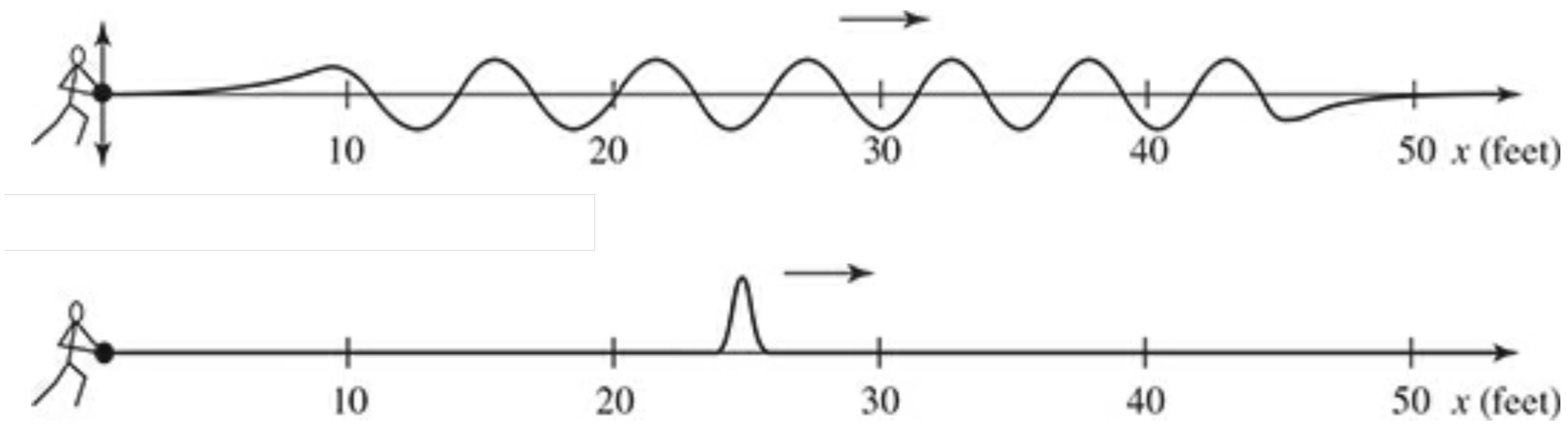
Today in Physics 237: operators and expectation values

- Probability density with continuous variables
- Normalization of Ψ
- Operators and their expectation values
- The Heisenberg uncertainty principle

No workshop Friday (“Rochester Monday”). Here are the next ones:

Today, 4:50pm, 305 Hylan, with Hifsa
Monday, 4:50pm, 102 Hylan, with Roshan

Office hours as regularly scheduled (e.g. Dan, Friday 3-5pm, 418 B&L)



Large x uncertainty, small p uncertainty

Small x uncertainty, large p uncertainty

Probability, with continuous variables

Again develop the relations we will use, with a concrete example. (Similar to the book's Example 1.2.)

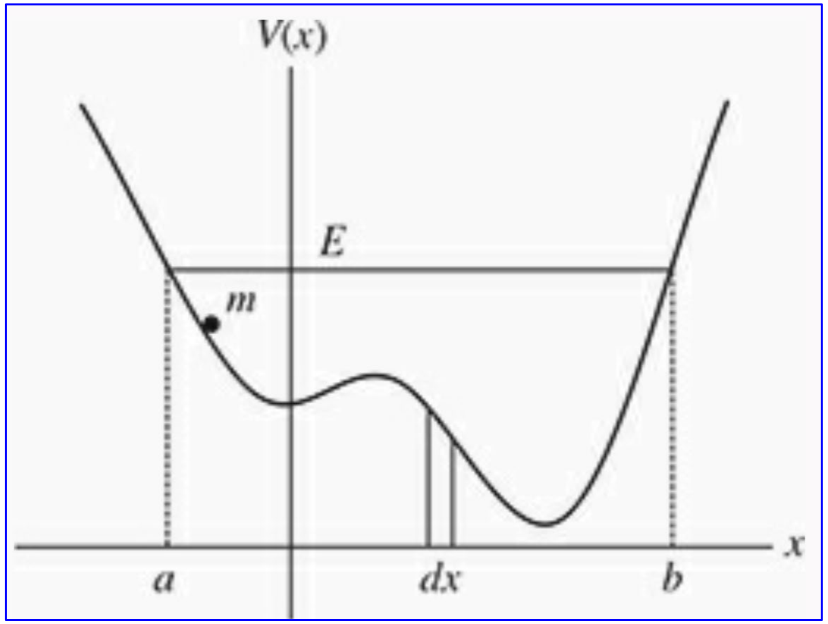
Problem 1.11. Imagine a particle of mass *m* and energy *E* in a potential well, sliding frictionlessly back and forth between the classical turning points *a* and *b*. Classically, the probability of finding the particle in the range *dx* (if, for example, you took a snapshot at a random time *t*) is equal to the fraction of the time *T* it takes to get from *a* to *b* that it spends in the interval *dx*. With $\rho(x)$ as the probability density – see Lecture 1, page 5 – we have

$$\rho(x)dx = \frac{dt}{T} = \frac{(dt/dx)dx}{T} = \frac{dx}{v(x)T} \quad ,$$

where *v*(*x*) is the speed, and $T = \int_0^T dt = \int_a^b \frac{1}{v(x)}dx \quad .$

Thus $\rho(x) = \dfrac{1}{v(x)T}$.

This is perhaps the closest classical analogue to $|\Psi|^2$.



Probability, with continuous variables (continued)

a. Use conservation of energy to express $v(x)$ in terms of E and $V(x)$.

$$E = \frac{1}{2}mv^2 + V \quad \Rightarrow \quad v = \sqrt{\frac{2(E-V)}{m}} \quad .$$

b. As an example, find $\rho(x)$ for the simple harmonic oscillator, $V(x) = kx^2/2$. Plot $\rho(x)$ and check that it is correctly normalized.

First find the turning points: b such that $v = 0$, i.e. $E = V$: $E = \frac{kb^2}{2} \Rightarrow b^2 = \frac{2E}{k}$. The other one is $a = -b$.

Then $\rho(x) = 1/v(x)T$, so we need to calculate T:

$$T = \int_0^T dt = \int_{-b}^{b} \frac{1}{v(x)} dx = \sqrt{m} \int_{-b}^{b} \frac{1}{\sqrt{2(E-V)}} dx = \sqrt{\frac{m}{k}} \int_{-b}^{b} \frac{1}{\sqrt{1-(x/b)^2}} \frac{dx}{b}$$

$$\left\{ \begin{array}{l} \frac{x}{b} = \sin u, \frac{dx}{b} = \cos u\, du, \sqrt{1-\left(u^2/b^2\right)} = \cos u \\ \frac{x}{b} = -1 \rightarrow 1 \Rightarrow u = -\frac{\pi}{2} \rightarrow \frac{\pi}{2} \end{array} \right.$$

$$= \sqrt{\frac{m}{k}} \int_{-\pi/2}^{\pi/2} du = \sqrt{\frac{m}{k}} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \pi \sqrt{\frac{m}{k}} \quad .$$

Yes, you should always carry out the integrals in detail.

Probability, with continuous variables (continued)

Then the probability density becomes $\rho(x) = \dfrac{1}{v(x)T} = \boxed{\dfrac{1}{v(x)\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{\pi\sqrt{b^2 - x^2}}}$.

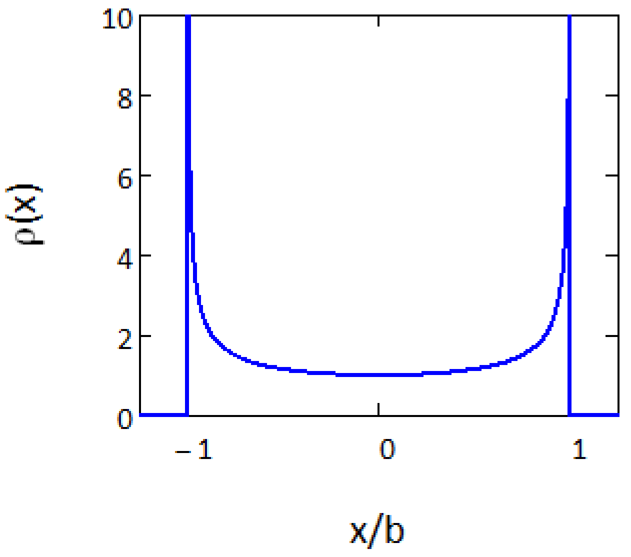
Check normalization: the particle lies somewhere between $-b$ and $+b$, leaving us with an integral we just calculated:

$$\int_{-b}^{b} \rho(x)dx = \frac{1}{\pi}\sqrt{\frac{k}{m}} \int_{-b}^{b} \frac{1}{v(x)}dx = \frac{1}{\pi}\sqrt{\frac{k}{m}}\pi\sqrt{\frac{m}{k}} = \boxed{1 \quad . \quad \checkmark\checkmark}$$

c. For the classical harmonic oscillator in part b, find $\langle x \rangle, \langle x^2 \rangle$, and σ_x.

The first one is easy: $\langle x \rangle = \displaystyle\int_{-b}^{b} x\rho(x)dx = \frac{b}{\pi}\int_{-b}^{b} \frac{(x/b)(dx/b)}{\sqrt{1-(x/b)^2}} = \frac{b}{2\pi}\int_{0}^{0} \frac{du}{\sqrt{u}} = \boxed{0}$.

Makes sense, too.



The sharp peaks in $\rho(x)$ should look familiar: the particle spends most of its time near the turning points, and very little time at its highest speeds near the bottom of the potential well.

Probability, with continuous variables (continued)

The next integral takes a few more steps:

$$\left\langle x^2\right\rangle = \int_{-b}^{b} x^2\rho(x)dx = \frac{b^2}{\pi}\int_{-b}^{b}\frac{(x/b)^2(dx/b)}{\sqrt{1-(x/b)^2}} = \frac{b^2}{\pi}\int_{-1}^{1}\frac{u^2du}{\sqrt{1-u^2}} \qquad \begin{cases} u=\sin w,\, du=\cos w\, dw, \\ w=-\pi/2\to\pi/2 \end{cases}$$

$$=\frac{b^2}{\pi}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 w\cos w\, du}{\cos w} = \frac{b^2}{\pi}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 w\cos w\, dw}{\cos w} = \frac{b^2}{\pi}\int_{-\pi/2}^{\pi/2}\sin^2 w\, dw \quad .$$

But it offers a trick worth remembering: integrate by parts, with $u=\sin w, du=\cos w\, dw, dv=\sin w dw, v=-\cos w$:

$$\int_{-\pi/2}^{\pi/2}\sin^2 w\, dw = uv-\int v du = \overbrace{-\sin w\cos w\Big|_{-\pi/2}^{\pi/2}}^{0} - \int_{-\pi/2}^{\pi/2}\left(-\cos^2 w\right)dw = \int_{-\pi/2}^{\pi/2}\cos^2 w\, dw \quad (!),$$

$$\text{so}\quad \pi = \int_{-\pi/2}^{\pi/2} dw = \int_{-\pi/2}^{\pi/2}\left(\sin^2 w+\cos^2 w\right)dw = 2\int_{-\pi/2}^{\pi/2}\sin^2 w\, dw \;\Rightarrow\; \int_{-\pi/2}^{\pi/2}\sin^2 w\, dw = \frac{\pi}{2} \quad .$$

Probability, with continuous variables (continued)

And so $\left\langle x^2 \right\rangle = \frac{b^2}{\pi}\frac{\pi}{2} = \boxed{\frac{b^2}{2}}$.

The first result makes the standard deviation come out easy: $\sigma_x = \sqrt{\left\langle x^2 \right\rangle - \langle x \rangle^2} = \sqrt{\left\langle x^2 \right\rangle} = \boxed{\frac{b}{\sqrt{2}}}$.

