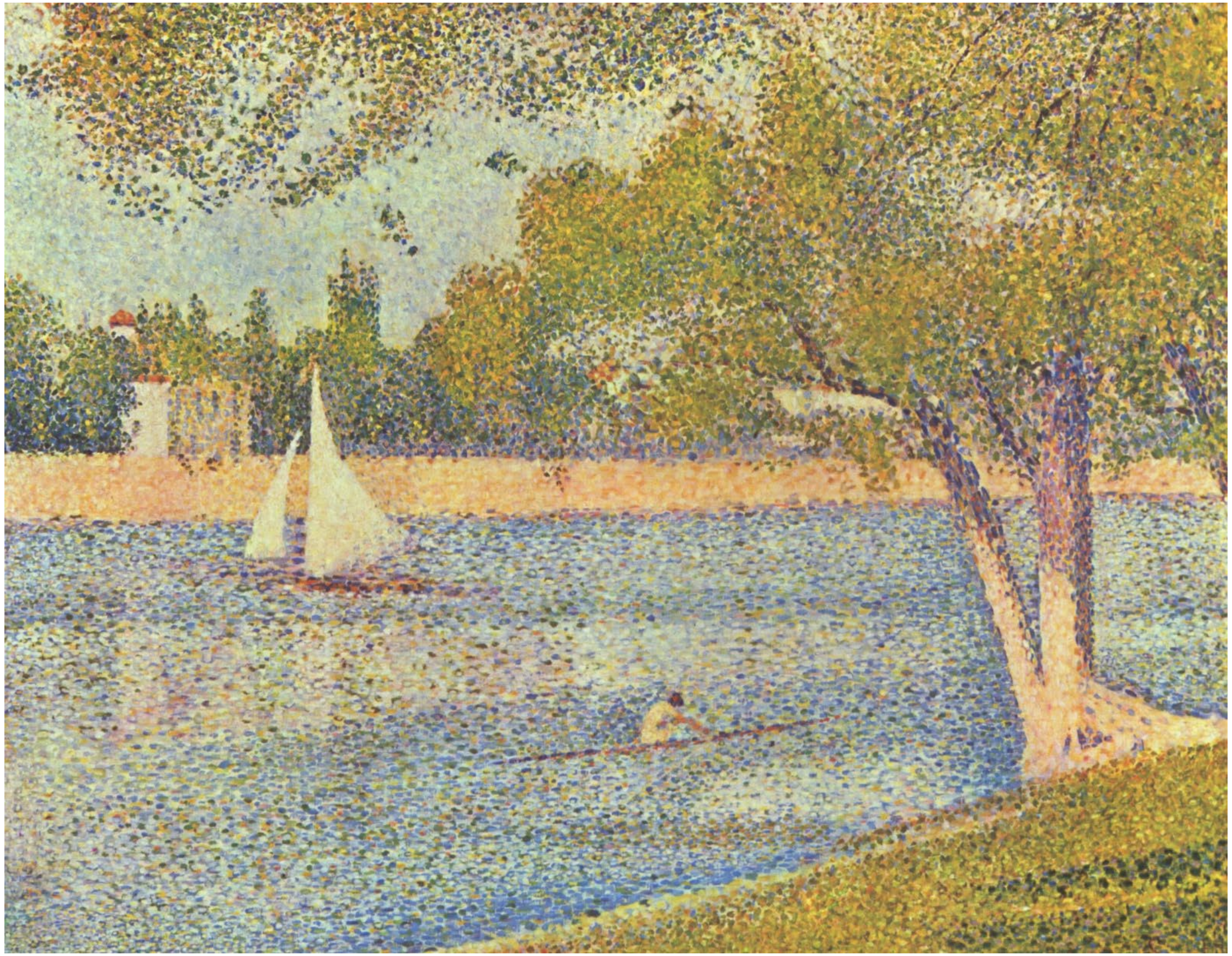
Today in Physics 237: the wavefunction as probability density

- The "Copenhagen" – probabilistic – interpretation of Schrodinger-equation wavefunctions
- Necessaries of probability, for discrete and continuous variables



Georges Seurat, *La Seine à la Grande-Jatte* (1888), Royal Museums of Fine Arts, Brussels

See Griffiths & Schroeter: p. 8, or search for "pointillist."

Review: the wave-particle duality

As you have heard in PHYS 143 or 123,

- Free electrons, however they are produced, are undeniably particles. But models of atoms based, solar-system-like, on orbits of pointlike electrons and nuclei do not work: the orbits decay in a very short time.

