Today in Physics 237: squeezed states

- Coherent states of the quantum simple harmonic oscillator
- Squeezing the coherent state
- The squeezed amplitude operator
- Gravity wave detection, quantum nondemolition, and LIGO

LIGO’s Hanford Observatory (Caltech/MIT/NSF)

Coherent states of the simple harmonic oscillator

- The ladder operators have eigenstates, which are linear superpositions of simple-harmonic-oscillator eigenstates.
- For the lowering operator, let's call that eigenstate $|\alpha\rangle$, so $\hat{a}_-|\alpha\rangle = \alpha|\alpha\rangle$, and calculate its *x*-*p* uncertainty product.
  - This is part of G&S problem 3.42, on this week's assignment. For example,

$$\langle x\rangle = \langle\alpha|\hat{x}|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|\hat{a}_+ + \hat{a}_-|\alpha\rangle$$

Recall that $\hat{a}_+ = (\hat{a}_-)^\dagger$, so $\langle\alpha|\hat{a}_+ = \langle\alpha|\alpha^*$

$$= \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|\alpha^* + \alpha|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\alpha^* + \alpha)\langle\alpha|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\alpha^* + \alpha) \quad ;$$

$$\left\langle x^2\right\rangle = \langle\alpha|\hat{x}^2|\alpha\rangle = \frac{\hbar}{2m\omega}\langle\alpha|\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2|\alpha\rangle$$

Use $[\hat{a}_-, \hat{a}_+] = 1$

$$= \frac{\hbar}{2m\omega}\langle\alpha|\underbrace{\hat{a}_+^2 + 2\hat{a}_+\hat{a}_- + 1 + \hat{a}_-^2}|\alpha\rangle = \frac{\hbar}{2m\omega}\left(\alpha^{*2} + 2\alpha^*\alpha + 1 + \alpha^2\right) = \frac{\hbar}{2m\omega}\left[(\alpha^* + \alpha)^2 + 1\right] \quad ;$$

Note **normal ordering**: each term has all factors of $\hat{a}_+$ to the **left** of the factors of $\hat{a}_-$.

Coherent states of the simple harmonic oscillator (continued)

$$\sigma_x = \sqrt{\left\langle x^2 \right\rangle - \left\langle x \right\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\left(\alpha^* + \alpha\right)^2 + 1 - \left(\alpha^* + \alpha\right)^2} = \sqrt{\frac{\hbar}{2m\omega}} \quad .$$

- Next you will find, as you solve the rest of this problem, that

$$\sigma_p = \sqrt{\left\langle p^2 \right\rangle - \left\langle p \right\rangle^2} = \sqrt{\frac{m\hbar\omega}{2}} \quad \Rightarrow \quad \sigma_x \sigma_p = \frac{\hbar}{2} \quad ,$$

  making $\left|\alpha\right\rangle$ a minimum-uncertainty state, as is the ground (v = 0) state of the SHO, unlike the excited (v > 0) states.

- This **coherent** state could be a superposition of all the single-quantum SHO states, so that its Fourier coefficients would be

$$c_v = \left\langle \psi_v \middle| \alpha \right\rangle = \frac{\alpha^v}{\sqrt{v!}} c_0 \quad , \quad \text{with} \quad c_0 = e^{-|\alpha|^2/2} \quad ,$$

Coherent states of the simple harmonic oscillator (continued)

as you will show when you solve G&S problem 3.42.

- Today, however, we consider the case of a simple harmonic oscillator for which only **two** states are available.
  - Instead of the infinity of states, we have a very large ensemble of two-state SHOs: $N \gg 1$ identically-prepared copies of one another, confined to exactly the same space in *x,* and involving *N* quanta.
  - Instead of being normalized to 1, $|c_0|^2 + |c_1|^2 = N$.
  - Otherwise the states, and the results of operations, are the same as before.
- This is not as artificial as our usual setups: it is a description of a **laser**.
  - We may regard a laser, simply, as a machine which maintains a large numbers of two-state oscillators with constant proportions of the oscillators in the two states.
  - They generate lots of monochromatic light, composed of so many photons that the light can be regarded for **most** purposes as a classical electromagnetic wave with constant phase and polarization: that is, **coherent light.**

Squeezing the coherent state

- Let us define this wave's amplitude as a complex operator,

$$\hat{X}_1 + i\hat{X}_2 = \hat{a}_- = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} + i\hat{p}) = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{1}{\sqrt{2m\hbar\omega}}\hat{p} \quad .$$