Normalization

- In the previous example, the probability density $\rho(x)$ came normalized, out of the box: that is, the integral of $\rho(x)$ over the whole domain of x is 1, so the particle is certain to be found within the domain.
- We can, and should, **demonstrate the terms under which it is possible to normalize** $\rho(x)$, and to **keep a normalized wavefunction normalized as time goes on**. In 1-D –
  - The first part is easy: if $\Psi'(x,t)$ is a solution to the Schrödinger equation, then so is $A\Psi'(x,t)$, where A is any constant, real or complex. Thus as long as $\Psi'(x,t)$ is **square-integrable** and is **nontrivial** (i.e. $\Psi'(x,t)\neq 0$), A can be chosen such that $\Psi(x,t)=A\Psi'(x,t)$ is normalized:

$$\int_{-\infty}^{\infty}\left|\Psi(x,t)\right|^2 dx=\int_{-\infty}^{\infty}\left|A\Psi'(x,t)\right|^2 dx=\left|A\right|^2\int_{-\infty}^{\infty}\left|\Psi'(x,t)\right|^2 dx=1\quad\Rightarrow\quad\boxed{\left|A\right|=\left(\int_{-\infty}^{\infty}\left|\Psi'(x,t)\right|^2 dx\right)^{-1/2}}\ .$$

  - Square-integrable means simply that the integral of $\left|\Psi\right|^2$ is finite. This in turn means that $\Psi(x,t)\rightarrow 0$ faster than $1/\sqrt{\left|x\right|}$ does.
    - Victims of PHYS 217 may recall similar conditions on $\boldsymbol{E}$ and $\boldsymbol{B}$ to keep electromagnetic energy finite.

Normalization (continued)

- For normalization to last, as time goes on, the time derivative of the square-integral of Ψ must be zero:

$$\frac{d}{dt}\int_{-\infty}^{\infty}\left|\Psi(x,t)\right|^2 dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left|\Psi(x,t)\right|^2 dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}\Psi^*\Psi dx = \int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial t}+\frac{\partial\Psi^*}{\partial t}\Psi\right)dx \quad .$$

- Use the Schrödinger equation for Ψ and for its complex conjugate:

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2}+V\right)\Psi \quad , \quad -i\hbar\frac{\partial\Psi^*}{\partial t} = \left(\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2}+V\right)\Psi^* \quad ;$$

and $1/i = -i$; producing

$$\frac{d}{dt}\int_{-\infty}^{\infty}\left|\Psi(x,t)\right|^2 dx = \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2}-\frac{\partial^2\Psi^*}{\partial x^2}\Psi\right)dx = \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)dx = \frac{i\hbar}{2m}\left[\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right]_{-\infty}^{\infty} = \boxed{0} \quad ,$$

as long as Ψ and $\Psi^* \rightarrow 0$ as $x \rightarrow \pm\infty$, consistent with the square-integrable condition for normalization.

Operators and their expectation values

- Given $\rho = |\Psi|^2$, the **expectation value** of $f(x,t)$ – the value which would be returned in a measurement – is, in 1-D,

$$\langle f \rangle = \int_{-\infty}^{\infty} f(x,t) |\Psi(x,t)|^2 \, dx \quad .$$

- So the expectation value of position is $\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x,t)|^2 \, dx$.
- That this is the same as the average value above is misleading: on a given quantum, the first measurement of *x* compels the wavefunction to assume a position *x*, after which the variance of *x* changes.
- Because of what ρ is in the Copenhagen interpretation, this expectation value really is the average value returned by measurements on an **ensemble** of identically-prepared quanta or systems. Here the meaning of ensemble, and the ensemble average, is the same as what may be familiar to you from PHYS 227.
- Since the integral is only over position, the expectation value can change over time: $\frac{d\langle x \rangle}{dt} = \int_{-\infty}^{\infty} x \frac{\partial}{\partial t} |\Psi(x,t)|^2 \, dx$. Manipulating the time derivative of $|\Psi|^2$ as we did two pages back, …

Operators and their expectation values (continued)

- … we get
$$\frac{d\langle x\rangle}{dt}=\frac{i\hbar}{2m}\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x}+\frac{\partial\Psi^*}{\partial x}\Psi\right)dx \quad .$$

- Integrate this by parts, with $u=x,\ du=dx,\ dv=\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x}+\frac{\partial\Psi^*}{\partial x}\Psi\right)dx$, and $v=\Psi^*\frac{\partial\Psi}{\partial x}+\frac{\partial\Psi^*}{\partial x}\Psi$:

$$\frac{d\langle x\rangle}{dt}=uv-\int vdu=\frac{i\hbar}{2m}x\left(\Psi^*\frac{\partial\Psi}{\partial x}+\frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{\infty}-\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x}+\frac{\partial\Psi^*}{\partial x}\Psi\right)dx \quad .$$

- As we noted three pages back, $\Psi(x,t)\rightarrow 0$ as $|x|\rightarrow\infty$ faster than $1\big/\sqrt{|x|}$ does, so the first term vanishes:

$$x\left(\Psi^*\frac{\partial\Psi}{\partial x}+\frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{\infty}=0 \quad \Rightarrow \quad \frac{d\langle x\rangle}{dt}=-\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x}+\frac{\partial\Psi^*}{\partial x}\Psi\right)dx \quad .$$

Operators and their expectation values (continued)

- One more integration by parts on the last term in the last integral, with $u = \Psi$, $du = \frac{\partial \Psi}{\partial x}dx$, $dv = -\frac{\partial \Psi^*}{\partial x}dx$, and $v = -\Psi^*$:

$$\frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial \Psi}{\partial x}dx - \frac{i\hbar}{2m}uv + \frac{i\hbar}{2m}\int v\,du = -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial \Psi}{\partial x}dx + \cancelto{0}{\frac{i\hbar}{2m}\Psi\Psi^*\Big|_{-\infty}^{\infty}} - \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial \Psi}{\partial x}dx = -\frac{i\hbar}{m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial \Psi}{\partial x}dx \quad .$$

- We might as well call the left-hand side $\langle v\rangle$.

- Since we're working at nonrelativistic speed, we can also define $\langle p\rangle = m\langle v\rangle$, the expectation value of momentum. So we have three expectation values, which we will write in suggestive form:

$$\langle x\rangle = \int_{-\infty}^{\infty}\Psi^*(x,t)[x]\Psi(x,t)dx \quad , \quad \langle v\rangle = \int_{-\infty}^{\infty}\Psi^*(x,t)\left[-\frac{i\hbar}{m}\frac{\partial}{\partial x}\right]\Psi(x,t)dx \quad , \quad \langle p\rangle = \int_{-\infty}^{\infty}\Psi^*(x,t)\left[-i\hbar\frac{\partial}{\partial x}\right]\Psi(x,t)dx \quad .$$

- Important to remind: these expressions tell you averages expected for measurements on a ensembles of identically-prepared systems. They do not pertain to any specific system! Neither $\langle x\rangle$, nor $\langle v\rangle$, nor $\langle p\rangle$ are the position, speed, or momentum of a specific quantum.

Operators and their expectation values (continued)

- In reflection of this, we refer to the objects sandwiched between Ψ^* and Ψ as **operators** which **represent** position, velocity and momentum: operators, because two of them need a function to their right to yield a result.

$$x \leftrightarrow x \quad , \quad v \leftrightarrow -\frac{i\hbar}{m}\frac{\partial}{\partial x} \quad , \quad p \leftrightarrow -i\hbar\frac{\partial}{\partial x} \quad .$$

- From now on we will indicate operators with a carat:

$$\hat{x} = x \quad , \quad \hat{v} = -\frac{i\hbar}{m}\frac{\partial}{\partial x} \quad , \quad \hat{p} = -i\hbar\frac{\partial}{\partial x} \quad .$$

- Other dynamical quantities which, classically, include position, velocity or momentum are therefore represented by the corresponding combinations of these operators: $Q(x,p) \leftrightarrow Q\left(x, -i\hbar\frac{\partial}{\partial x}\right)$. Their expectation values are calculated as above. Kinetic energy, for example:

$$T = \frac{1}{2}mv^2 = \frac{p^2}{2m} \leftrightarrow \hat{T} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \quad , \quad \langle T \rangle = \int_{-\infty}^{\infty} \Psi^* \hat{T} \Psi dx = \int_{-\infty}^{\infty} \Psi^* \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \right] \Psi dx \quad .$$

The uncertainty principle

As you have learned in PHYS 123 or 143, the most important feature of the position and momentum operators is that their standard deviations are related by the Heisenberg uncertainty principle: $\sigma_x \sigma_p \geq \hbar/2$.