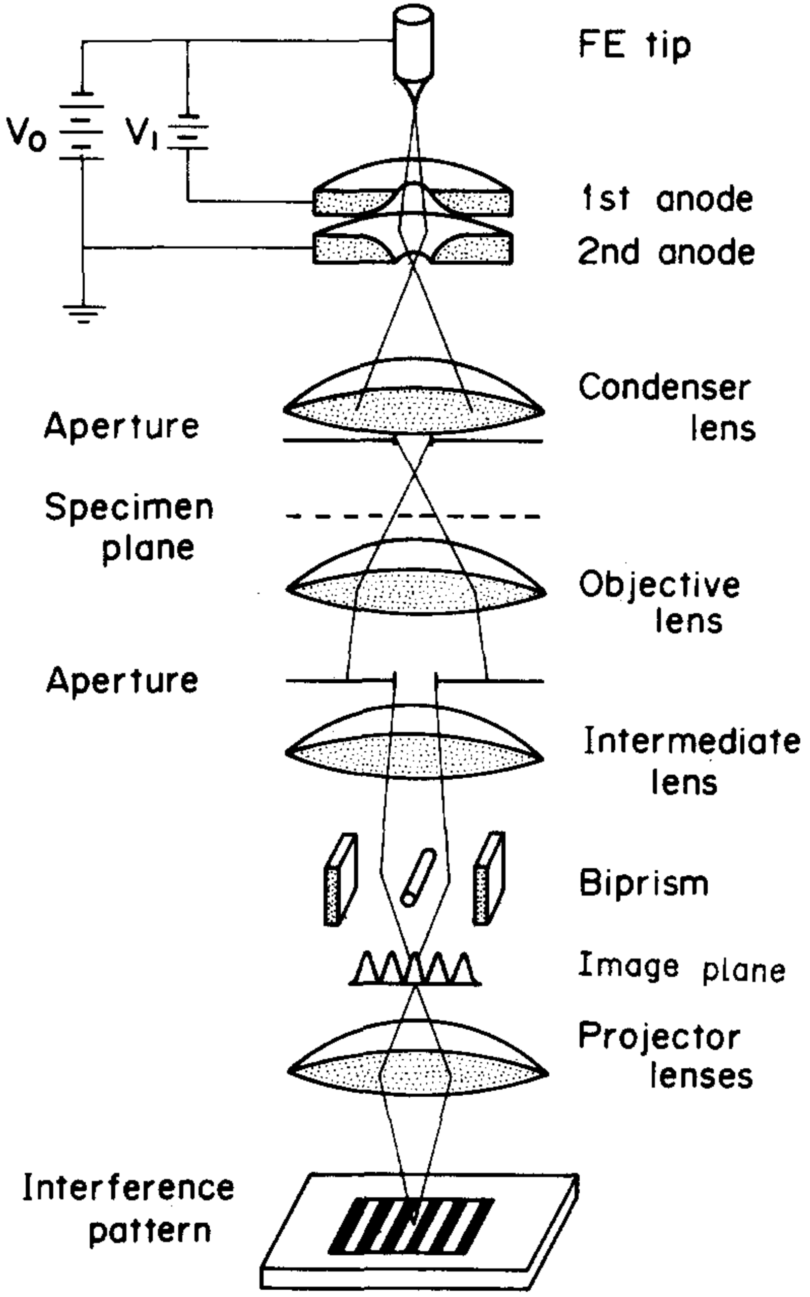
- And models based upon a wave representation of electrons lead to indefinitely-long-lived atoms.

  - When an electron is bound or otherwise confined within a space of dimensions similar to its **de Broglie** (1925) **wavelength**, $\lambda_{dB} = h/m_e v$, it exhibits wave properties, including constructive and destructive interference.

  - Otherwise it acts as a particle, as all unbound/unconfined electrons do.

- Similar to light, which acts as a wave when its propagation path lengths differ by amounts similar to its wavelength, as in the double-slit experiment (Young 1803) and as a particle otherwise, as in the photoelectric effect (Einstein 1905).

- It took til 1988 to perform the Young double-slit experiment on electrons, under conditions in which only one electron is in the apparatus at a time, but the result is definitive …



Electron microscope

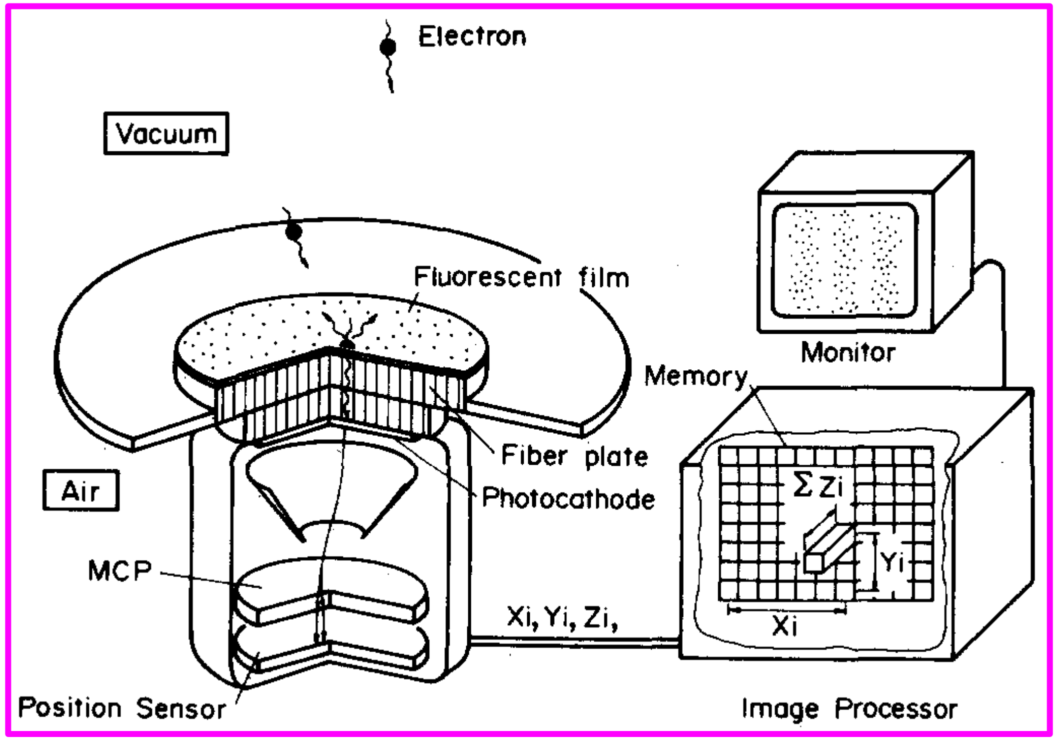
Electrostatic biprism

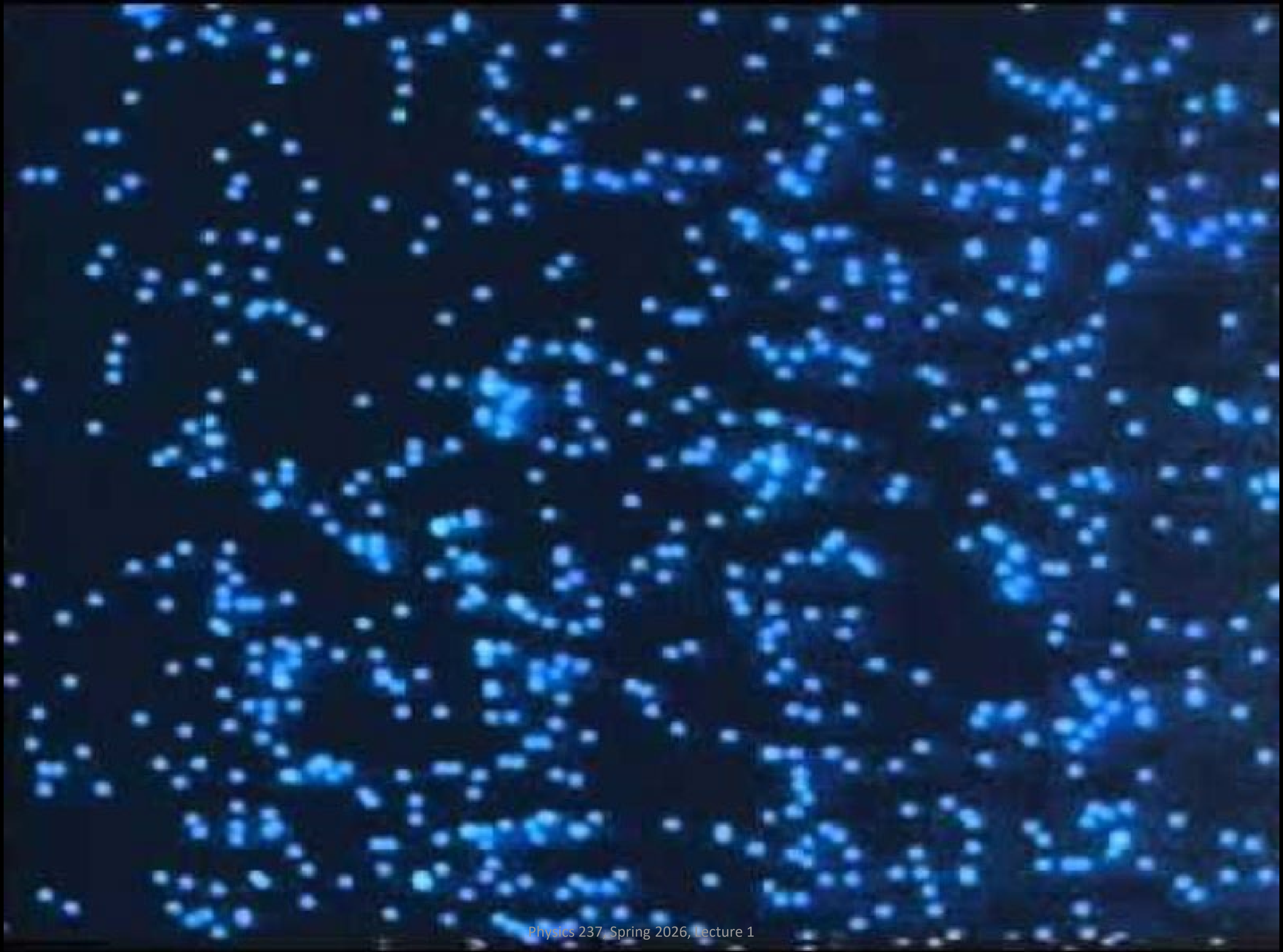
Position-sensitive single-electron detector