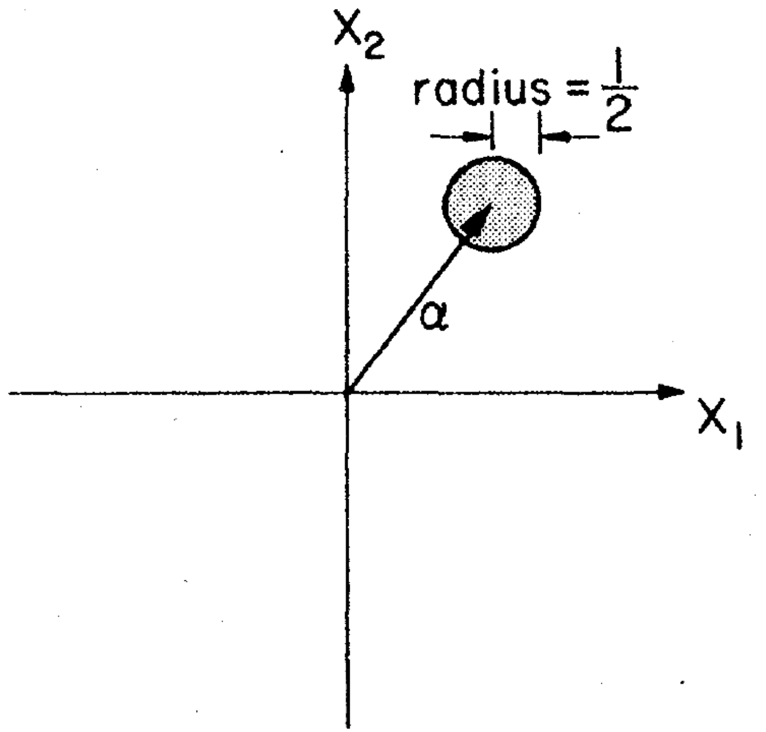
- As they're just made up of $\hat{x}$ and $\hat{p}$, $\hat{X}_1$ and $\hat{X}_2$ do not commute, and are subject to an uncertainty principle. From page 3,

$$\left[\hat{X}_1, \hat{X}_2\right] = \frac{i}{2} \quad , \quad \left\langle \hat{X}_1 + i\hat{X}_2 \right\rangle = \alpha \quad ;$$

$$\sigma_{X_1} = \frac{1}{2} \quad , \quad \sigma_{X_2} = \frac{1}{2} \quad , \quad \sigma_{X_1}\sigma_{X_2} = \frac{1}{4} \quad .$$

  - As a wave, the coherent state's phase is $\varphi = \arctan\left(\langle X_2 \rangle / \langle X_1 \rangle\right)$.

- Plotting $\mathrm{Im}\alpha$ vs. $\mathrm{Re}\alpha$, the uncertainties appear as a circle of radius ½ about the point α.



Caves 1981

Squeezing the coherent state (continued)

- Next define the **squeeze operator**, $S(z)=e^{z^*\hat{a}_-^2/2}e^{-z\hat{a}_+^2/2}$, $z=re^{i\vartheta}$, a function of a complex number z.

- The squeeze operator is unitary, as demonstrated by the product of S with its adjoint:

$$S^\dagger S=\left(e^{z^*\hat{a}_-^2/2}e^{-z\hat{a}_+^2/2}\right)^\dagger\left(e^{z^*\hat{a}_-^2/2}e^{-z\hat{a}_+^2/2}\right)=\left(e^{z\hat{a}_+^2/2}e^{-z^*\hat{a}_-^2/2}\right)\left(e^{z^*\hat{a}_-^2/2}e^{-z\hat{a}_+^2/2}\right)=1 \quad ,$$

  so in transforming state vectors it preserves the values of all the inner products. Also $S(z)=S(z)^{-1}=S(-z)$.

- The squeeze operator transforms the coherent state to a phase-rotated new basis: $\boxed{|\alpha,z\rangle=S(z)|\alpha\rangle.}$ And in this new basis the expectation values of the operators are different than before; for instance,

$$\langle\alpha,z|\hat{a}_-|\alpha,z\rangle=\langle\alpha|S(z)^\dagger\hat{a}_-S(z)|\alpha\rangle=\langle\alpha|\left(e^{z\hat{a}_+^2/2}e^{-z^*\hat{a}_-^2/2}\right)\hat{a}_-\left(e^{z^*\hat{a}_-^2/2}e^{-z\hat{a}_+^2/2}\right)|\alpha\rangle \quad .$$

- The integrand in this matrix element is of the form $e^{\hat{A}}\hat{B}e^{-\hat{A}}$, which we worked out in Lecture 12, p. 5. In terms of r and ϑ, this comes out to

$$S(z)^\dagger\hat{a}_-S(z)=\hat{a}_-\cosh r-\hat{a}_+e^{i\vartheta}\sinh r \quad .$$

The squeezed amplitude operator

- First calculate a couple commutators of the ladder operators, using $\left[\hat{a}_-,\hat{a}_+\right]=\hat{a}_-\hat{a}_+-\hat{a}_+\hat{a}_-=1$ from Lecture 5, p.6:

$$\left[\hat{a}_+^2,\hat{a}_-\right]=\hat{a}_+^2\hat{a}_--\hat{a}_-\hat{a}_+^2=\hat{a}_+^2\hat{a}_--\hat{a}_+\hat{a}_-\hat{a}_+-\hat{a}_+=\hat{a}_+^2\hat{a}_--\hat{a}_+^2\hat{a}_--2\hat{a}_+=-2\hat{a}_+ \quad ;$$

$$\left[\hat{a}_-^2,\hat{a}_+\right]=\hat{a}_-^2\hat{a}_+-\hat{a}_+\hat{a}_-^2=\hat{a}_-\hat{a}_+\hat{a}_-+\hat{a}_--\hat{a}_+\hat{a}_-^2=\hat{a}_+\hat{a}_-^2+2\hat{a}_--\hat{a}_+\hat{a}_-^2=2\hat{a}_- \quad .$$

- Then take $-\hat{A}=e^{z^*\hat{a}_-^2/2-z\hat{a}_+^2/2}$ and $\hat{B}=\hat{a}_-$, and calculate the commutators in

$$e^A\hat{B}e^{-A}=\hat{B}+[\hat{A},\hat{B}]+\frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]]+\ldots+\frac{1}{n!}\overbrace{\left[\hat{A},\left[\hat{A},\ldots\left[\hat{A},[\hat{A},\hat{B}]\right]\ldots\right]\right]}^{n\text{ brackets}}+\ldots$$

from Lecture 12, p. 5).