- This of course means that the more precisely determined the position of an ensemble of identically-prepared systems is, the less precisely determined is the momentum, and vice versa.
- We won't get to prove the uncertainty principle for a few weeks, but with wavefunctions and basic probability tools we can now begin to demonstrate its effects.

Problem 1.9. A particle of mass *m* has the wave function $\Psi(x,t) = Ae^{-a\left[\left(mx^2/\hbar\right)+it\right]}$, where *A* and *a* are positive real constants.

a. Find *A*.

That is, normalize the wavefunction: $1 = \int_{-\infty}^{\infty} \Psi^* \Psi dx = A^2 \int_{-\infty}^{\infty} e^{-a\left[\left(mx^2/\hbar\right)-it\right]} e^{-a\left[\left(mx^2/\hbar\right)+it\right]} dx = 2A^2 \int_0^{\infty} e^{-(2am/\hbar)x^2} dx$.

The uncertainty principle (continued)

- The integrand is a Gaussian. Some might not have seen it integrated before, but for now I'll just use the answer which I'll work it out a couple pages hence. Note that the wavefunction is symmetrical about x = 0, so reduce the integration bounds to $0 \rightarrow \infty$ and multiply by two:

$$2A^2 \int_0^\infty e^{-(2am/\hbar)x^2} dx = A^2 \sqrt{\frac{\pi\hbar}{2am}} = 1 \quad \Rightarrow \quad \boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}} \; .$$

b. For what potential function $V(x)$ is this a solution to the Schrödinger equation?

Take the t and x derivatives, and plug it into the equation to see:

$$\frac{\partial\Psi}{\partial t} = -iaAe^{-a\left[\left(mx^2/\hbar\right)+it\right]} = -ia\Psi \quad , \quad \frac{\partial\Psi}{\partial x} = Ae^{-a\left[\left(mx^2/\hbar\right)+it\right]} \frac{\partial}{\partial x}\left(-\frac{amx^2}{\hbar} - iat\right) = -\frac{2amx}{\hbar}\Psi \quad ,$$

$$\frac{\partial^2\Psi}{\partial x^2} = -\frac{2amx}{\hbar}\frac{\partial\Psi}{\partial x} - \frac{2am}{\hbar}\Psi = \left(\frac{2amx}{\hbar}\right)^2 \Psi - \frac{2am}{\hbar}\Psi = -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi \quad ;$$

The uncertainty principle (continued)

$$i\hbar\frac{\partial\Psi}{\partial t}=\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V\right)\Psi \quad\Rightarrow\quad V\Psi=i\hbar\frac{\partial\Psi}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$$

$$V=i\hbar(-ia)+\frac{\hbar^2}{2m}\left[-\frac{2am}{\hbar}\left(1-\frac{2amx^2}{\hbar}\right)\right]=\hbar a-\hbar a\left(1-\frac{2amx^2}{\hbar}\right)=\boxed{2a^2mx^2}\ .$$

So it's a harmonic-oscillator potential $\left(\propto x^2\right)$.

c. Calculate the expectation values of x, x^2, p, and p^2.

First: the expectation values of x and p are both $\boxed{\text{zero,}}$ because $\left|\Psi\right|^2$ is symmetrical about $x = 0$ but both x and p are antisymmetrical, changing signs at $x = 0$. The $-\infty \rightarrow 0$ and $0 \rightarrow \infty$ portions of the expectation-value integrals cancel each other.

Now the other values. Again, see below for the details of the integrals.

The uncertainty principle (continued)

$$\left\langle x^2 \right\rangle = 2A^2 \int_0^{\infty} x^2 e^{-2amx^2/\hbar} dx = \left(2\sqrt{\frac{2am}{\pi\hbar}} \right) \left(\frac{1}{4} \frac{\hbar}{2am} \sqrt{\frac{\pi\hbar}{2am}} \right) = \boxed{\frac{\hbar}{4am}} \ ,$$

$$\left\langle p^2 \right\rangle = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{\partial^2 \Psi}{\partial x^2} dx = \hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{2am}{\hbar} \left(1 - \frac{2amx^2}{\hbar} \right) \Psi dx$$

$$= 2am\hbar \int_{-\infty}^{\infty} \Psi^* \Psi dx - 4a^2 m^2 \int_{-\infty}^{\infty} \Psi^* x^2 \Psi dx = 2am\hbar - 4a^2 m^2 \frac{\hbar}{4am} = \boxed{am\hbar} \ .$$

c. Find the standard deviations of *x* and *p.* Is the product consistent with the uncertainty principle?