Electron double-slit experiment

Only one electron in the apparatus at a time!

- Paper: [Tonomura et al. 1989]
- Video, narrated by Tonomura: [YouTube]





Review: the Schrödinger equation

- As you have also heard, the wave properties of electrons – or any other quanta confined to de Broglie wavelength dimensions – are described by the wave equation invented by Schrödinger (1926):

$$i\hbar\frac{\partial\Psi}{\partial t}=\left(-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2}+V\right)\Psi \quad \text{in 1-D;} \qquad i\hbar\frac{\partial\Psi}{\partial t}=\left(-\frac{\hbar^2}{2m_e}\nabla^2+V\right)\Psi \quad \text{in 3-D.}$$

- The solution to the Schrödinger equation – the **wave function** $\Psi(\boldsymbol{r},t)$ – is directly related to the **probability** that the quantum, an electron in this case, would be found by measurement to lie at position $\boldsymbol{r}$ at time t.

- In particular, the probability P that the quantum would be found at time t along a line $\mathcal{C}$ (1-D) or within a volume $\mathcal{V}$ (3-D) is

$$P=\int_{\mathcal{C}}|\Psi|^2\,dx=\int_{\mathcal{C}}\Psi^*\Psi\,dx\equiv\int_{\mathcal{C}}\rho(x,t)\,dx \quad \text{(1-D), or} \quad P=\int_{\mathcal{V}}|\Psi|^2\,d\tau\equiv\int_{\mathcal{V}}\rho(\boldsymbol{r},t)\,d\tau \quad \text{(3-D),}$$

where $\rho=|\Psi|^2$ is the **probability density**: probability per unit length (1-D) or per unit volume (3-D).

  - This interpretation of $\Psi(\boldsymbol{r},t)$ is usually credited to Born (c. 1926), who said (Born 1955) that Einstein had also been working along those lines earlier.

Review? Indeterminacy, and the effect of (objective) measurements

Physicists and philosophers have had fun with the probabilistic interpretation of $\Psi(\boldsymbol{r},t)$ for precisely a century now. Does pre-measurement indeterminacy of quantum states mean that

- quantum mechanics is incomplete, and that discovery of suitable **local hidden variables** could allow formulation of an exact and **deterministic** theory?
  - This was Einstein's position, a.k.a. "God does not play dice," which he said several times in public, e.g. at the 1927 Solvay conference. Or is it that
- the quantum state – its position at time *t,* in this case – is by nature **undetermined** until a measurement is made, at which point the quantum would assume a state? And that there are no local hidden variables?
  - This is the **Copenhagen interpretation**, adopted by Bohr, Schrödinger, Heisenberg, Born, and Pauli, among others.
  - Bohr's position, also delivered at Solvay in the same conversation as Einstein's: "Stop telling God what to do!"