The squeezed amplitude operator (continued)

- Here are the first six. See the pattern?

$$[\hat{A},\hat{B}] = \frac{1}{2}z\left[\hat{a}_+^2,\hat{a}_-\right] - \frac{1}{2}z^*\cancelto{0}{\left[\hat{a}_-^2,\hat{a}_-\right]} = -z\hat{a}_+$$

$$\left[\hat{A},[\hat{A},\hat{B}]\right] = -\frac{1}{2}z^2\cancelto{0}{\left[\hat{a}_+^2,\hat{a}_+\right]} + \frac{1}{2}z^*z\left[\hat{a}_-^2,\hat{a}_+\right] = z^*z\hat{a}_-$$

$$\left[\hat{A},\left[\hat{A},[\hat{A},\hat{B}]\right]\right] = \frac{1}{2}zz^*z\left[\hat{a}_+^2,\hat{a}_-\right] - \frac{1}{2}z^{*2}z\cancelto{0}{\left[\hat{a}_-^2,\hat{a}_-\right]} = -zz^*z\hat{a}_+$$

$$\left[\hat{A},\left[\hat{A},\left[\hat{A},[\hat{A},\hat{B}]\right]\right]\right] = -\frac{1}{2}z^2z^*z\cancelto{0}{\left[\hat{a}_+^2,\hat{a}_+\right]} + \frac{1}{2}z^*zz^*z\left[\hat{a}_-^2,\hat{a}_+\right] = z^*zz^*z\hat{a}_-$$

$$\left[\hat{A},\left[\hat{A},\left[\hat{A},\left[\hat{A},[\hat{A},\hat{B}]\right]\right]\right]\right] = \frac{1}{2}zz^*zz^*z\left[\hat{a}_+^2,\hat{a}_-\right] - \frac{1}{2}z^{*2}zz^*z\cancelto{0}{\left[\hat{a}_-^2,\hat{a}_-\right]} = -zz^*zz^*z\hat{a}_+$$

$$\left[\hat{A},\left[\hat{A},\left[\hat{A},\left[\hat{A},\left[\hat{A},[\hat{A},\hat{B}]\right]\right]\right]\right]\right] = -\frac{1}{2}z^2z^*zz^*z\cancelto{0}{\left[\hat{a}_+^2,\hat{a}_+\right]} + \frac{1}{2}z^*zz^*zz^*z\left[\hat{a}_-^2,\hat{a}_+\right] = z^*zz^*zz^*z\hat{a}_-$$

The squeezed amplitude operator (continued)

- Substitute $\hat{A}$ and $\hat{B}$ back out; use the definition of z, $z = re^{i\vartheta}, z^* z = r^2$; and gather terms in $\hat{a}_-$ and $\hat{a}_+$:

$$\left(e^{z\hat{a}_+^2/2}e^{-z^*\hat{a}_-^2/2}\right)\hat{a}_-\left(e^{z^*\hat{a}_-^2/2}e^{-z\hat{a}_+^2/2}\right) = \hat{a}_- - z\hat{a}_+ + \frac{1}{2!}z^*z\hat{a}_- - \frac{1}{3!}zz^*z\hat{a}_+ + \frac{1}{4!}z^*zz^*z\hat{a}_- - \frac{1}{5!}zz^*zz^*z\hat{a}_+ + \frac{1}{6!}z^*zz^*zz^*z\hat{a}_- + \dots$$

$$= \left(1 + \frac{1}{2!}r^2 + \frac{1}{4!}r^4 + \frac{1}{6!}r^6 + \dots\right)\hat{a}_- - e^{i\vartheta}\left(r + \frac{1}{3!}r^3 + \frac{1}{5!}r^5 + \dots\right)\hat{a}_+ \quad .$$

- Finally, recall the infinite-series expressions for the hyperbolic cosine and sine:

$$\cosh r = \sum_{n=0}^{\infty}\frac{r^{2n}}{(2n)!} \quad \text{and} \quad \sinh r = \sum_{n=0}^{\infty}\frac{r^{2n+1}}{(2n+1)!} \quad ,$$

to produce

$$S(z)^\dagger \hat{a}_- S(z) = \left(e^{z\hat{a}_+^2/2}e^{-z^*\hat{a}_-^2/2}\right)\hat{a}_-\left(e^{z^*\hat{a}_-^2/2}e^{-z\hat{a}_+^2/2}\right) = \boxed{\hat{a}_-\cosh r - \hat{a}_+ e^{i\vartheta}\sinh r} \quad .$$

Squeezing the coherent state (continued)

- It takes a lot of writing, but, following the same methods – application of the squeeze operator, and calculation of lots of commutators using the ladder operators – we can get the full set of properties, expectation values, and uncertainties for the squeezed state as we did for the original coherent state.
- And the difference is interesting and extremely useful. Let $\hat{Y}_1 + i\hat{Y}_2 = \left(\hat{X}_1 + \hat{i}X_2\right)e^{-i\vartheta/2}$;

$$S(z)^\dagger \hat{a}_+ S(z) = \hat{a}_+ \cosh r - \hat{a}\_e^{-i\vartheta} \sinh r$$

Adjoint of green-pages result

$$S(z)^\dagger \left(\hat{Y}_1 + i\hat{Y}_2\right) S(z) = \hat{Y}_1 \left(\cosh r - \sinh r\right) + i\hat{Y}_2 \left(\cosh r + \sinh r\right) = \boxed{\hat{Y}_1 e^{-r} + i\hat{Y}_2 e^{r}}$$

r = squeeze factor

$$\left(\hat{Y}_1 + i\hat{Y}_2\right) e^{i\vartheta/2} \left|\alpha, z\right\rangle = \left(\hat{X}_1 + i\hat{X}_2\right) \left|\alpha, z\right\rangle = \alpha$$

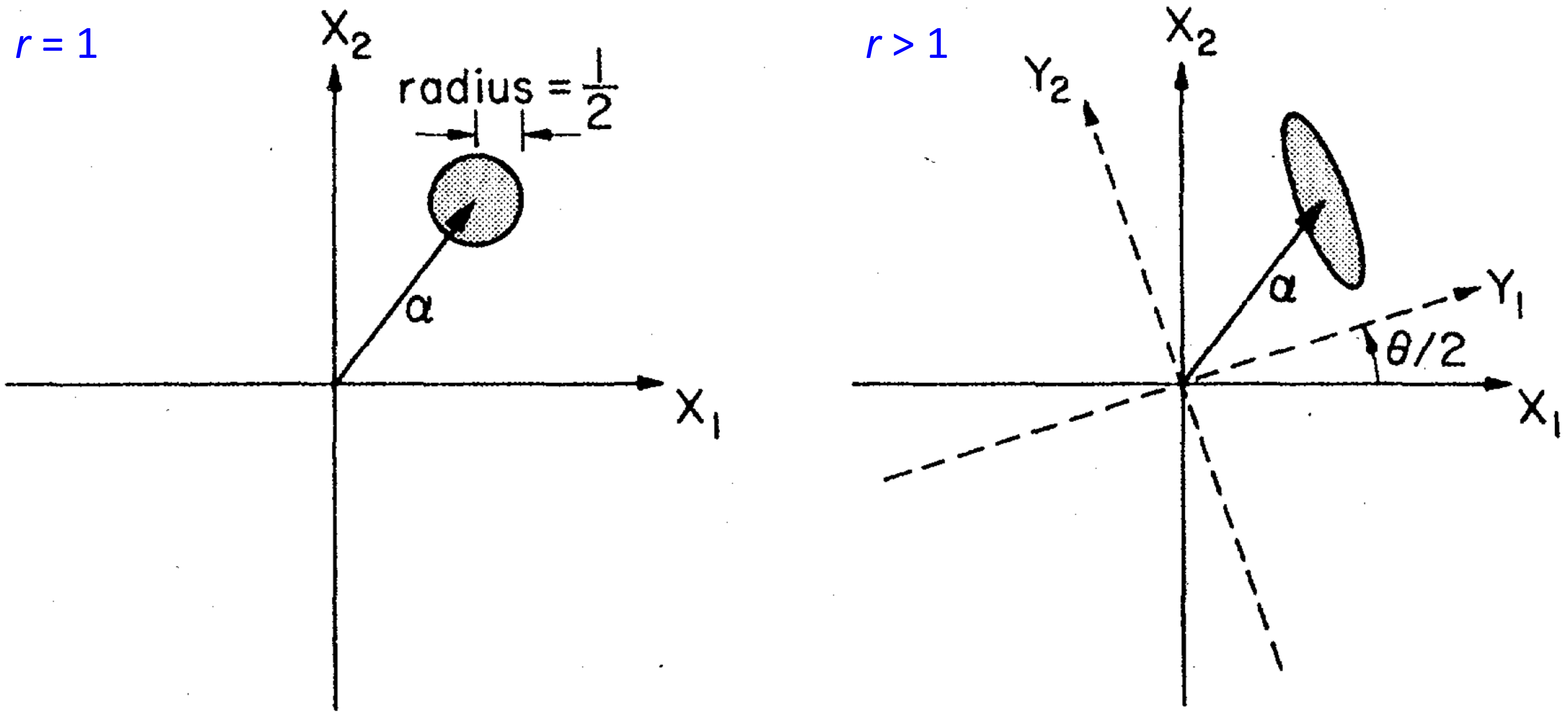
Still have the same eigenstates and eigenvalues

$$\boxed{\sigma_{Y_1} = \frac{1}{2} e^{-r} \quad , \quad \sigma_{Y_2} = \frac{1}{2} e^{r} \quad , \quad \sigma_{Y_1}\sigma_{Y_2} = \frac{1}{4}} \; .$$

Still minimum uncertainty product, but now $\sigma_{Y_1} \neq \sigma_{Y_2}$.

Squeezing the coherent state (continued)

- The upshot is that the squeeze factor *r* and rotation angle ϑ can be chosen to make the quantum uncertainty in $\hat{Y}_1$ or $\hat{Y}_2$ **arbitrarily small**, at the expense of making the uncertainty in the other one large.
- If you only need to measure one of them in your experiment, but you need even better sensitivity than the standard, uncertainty-principle quantum limit, then you should use squeezed states. Good example: **gravity-wave detection**.

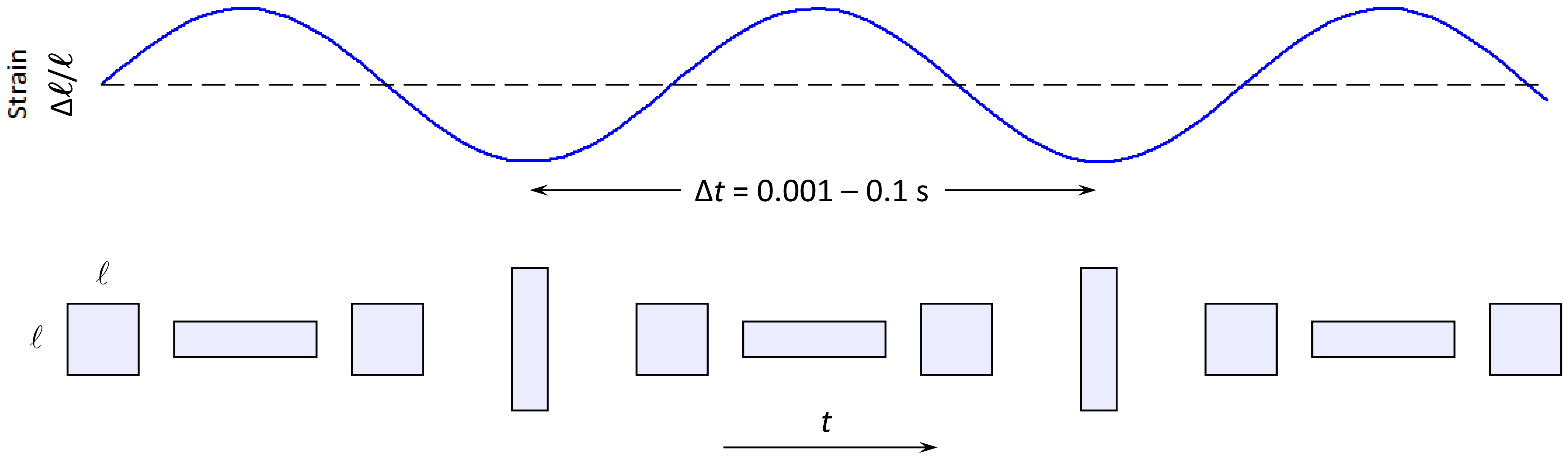


Caves 1981

Gravity waves

As you know, gravity waves were detected directly for the first time in September 2015, and have been detected at increasing rates since then, as upgrades to the Laser Interferometer Gravitational-wave Observatory (LIGO) improve its sensitivity. All the waves are from processes attending **black-hole creation or merger** in distant galaxies.

- Even the strongest gravity waves in nature are **extremely** weak. They produce a small **strain**, of order $\Delta\ell/\ell = 10^{-18}$, in a **quadrupolar** pattern, on the objects they pass through. Effect on a small cube, exaggerated by 10^{18}:

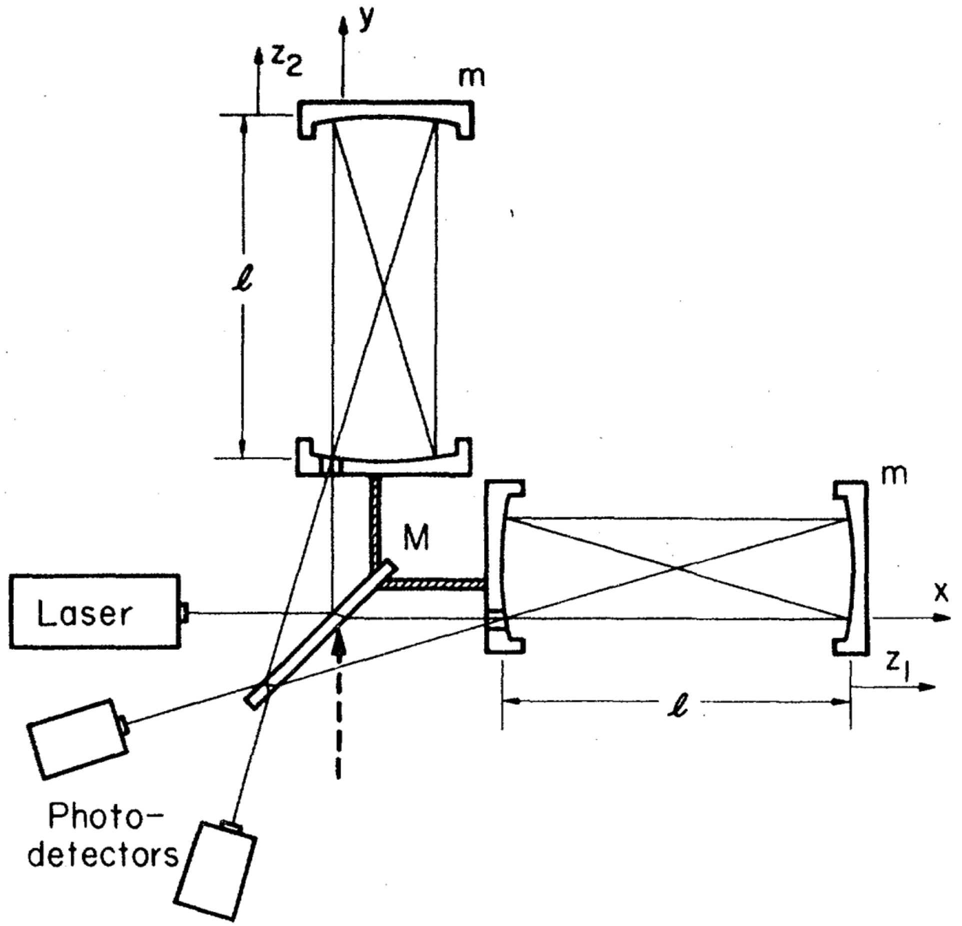


Gravity waves (continued)

- The best way to measure a difference in strain between perpendicular directions is with the classic **Michelson interferometer**, which you may have heard about being used to show that there's no such thing as the aether, and which led directly to Einstein's special theory of relativity.

  - Here one sends the beam from a stable high-power laser – the simple harmonic oscillator in this setup – through the interferometer, to a set of imaging detectors, and tunes the interferometer to sit a destructive-interference minimum on the detectors.

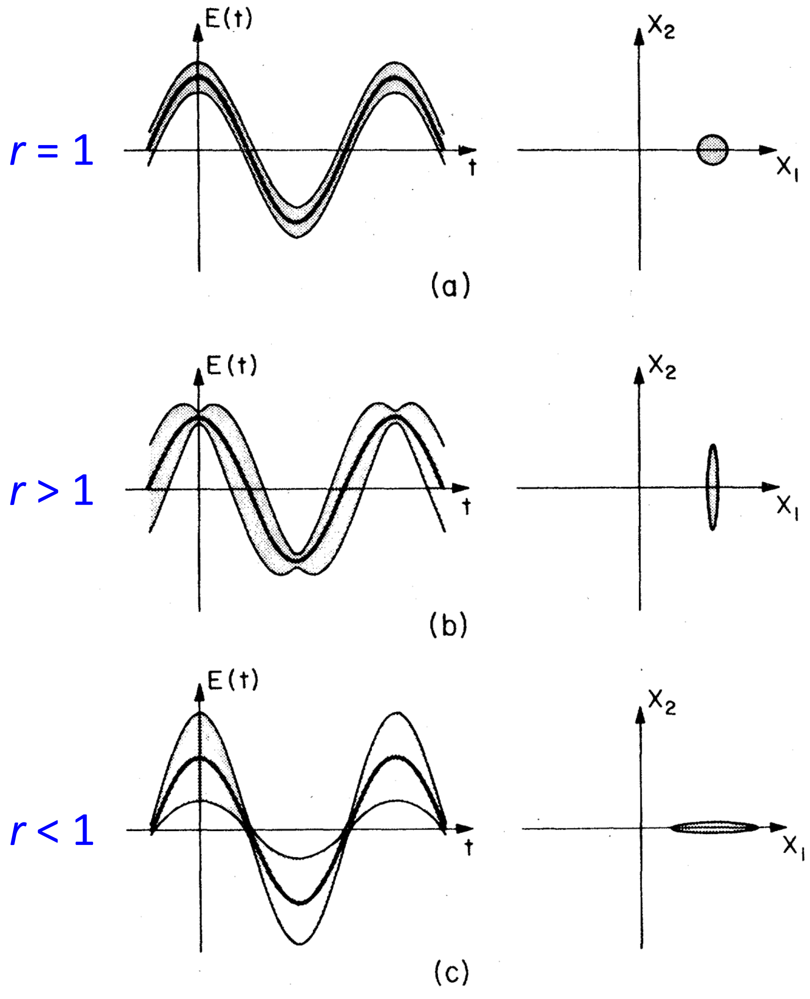
  - If a gravity wave goes past, straining one arm the opposite of the other, then the detector sees a sharp increase in signal.



Caves 1981

Gravity waves (continued)

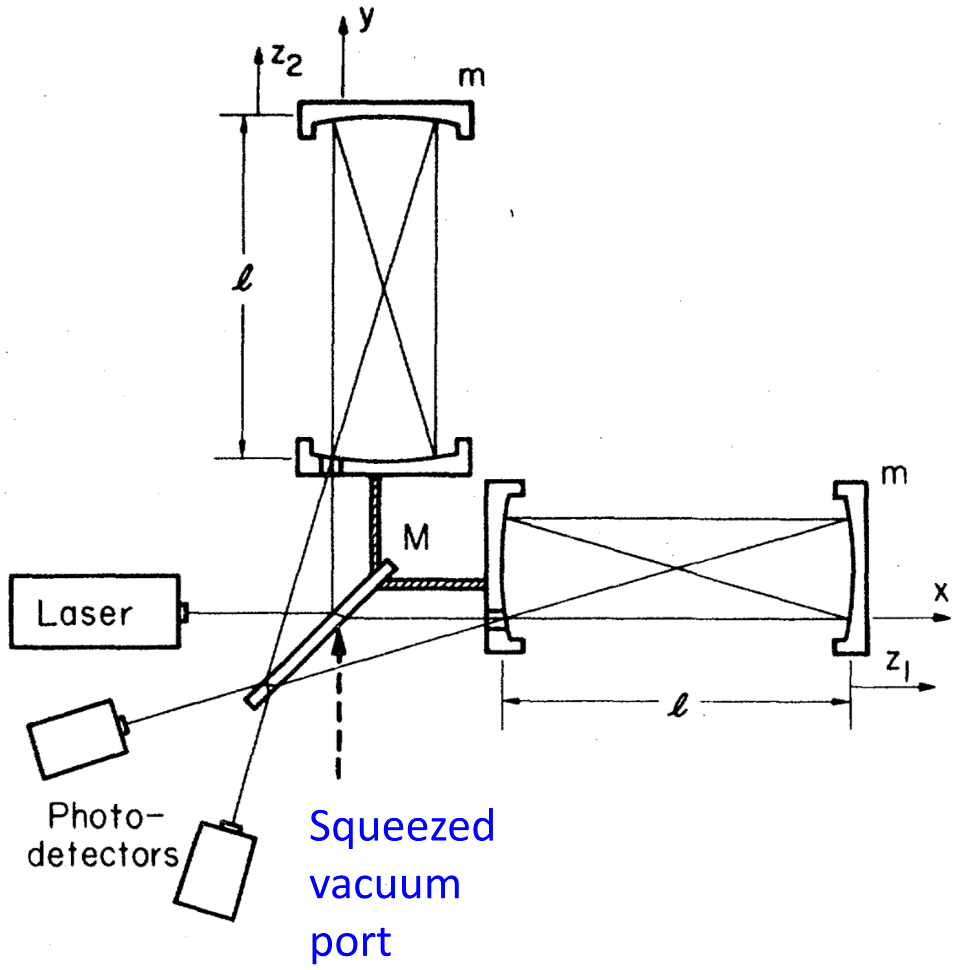
- Shifts in the interference pattern measure differences in lengths – thus the detectors monitor the Hermitian operator $\hat{X}_1$ or $\hat{Y}_1$.
- The other **quadrature** – operators $\hat{X}_2$ or $\hat{Y}_2$ – is not necessary for detecting gravity waves, so LIGO can afford to improve the sensitivity to $\hat{X}_1$ or $\hat{Y}_1$ at the expense of $\hat{X}_2$ or $\hat{Y}_2$.
- This use of squeezed states – using one operator and avoiding measurements of another non-commuting one – is often called **quantum nondemolition**.



