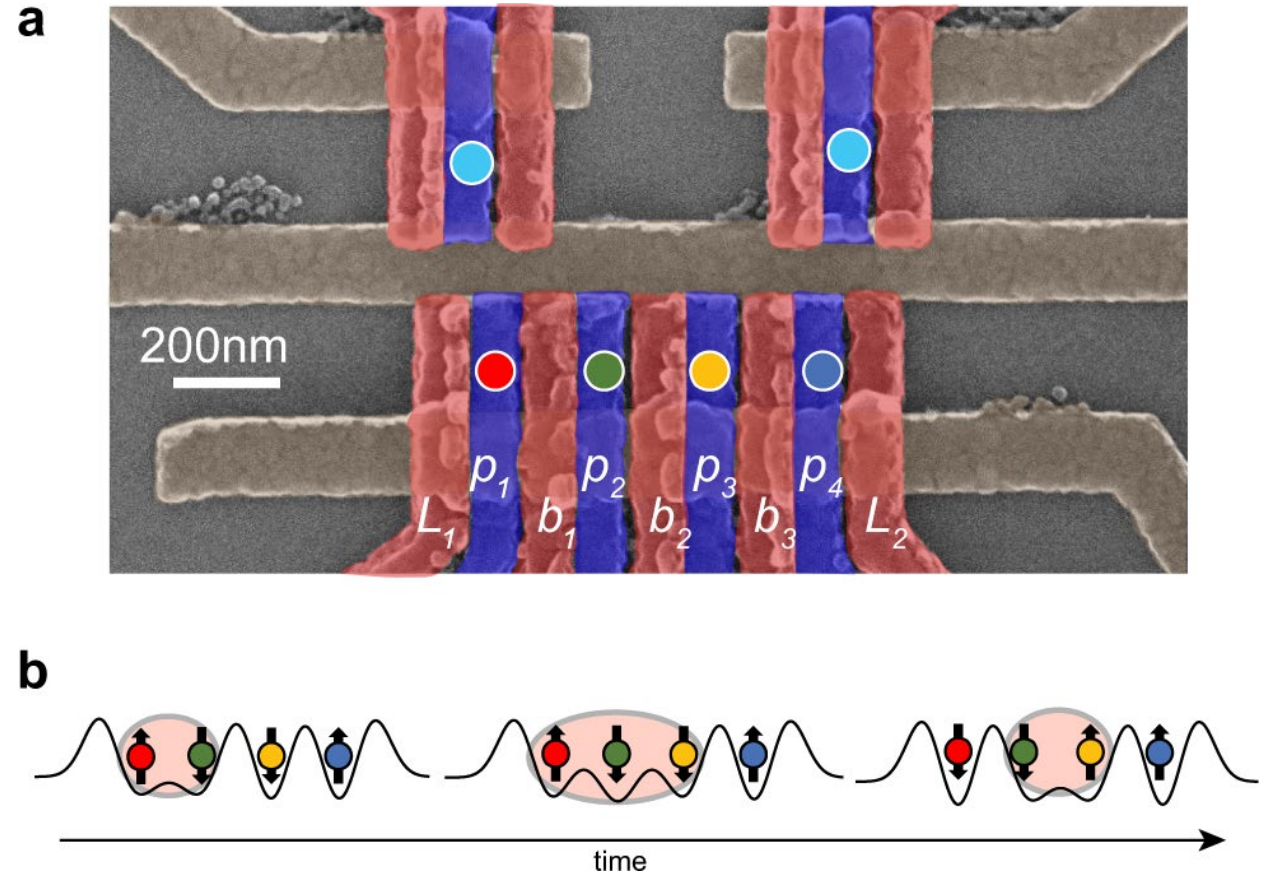


Today in Physics 237: spin

- Orbital angular momentum
- Spin
- Spin $\frac{1}{2}$, spinors, and the Pauli spin matrices



Quantum-dot spin chain: diagram (b) and electron micrograph of device (a). From [Kandel et al. 2021](#) (Prof. John Nichol's group).

Orbital angular momentum

A couple important new ingredients lie within our new ladder-operator-powered view of the angular part of the 3-D Schrödinger equation in spherical geometry.