(Both were using "God" metaphorically, btw…)

Indeterminacy, and the effect of (objective) measurements, continued

The proof by Bell (1964) that quantum mechanics is incompatible with local hidden variables, and the experimental validations of Bell's theorem (e.g. Freedman & Clauser 1972, Aspect et al. 1982), decided the Solvay 1927 issue **decisively in favor of the Copenhagen interpretation**.

- Thus in PHYS 237 we will adopt without further ado the probabilistic interpretation of $\Psi(\boldsymbol{r},t)$.

- Which would meet with approval in a slightly different school of thought, exemplified by Feynman:

> We have implied that in our [electron double-slit] experimental arrangement … it would be impossible to predict exactly what would happen. We can only predict the odds! This would mean, if it were true, that physics has given up on the problem of trying to predict exactly what will happen in a definite circumstance. Yes! physics *has* given up. *We do not know how to predict what would happen in a given circumstance*, and we believe now that it is impossible—that the only thing that can be predicted is the probability of different events.

Apparently Feynman never in fact said "shut up and calculate," but that's what he meant.

- I hope that you get to watch Prof. Blok return to this topic in PHYS 246, to discuss the ongoing closing of loopholes in the experimental validation of Bell's theorem.

Probability, with discrete variables

Clearest by means of a simple example from our textbook, to introduce some terminology and concepts of probability.

A room contains fourteen people, with ages as follows:

Age	14	15	16	22	24	25	Other
Number of people	1	1	3	2	2	5	0

a. If you select one individual at random from this group, what is the probability that their age is 15?

   1/14, since only one of the 14 is 15 years old. But we can generalize in case we need another age's probability:

$$N = \sum_j N_j = 14 \quad , \quad P_j = \frac{N_j}{N} = \boxed{\frac{1}{14}} \quad , \quad P = \sum_j P_j = 1 \quad .$$

   So $P_j = \frac{N_j}{N} = \frac{3}{14}$ for drawing a 16-year-old.

Probability, with discrete variables (continued)

b. What is the most probable age?

Obviously 25, with $P_j = \frac{N_j}{N} = \boxed{\frac{5}{14}}$.

c. What is the **median** age?

Median means there are as many smaller than there are larger. Here 7 are older than 23 and 7 are younger than 23, so the median is $\boxed{23.}$ For *N* large enough that you can't count at a glance: it's the value of *J* that makes

$$\sum_{j=0}^{J} N_j = \frac{N}{2} \quad .$$

4. What is the **average** age, which we would usually call $\langle A \rangle$?

Let the age for each bin *j* be A_j; then $\langle A \rangle = \sum_j A_j P_j = \frac{1}{14}\left(14 + 15 + 3\cdot 16 + 2\cdot 22 + 2\cdot 24 + 5\cdot 25\right) = \boxed{21}$.

In general, the average value of any function *f* with values over the discrete set *j* is $\boxed{\langle f \rangle = \sum_j f_j P_j}$.

Probability, with discrete variables (continued)

e. What is the average of the squares of the ages?

From what we just noted: $\langle A^2 \rangle = \sum_j A_j^2 P_j = \frac{1}{14}\left(14^2 + 15^2 + 3\cdot 16^2 + 2\cdot 22^2 + 2\cdot 24^2 + 5\cdot 25^2\right) = \boxed{459.6}$.

But $\langle A^2 \rangle = 459.6 \neq \langle A \rangle^2 = 441$.

f. What is the **variance**, $\sigma_A^2 = \left\langle \left(A - \langle A \rangle\right)^2 \right\rangle$, of the ages?

First, derive a useful result, using the fact that constants like $\langle f \rangle$ factor out of sums:

$$\sigma_f^2 = \left\langle \left(f - \langle f \rangle\right)^2 \right\rangle = \sum_j \left(f_j - \langle f \rangle\right)^2 P_j = \sum_j \left(f_j^2 - 2 f_j \langle f \rangle + \langle f \rangle^2\right) P_j = \langle f^2 \rangle - 2\langle f \rangle \langle f \rangle + \langle f \rangle^2 \boxed{= \langle f^2 \rangle - \langle f \rangle^2} \ .$$

So the variance of A is $\sigma_A^2 = \langle A^2 \rangle - \langle A \rangle^2 = \boxed{18.6.}$ Its **standard deviation**, $\sigma_A = \sqrt{\sigma_A^2}$, is 4.3.

Squares are positive definite, so $\boxed{\langle f^2 \rangle \geq \langle f \rangle^2.}$