Caves 1981

Gravity waves (continued)

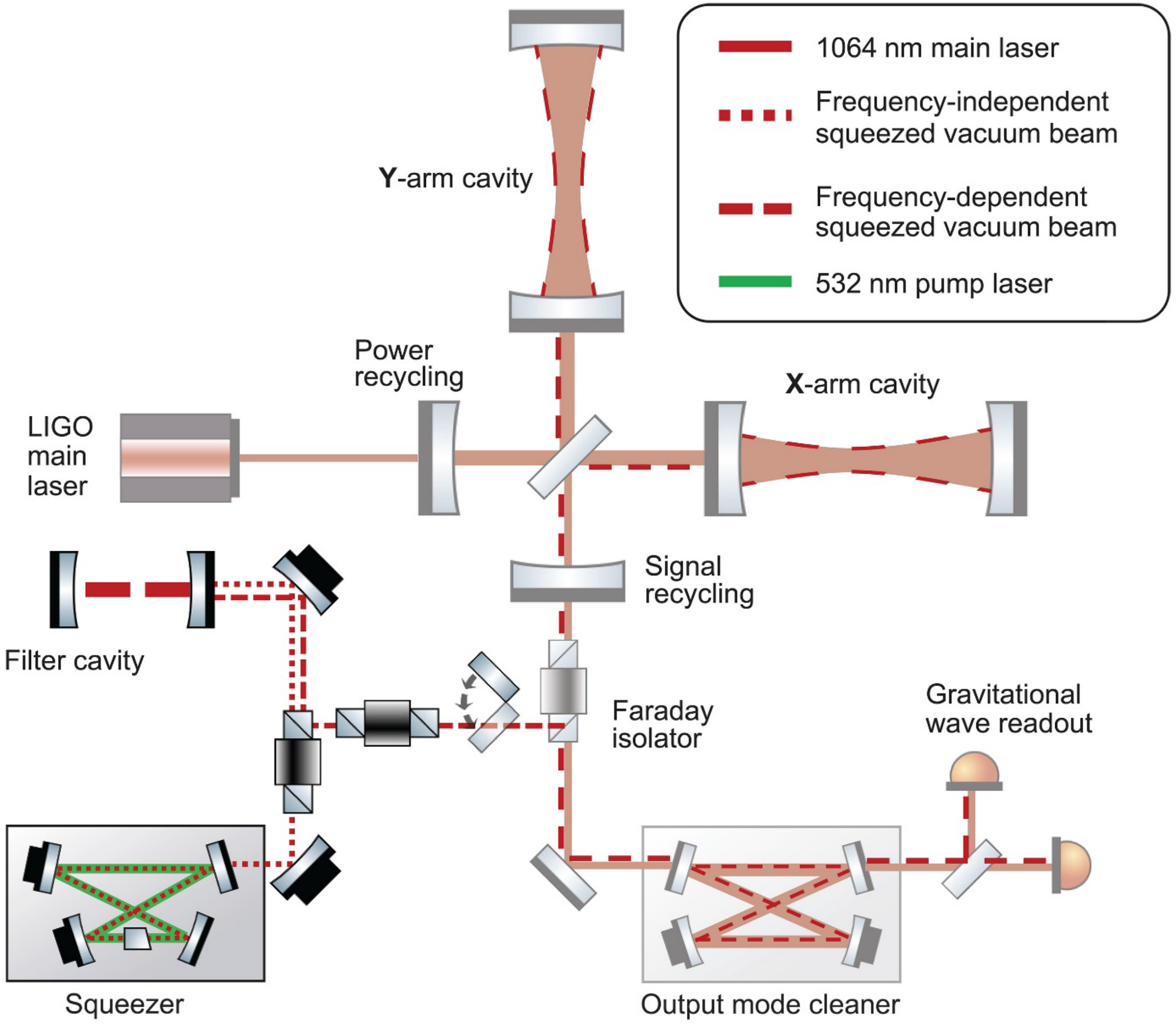
- Without squeezed states, the highly-optimized laser, structure, mirrors and detectors of LIGO developed over the course of > 40 years could detect the first few gravity-wave (GW) events from black-hole mergers. There is little room for improvement in these aspects, but …
- with squeezed states of $r > 1$, LIGO is much more sensitive.
- It turns out that the most important states on which to employ squeezing are those of the oscillator's ground state: the vacuum. LIGO currently combines squeezed vacuum with the laser beam; squeezing the laser is in the plans.
  - The whole scheme, including the importance of squeezing the vacuum, was completely worked out by 1981, by (2017 physics Nobel prizewinner) Kip Thorne's grad student Carl Caves, in his Ph.D. thesis.



