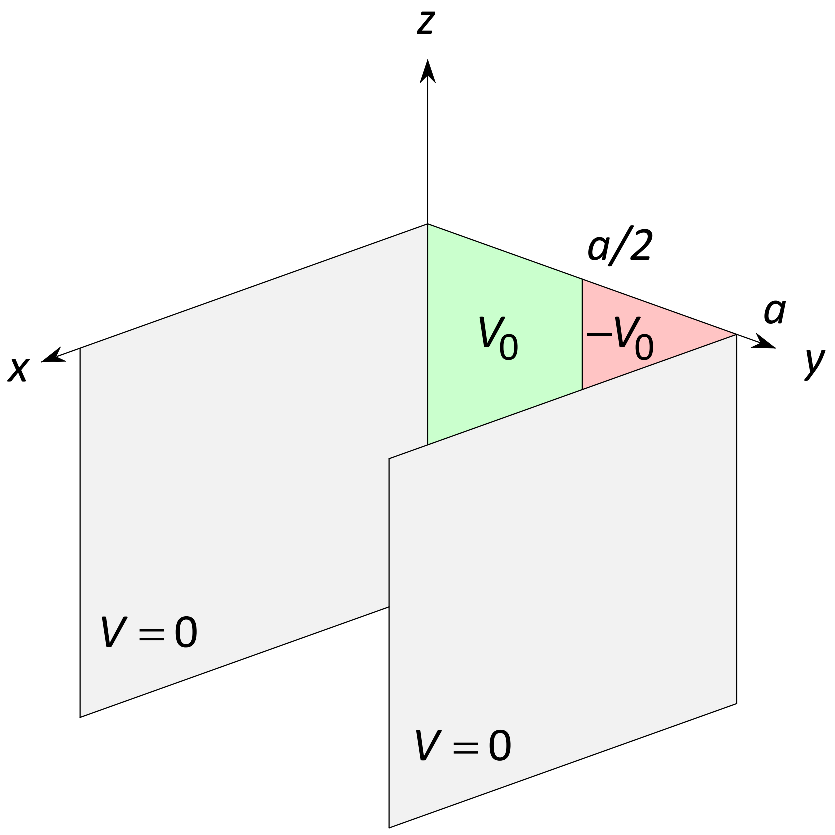
- *G:* $$G_n = \frac{2}{a}\int_0^a V(y)\sin\frac{n\pi y}{a}dy = \frac{2V_0}{a}\int_0^{a/2}\sin\frac{n\pi y}{a}dy - \frac{2V_0}{a}\int_{a/2}^{a}\sin\frac{n\pi y}{a}dy$$

$$= -\frac{2V_0}{n\pi}\cos\frac{n\pi y}{a}\bigg|_0^{a/2} + \frac{2V_0}{n\pi}\cos\frac{n\pi y}{a}\bigg|_{a/2}^{a}$$

$$= \frac{2V_0}{n\pi}\left(1 + \cos n\pi - 2\cos\frac{n\pi}{2}\right)$$

$$= \frac{8V_0}{n\pi} \quad , n = 2,6,10,14,...:$$ the even integers not divisible by 4;

$$= \frac{8V_0}{(4m+2)\pi} \quad , m = 0,1,2,...$$



Two concrete examples of the infinite slot (continued)

- Plug the Gs into the general solution:

$$V(x,y) = \sum_{n=1}^{\infty} G_n e^{-n\pi x/a} \sin\frac{n\pi y}{a}$$

$$\boxed{= \frac{8V_0}{\pi} \sum_{m=0}^{\infty} \frac{1}{(4m+2)} e^{-(4m+2)\pi x/a} \sin\frac{(4m+2)\pi y}{a}} \quad .$$

- Plot: