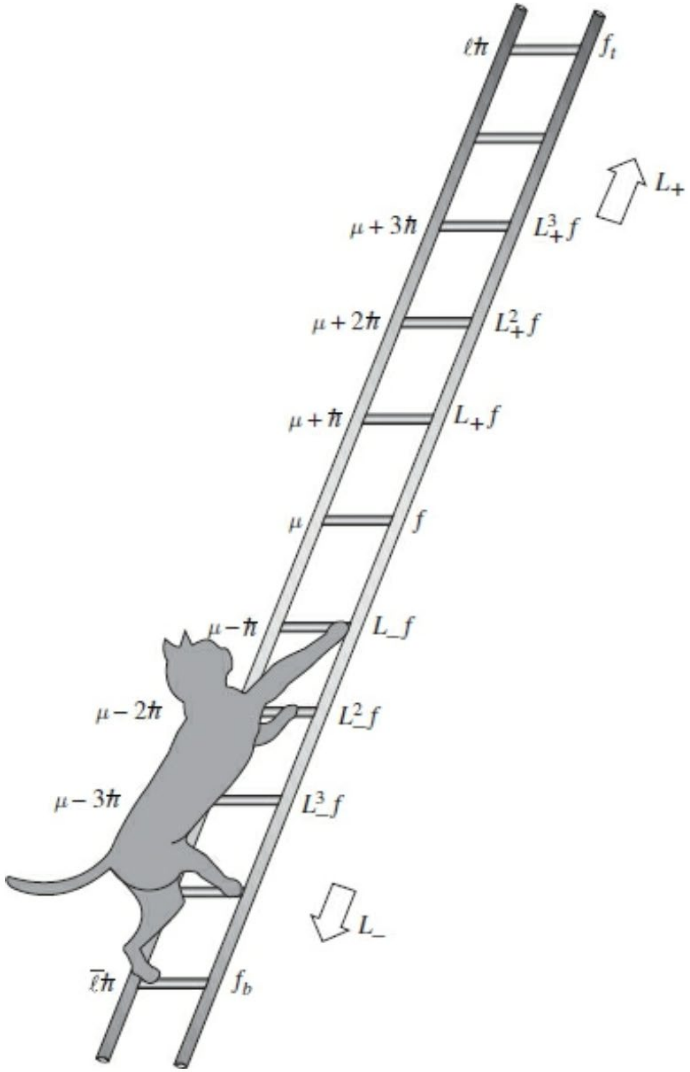
# Today in Physics 237: angular momentum in quantum mechanics

- Orbital angular momentum ***L*** in Cartesian coordinates
- Ladder operators for ***L***
- Eigenvalues and eigenfunctions for $\hat{L}^2$ and $\hat{L}_z$
- The ladder operators in spherical coordinates



# Angular momentum

- You already know how to include angular momentum in quantum mechanics:

Bold blue = unit vectors,
not operators

$$\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p}=\begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ x & y & z\\ p_x & p_y & p_z\end{vmatrix}=\begin{bmatrix}yp_z-zp_y\\ zp_x-xp_z\\ xp_y-yp_x\end{bmatrix}\leftrightarrow\hat{\boldsymbol{L}}=-i\hbar\begin{bmatrix}y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\\ z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\\ x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\end{bmatrix}$$

- And as is now usual, the first things we need to know about it are the commutators:

$$\frac{1}{(-i\hbar)^2}\left[\hat{L}_x,\hat{L}_y\right]=\left[y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y},z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right]=\left[y\frac{\partial}{\partial z},z\frac{\partial}{\partial x}\right]-\left[y\frac{\partial}{\partial z},x\frac{\partial}{\partial z}\right]-\left[z\frac{\partial}{\partial y},z\frac{\partial}{\partial x}\right]+\left[z\frac{\partial}{\partial y},x\frac{\partial}{\partial z}\right]$$

$$=y\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z},z\right]-\underset{0}{\cancel{[y,x]}}\frac{\partial}{\partial z}-z\left[\underset{0}{\cancel{\frac{\partial}{\partial y},\frac{\partial}{\partial x}}}\right]+x\frac{\partial}{\partial y}\left[z,\frac{\partial}{\partial z}\right]=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\quad\Rightarrow\quad\boxed{\left[\hat{L}_x,\hat{L}_y\right]=(-i\hbar)^2\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)=i\hbar\hat{L}_z}$$