Caves 1981

LIGO's interferometers, with squeezed vacuum states

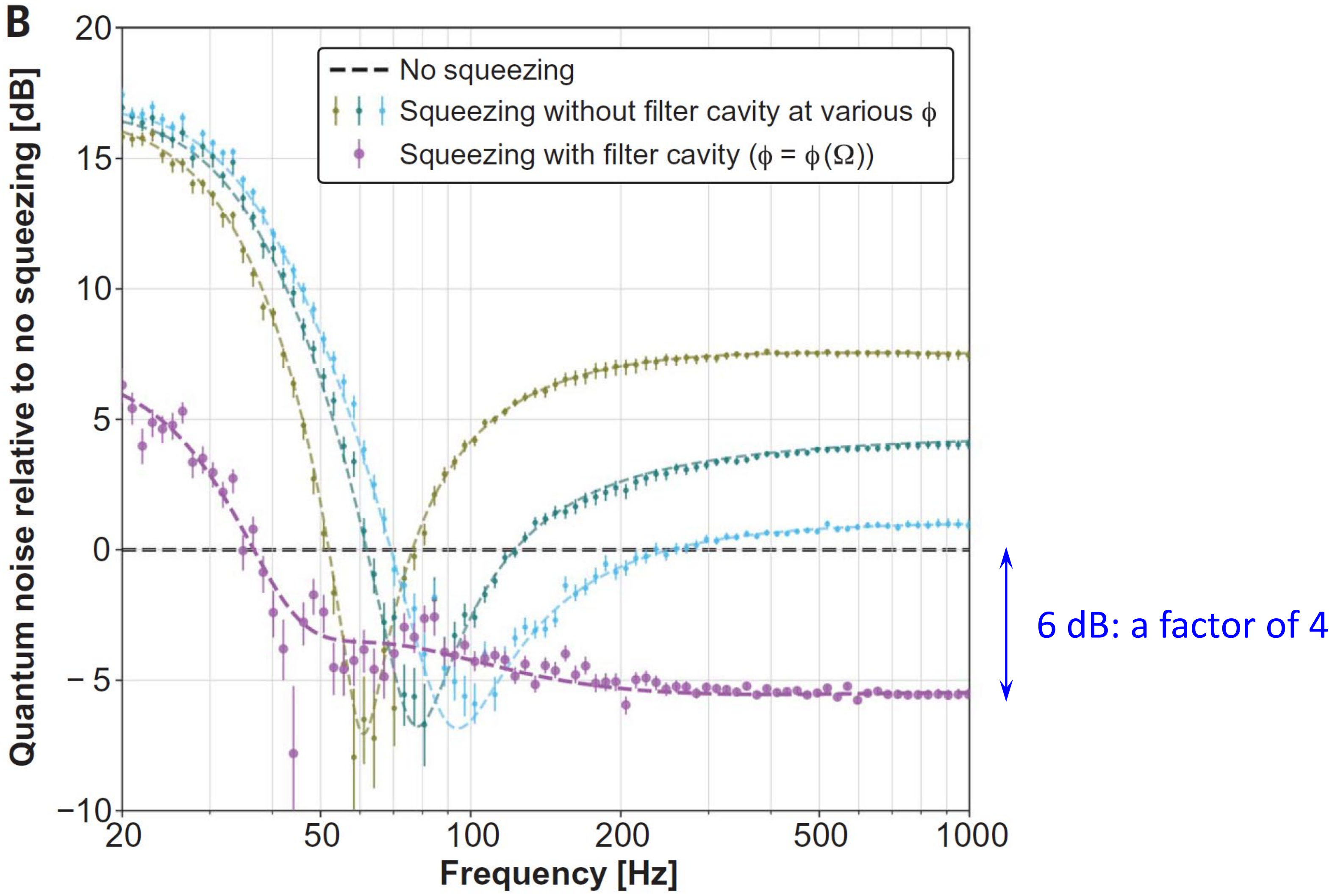
- One in Hanford, WA; the other in Livingston, LA.
  - Similar instruments have since been built in Italy (VIRGO) and Japan (KAGRA), and one is under way in India (IndIGO). They observe in tandem and triangulate their signals to locate the events on the sky.
- The X and Y arm cavities are about 4 km long; the beams travel in evacuated pipes.



Jia+2024

LIGO's interferometers, with squeezed vacuum states

Squeezed states have increased the volume of space probed by LIGO by a factor of about 60, still with lots of room for improvement.



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