Because the expectation values of *x* and *p* are zero, $\sigma_x \sigma_p = \sqrt{\left\langle x^2 \right\rangle \left\langle p^2 \right\rangle} = \sqrt{\frac{\hbar}{4am} am\hbar} = \boxed{\frac{\hbar}{2}}$.

So, $\boxed{\text{Yes,}}$ it is consistent with the uncertainty principle. It is what we would call a **minimum-uncertainty state**.

Integration of Gaussians

- Our first integral is of the form $I=\int\limits_{-\infty}^{\infty}e^{-x^2/2\sigma^2}dx$. It's far easier to integrate its square, switching to polar coordinates:

$$I^2=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}e^{-\left(x^2+y^2\right)/2\sigma^2}dxdy=\int\limits_{0}^{2\pi}\int\limits_{0}^{\infty}e^{-s^2/2\sigma^2}sdsd\varphi=2\pi\int\limits_{0}^{\infty}e^{-s^2/2\sigma^2}sds=4\pi\sigma^2\int\limits_{0}^{\infty}e^{-u^2}udu=2\pi\sigma^2\int\limits_{0}^{\infty}e^{-v}dv=2\pi\sigma^2 \quad .$$

So $\boxed{I=\sigma\sqrt{2\pi};}$ that is, the normalized Gaussian function is $g(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$. Above, $2\sigma^2=\frac{\hbar}{2am}$.

- Our second integral is of the form $J=\int\limits_{-\infty}^{\infty}x^2e^{-x^2/2\sigma^2}dx$. It can be integrated the same way, though with an angular integral which is worse than the radial part. Summary and result, omitting the boring details of the angular integral:

$$J^2=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}x^2y^2e^{-\left(x^2+y^2\right)/2\sigma^2}dxdy=\int\limits_{0}^{2\pi}\cos^2\varphi\sin^2\varphi d\varphi\int\limits_{0}^{\infty}s^4e^{-s^2/2\sigma^2}sds=\left(\sigma\sqrt{2}\right)^6\int\limits_{0}^{2\pi}\cos^2\varphi\sin^2\varphi d\varphi\int\limits_{0}^{\infty}u^4e^{-u^2}udu$$

$$=\frac{1}{2}\left(\sigma\sqrt{2}\right)^6\int\limits_{0}^{2\pi}\cos^2\varphi\sin^2\varphi d\varphi\int\limits_{0}^{\infty}v^2e^{-v}dv=\frac{1}{2}\left(\sigma\sqrt{2}\right)^6\left(\frac{\pi}{4}\right)(2!)=2\pi\sigma^6 \quad\Rightarrow\quad \boxed{J=\sqrt{2\pi}\sigma^3} \quad .$$

Boring details of angular integral

- When integrating powers of sines and cosines between bounds for which one or other is zero, there is usually (always?) a way *via* integration by parts to make the original integral recur in the results, as we did above:

$$\int_0^{2\pi} \cos^2\varphi \sin^2\varphi\, d\varphi = \int_0^{2\pi} \left(1-\sin^2\varphi\right)\sin^2\varphi\, d\varphi = \pi - \int_0^{2\pi} \sin^4\varphi\, d\varphi$$

$$\begin{cases} \text{First term: see page 5} \\ u = \sin^3\varphi,\, du = 3\sin^2\varphi\, d\varphi \\ dv = \sin\varphi\, d\varphi,\, v = -\cos\varphi \end{cases}$$

$$= \pi + \underbrace{\sin^3\varphi\cos\varphi\Big|_0^{2\pi}}_{0} - 3\int_0^{2\pi} \sin^2\varphi\cos^2\varphi\, d\varphi \quad ;$$

$$\int_0^{2\pi} \cos^2\varphi \sin^2\varphi\, d\varphi = \boxed{\frac{\pi}{4}} \; .$$