# Angular momentum (continued)

- Similarly, $\boxed{\left[\hat{L}_y,\hat{L}_z\right]=i\hbar\hat{L}_x \quad \text{and} \quad \left[\hat{L}_z,L_x\right]=i\hbar\hat{L}_y \quad .}$
- That is, different components of ***L*** are **incompatible** observables (c.f. Lecture 10, pp. 11-14), in which case each pair of components is subject to an uncertainty principle, e.g.

$$\sigma_{L_x}^2\sigma_{L_y}^2 \ge \left(\frac{1}{2i}\left\langle i\hbar L_z\right\rangle\right)^2 = \frac{\hbar^2}{4}\left\langle L_z\right\rangle^2 \quad .$$

- Each component does turn out to be compatible with $\hat{L}^2$ though; for example:

$$\left[\hat{L}^2,\hat{L}_z\right]=\left[\hat{L}_x^2,\hat{L}_z\right]+\left[\hat{L}_y^2,\hat{L}_z\right]+\left[\hat{L}_z^2,\hat{L}_z\right]=\hat{L}_x\left[\hat{L}_x,\hat{L}_z\right]+\left[\hat{L}_x,\hat{L}_z\right]\hat{L}_x+\hat{L}_y\left[\hat{L}_y,\hat{L}_z\right]+\left[\hat{L}_y,\hat{L}_z\right]\hat{L}_y$$

$$=\hat{L}_x\left(-i\hbar\hat{L}_y\right)+\left(-i\hbar\hat{L}_y\right)\hat{L}_x+\hat{L}_y\left(i\hbar\hat{L}_x\right)+\left(i\hbar\hat{L}_x\right)\hat{L}_y=0 \quad \Rightarrow \quad \left[\hat{L}^2,\hat{L}_x\right]=\left[\hat{L}^2,\hat{L}_y\right]=0$$

$$\Rightarrow \quad \boxed{\left[\hat{L}^2,\hat{\boldsymbol{L}}\right]=0} \; .$$

# Ladder operators for angular momentum

- Thus we can find states which are eigenstates of both $\hat{L}^2$ and any component of $\hat{\boldsymbol{L}}$. Like, say, $\hat{L}^2 f = \lambda f \quad , \quad \hat{L}_z f = \mu f \quad .$
- What kinds of states are eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$? We could find out by solving these eigenvalue problems, but there's an easier way, similar to the use of the ladder operators $\hat{a}_\pm$ for the simple harmonic oscillator (Lecture 5).
- Let $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$:

$$\left[\hat{L}_z, \hat{L}_\pm\right] = \left[\hat{L}_z, \hat{L}_x\right] \pm i\left[\hat{L}_z, \hat{L}_y\right] = i\hbar\hat{L}_y \pm i\left(-i\hbar\hat{L}_x\right) = \pm\hbar\left(\hat{L}_x \pm i\hat{L}_y\right) = \pm\hbar\hat{L}_\pm \quad ,$$

$$\left[\hat{L}^2, \hat{L}_\pm\right] = \left[\hat{L}^2, \hat{L}_x\right] \pm i\left[\hat{L}^2, \hat{L}_y\right] = 0 \quad .$$

- If *f* is an eigenfunction of $\hat{L}^2$ and $\hat{L}_z$, then $\hat{L}_\pm f$ is also an eigenfunction of $\hat{L}_z$:

$$\hat{L}^2\left(\hat{L}_\pm f\right) = \hat{L}_\pm\left(\hat{L}^2 f\right) = \lambda\left(\hat{L}_\pm f\right) \quad \text{, and}$$

$$\hat{L}_z\left(\hat{L}_\pm f\right) = \hat{L}_\pm\left(\hat{L}_z f\right) \pm \hbar\hat{L}_\pm f = \mu\hat{L}_\pm f \pm \hbar\hat{L}_\pm f = \left(\mu \pm \hbar\right)\hat{L}_\pm f \quad .$$

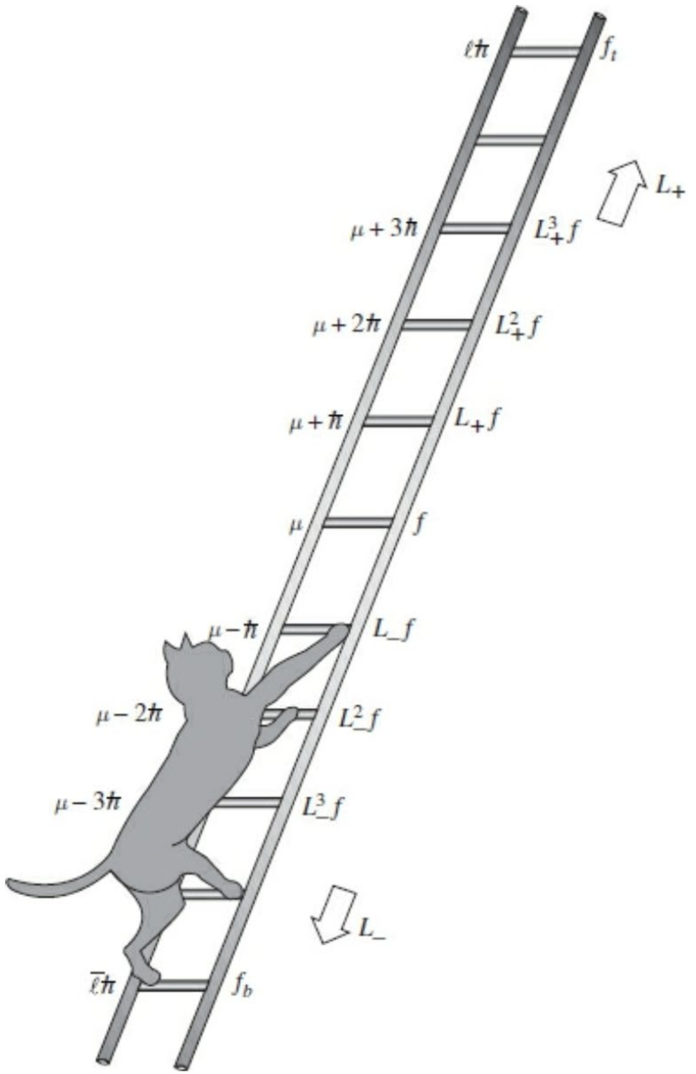
# Ladder operators for angular momentum (continued)

- So $\hat{L}_\pm$ act on eigenfunctions of $\hat{L}_z$ the way $\hat{a}_\pm$ act on eigenfunctions of $\hat{H} = \hbar\omega\left(\hat{a}_\pm\hat{a}_\mp \pm \frac{1}{2}\right)$ in the case of the simple harmonic oscillator:

$$\hat{L}_z f = \mu f \quad , \quad \hat{L}_z\left(\hat{L}_\pm f\right) = \left(\mu \pm \hbar\right)\left(\hat{L}_\pm f\right) \quad ;$$

$$\hat{H}\psi_v = E\psi_v \quad , \quad \hat{H}\left(\hat{a}_\pm\psi_v\right) = \left(E \pm \hbar\omega\right)\left(\hat{a}_\pm\psi_v\right) \quad .$$

- $\hat{L}_\pm$ are raising and lowering operators for the *z* component of angular momentum, as $\hat{a}_\pm$ are for the energy of a quantum simple harmonic oscillator.
- Note that there is nothing special about *z* here. For every Cartesian component of ***L***, there are raising and lowering operators built out of the other two components. Conventionally we only discuss one, choosing *z*.



# Ladder operators for angular momentum (continued)

- But $L_z$ cannot exceed $\sqrt{\left|L^2\right|}$. So there must be a state with a maximum value of $L_z$ – call it $f_\ell$, and call the corresponding eigenvalue $\mu = \ell$:

$$\hat{L}_+ f_\ell = 0 \quad , \quad \hat{L}_z f_\ell = \hbar\ell f_\ell \quad , \quad \hat{L}^2 f_\ell = \lambda f_\ell \quad .$$

- Also $\hat{L}_\pm \hat{L}_\mp = \left(\hat{L}_x \pm i\hat{L}_y\right)\left(\hat{L}_x \mp i\hat{L}_y\right) = \hat{L}_x^2 + \hat{L}_y^2 \mp i\left(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x\right) = \hat{L}^2 - \hat{L}_z^2 \mp i\left(i\hbar\hat{L}_z\right)$

$$\Rightarrow \quad \hat{L}^2 = \hat{L}_\pm \hat{L}_\mp + \hat{L}_z^2 \mp \hbar\hat{L}_z \quad .$$

- And $\hat{L}^2 f_\ell = \left(\hat{L}_- \hat{L}_+ + \hat{L}_z^2 + \hbar\hat{L}_z\right) f_\ell = \left(0 + \hbar^2\ell^2 + \hbar^2\ell\right) f_\ell = \hbar^2\ell(\ell+1) f_\ell = \lambda f_\ell$

$$\Rightarrow \quad \lambda = \hbar^2\ell(\ell+1) \quad .$$

# Ladder operators for angular momentum (continued)

- Similarly there is a lowest- $L_z$ state; call it $f_b$, which is such that $\hat{L}_- f_b = 0$ .

$$\hat{L}_z f_b = \hbar\bar{\ell} f_b \quad , \quad \hat{L}^2 f_b = \lambda f_b \quad :$$

$$\hat{L}^2 f_b = \left(\hat{L}_+\hat{L}_- + \hat{L}_z^2 - \hbar\hat{L}_z\right) f_b = \left(0+\hbar^2\bar{\ell}^2 - \hbar^2\bar{\ell}\right) f_b = \hbar^2\bar{\ell}(\bar{\ell}-1) f_b = \lambda f_b$$

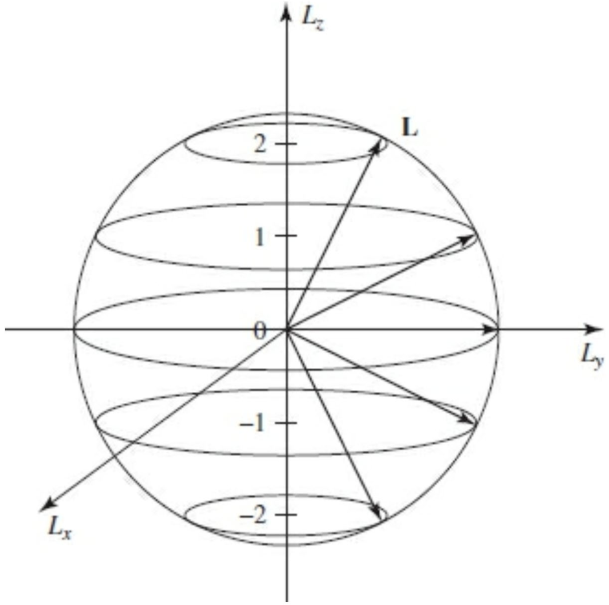
$$\Rightarrow \quad \lambda = \hbar^2\bar{\ell}(\bar{\ell}-1) \quad .$$

- But from the highest ladder-rung we got $\lambda = \hbar^2\ell(\ell+1)$ so

$$\bar{\ell}(\bar{\ell}-1) = \ell(\ell+1) \begin{cases} \bar{\ell} = \ell+1 \quad ??? \quad (\bar{\ell} > \text{maximum?}) \\ \boxed{\bar{\ell} = -\ell} \end{cases}$$

# Ladder operators for angular momentum (continued)

- Summary: the eigenvalues of $\hat{L}_z$ – call them $m\hbar$ – run $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$: a total of $2\ell+1$ rungs.
- From lowest to highest rung is *N* steps: $-\ell + N = \ell \quad \Rightarrow \quad \ell = N/2$.
  - Thus $\ell$ can only be an integer or a half-integer.
- And the (common) eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$ – call them $f_\ell^m$ – are such that



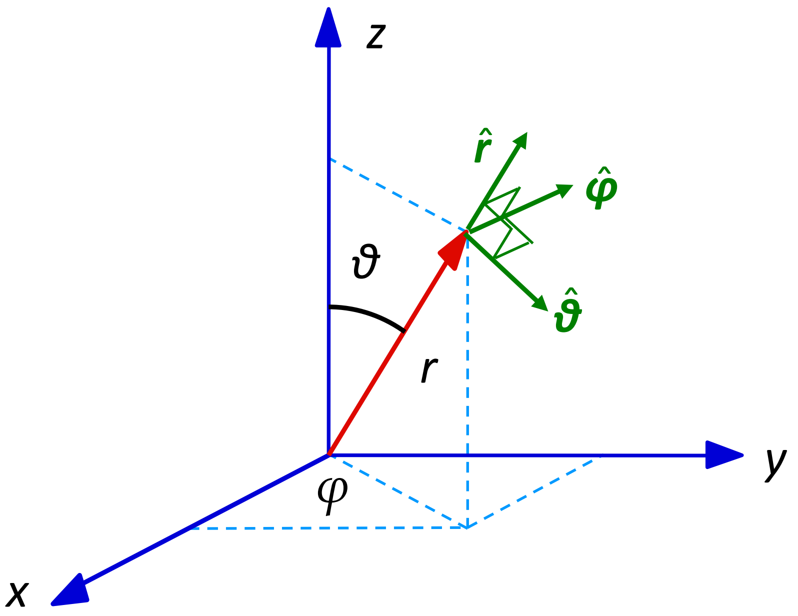
$$\boxed{\begin{gathered}\hat{L}^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m \quad , \quad \hat{L}_z f_\ell^m = m\hbar f_\ell^m \quad , \\[2em] \ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \quad \text{and} \quad m = -\ell, -\ell+1, \ldots, \ell\end{gathered}}$$

# Now in spherical coordinates

- Recall $\nabla = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\vartheta}}\frac{1}{r}\frac{\partial}{\partial\vartheta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}$ : Bold blue = unit vectors again

$$\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}}\times\hat{\boldsymbol{p}} = -i\hbar\hat{\boldsymbol{r}}\times\nabla = -i\hbar\left[r\left(\hat{\boldsymbol{r}}\times\hat{\boldsymbol{r}}\right) + \left(\hat{\boldsymbol{r}}\times\hat{\boldsymbol{\vartheta}}\right)\frac{\partial}{\partial\vartheta} + \left(\hat{\boldsymbol{r}}\times\hat{\boldsymbol{\varphi}}\right)\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right]$$

$$= -i\hbar\left[\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\vartheta} - \hat{\boldsymbol{\vartheta}}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right]$$



- Look up the unit vectors' Cartesian components in [PHYS 217, Lecture 3:]()

$$\hat{\boldsymbol{\vartheta}} = \cos\vartheta\cos\varphi\hat{\boldsymbol{x}} + \cos\vartheta\sin\varphi\hat{\boldsymbol{y}} - \sin\vartheta\hat{\boldsymbol{z}} \quad ,$$

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\boldsymbol{x}} + \cos\varphi\hat{\boldsymbol{y}} \quad ,$$

to get

$$\hat{\boldsymbol{L}} = -i\hbar\left[\left(-\sin\varphi\hat{\boldsymbol{x}} + \cos\varphi\hat{\boldsymbol{y}}\right)\frac{\partial}{\partial\vartheta} - \left(\cos\vartheta\cos\varphi\hat{\boldsymbol{x}} + \cos\vartheta\sin\varphi\hat{\boldsymbol{y}} - \sin\vartheta\hat{\boldsymbol{z}}\right)\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right] \quad .$$

# Now in spherical coordinates (continued)

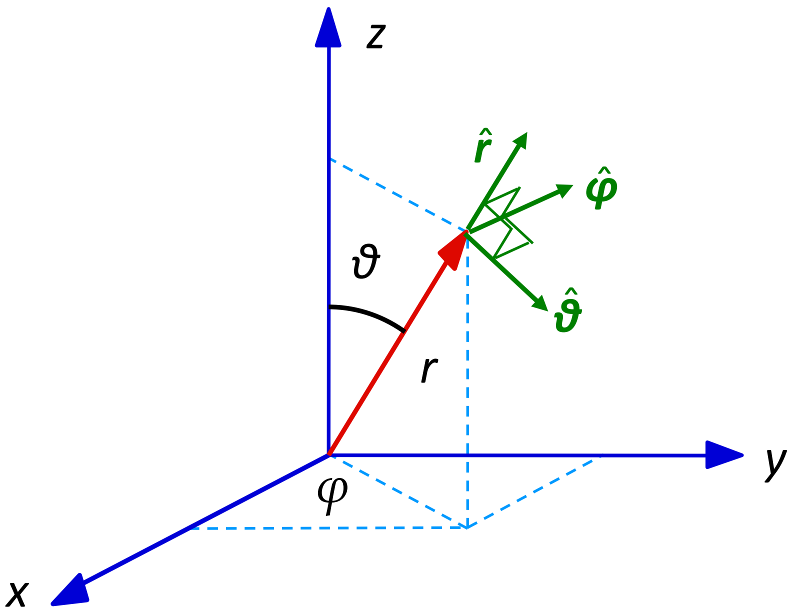
- So

$$\hat{L}_x = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\vartheta} - \cos\varphi\cot\vartheta\frac{\partial}{\partial\varphi}\right) \quad ,$$

$$\hat{L}_y = -i\hbar\left(+\cos\varphi\frac{\partial}{\partial\vartheta} - \sin\varphi\cot\vartheta\frac{\partial}{\partial\varphi}\right) \quad ,$$

$$\boxed{\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}} \quad .$$

(especially noteworthy)



- This makes the new ladder operators

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\vartheta} - \cos\varphi\cot\vartheta\frac{\partial}{\partial\varphi}\right)$$

$$\mp\hbar\left(+\cos\varphi\frac{\partial}{\partial\vartheta} - \sin\varphi\cot\vartheta\frac{\partial}{\partial\varphi}\right) \quad .$$

# Now in spherical coordinates (continued)

- Use the Euler formula $\cos\varphi \pm i\sin\varphi = e^{\pm i\varphi}$ :

$$\hat{L}_\pm = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\vartheta} - \cos\varphi\cot\vartheta\frac{\partial}{\partial\varphi}\right) \pm i(-i\hbar)\left(+\cos\varphi\frac{\partial}{\partial\vartheta} - \sin\varphi\cot\vartheta\frac{\partial}{\partial\varphi}\right)$$

$$= -i\hbar\left[-\sin\varphi\frac{\partial}{\partial\vartheta} - \cos\varphi\cot\vartheta\frac{\partial}{\partial\varphi} \pm i\cos\varphi\frac{\partial}{\partial\vartheta} \mp i\sin\varphi\cot\vartheta\frac{\partial}{\partial\varphi}\right] = -i\hbar\left[\pm i(\cos\varphi \pm i\sin\varphi)\frac{\partial}{\partial\vartheta} - (\cos\varphi \pm i\sin\varphi)\cot\vartheta\frac{\partial}{\partial\varphi}\right]$$

$$= -i\hbar e^{\pm i\varphi}\left(\pm i\frac{\partial}{\partial\vartheta} - \cot\vartheta\frac{\partial}{\partial\varphi}\right) = \hbar e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\vartheta} + i\cot\vartheta\frac{\partial}{\partial\varphi}\right) = \boxed{\pm\hbar e^{\pm i\varphi}\left(\frac{\partial}{\partial\vartheta} \pm i\cot\vartheta\frac{\partial}{\partial\varphi}\right)} \quad .$$

- For practice this week, you will show that (G&S problem 4.24)

$$\boxed{\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right]}$$

# Now in spherical coordinates (continued)

- This determines $f_\ell^m$ :

$$\hat{L}^2 f_\ell^m = -\hbar^2 \left[ \frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \left( \sin\vartheta \frac{\partial}{\partial\vartheta} \right) + \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial\varphi^2} \right] f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m \quad ,$$

$$\hat{L}_z f_\ell^m = -i\hbar \frac{\partial}{\partial\varphi} f_\ell^m = \hbar m f_\ell^m \quad .$$

See Lecture 14, pp. 9-12, for the solution: these equations are the same as the polar and azimuthal equations, the angular parts into which we separated the time-independent Schrödinger equation, and we found that the solutions comprise the spherical harmonics:

$$\boxed{f_\ell^m = Y_\ell^m(\vartheta,\varphi) \quad .}$$

- In separating and solving the Schrödinger equation in spherical geometry, we were (unintentionally) obtaining eigenfunctions of $\hat{H}$ which also happen to be eigenfunctions of the $\hat{H}$-compatible observables $\hat{L}^2$ and $\hat{L}_z$.