- In G&S problem 4.24, on this week's assignment, you will show that \hat{L}^2 can be written as

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \left(\sin\vartheta \frac{\partial}{\partial\vartheta} \right) + \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial\varphi^2} \right] ,$$

whence the time-independent Schrödinger equation becomes

$$\frac{1}{2mr^2} \left[-\hbar^2 \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \hat{L}^2 \right] \psi + V\psi = E\psi .$$

This form of the time-independent Schrödinger equation has many uses. An introductory-example use appears as G&S problem 4.27, also on this week's assignment.

Orbital angular momentum (continued)

- In its previous appearance as the genesis to the ϑ - φ part of the hydrogen wavefunction, the angular equation resulted in whole-integer values of the indices ℓ and m . But half-integer values are allowed in the operator-algebra solution ([Lecture 17](#), p. 8). This is a fortuitous coincidence:
 - The angular solution we obtained for the hydrogen atom has only whole-integer indices because the Hamiltonian only included terms which correspond to **orbital angular momentum**: the quantum analogue of kinetic energy of **revolving** bodies, and what we have called L so far.
 - As such, it evokes the Bohr model of the H atom, in which the electron **orbits** the proton in solar-system fashion.
 - Bohr knew that this generally would not be stable, owing to continuous electromagnetic radiation (emission of light) by the orbiting electron.
 - Thus he hypothesized that there is something special about orbits with quantized angular momentum: $L_n = n\hbar$, $n = 1, 2, \dots$. These orbits were mysteriously allowed to violate energy conservation. Radiation at discrete wavelengths was allowed for electrons jumping between such orbits, *this* obeying energy conservation.
 - This brings us to **spin**, which needs the half-integer indices.

Spin

- As orbital angular momentum is the quantum analogue of revolving bodies, spin is often taken as the analogue of rotating bodies. Electrons, protons, and neutrons all have spin angular momentum $S = \hbar/2$ (**spin ½**).
- This analogy really doesn't work. Consider [G&S problem 4.28](#), done here in cgs units:

If the electron were a classical solid sphere, with radius $r_c = e^2/m_e c^2$ – the classical electron radius, obtained by assuming the electron's mass to be attributable to energy stored in the electric field – and if its angular momentum were $\hbar/2$, then how fast would a point on its “equator” be moving? Does this model of electron spin make sense?

- In PHYS 141 and 142 (or ASTR 111 and 142) you learned that a uniform sphere with radius r_c , carrying mass m_e and charge e , has moment of inertia I and electrostatic potential energy U given by

$$I = \frac{2}{5} m_e r_c^2 = \frac{L}{\omega} = \frac{\hbar}{2\omega} = \frac{\hbar r_c}{2v} \quad \text{and} \quad U = \frac{3e^2}{5r_c} = m_e c^2 \quad \Rightarrow \quad v = \frac{\hbar r_c}{2} \frac{5}{2m_e r_c^2} = \frac{5\hbar}{4m_e} \frac{5m_e c^2}{3e^2} = \frac{25\hbar c}{12e^2} = \frac{25}{12} 137c = \boxed{285c} .$$

Violates special relativity, egregiously, so No.

$\alpha = e^2/\hbar c \cong 1/137$ is called the **fine - structure constant**.

Spin (continued)

- So regard spin \mathbf{S} as an intrinsic angular momentum (and as a vector), built into the elementary quanta, unrelated to a physical rotation.
- Good news: \mathbf{S} works just like orbital angular momentum. The same operator scheme can be used as with \mathbf{L} .

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z \quad , \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x \quad , \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y \quad ,$$

$$\hat{S}^2|sm\rangle = \hbar^2s(s+1)|sm\rangle \quad , \quad \hat{S}_z|sm\rangle = \hbar m|sm\rangle \quad ,$$

$$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y \quad , \quad \hat{S}^2(\hat{S}_\pm|sm\rangle) = \hbar^2s(s+1)(\hat{S}_\pm|sm\rangle) \quad , \quad \hat{S}_z(\hat{S}_\pm|sm\rangle) = \hbar(m\pm 1)(\hat{S}_\pm|sm\rangle) \quad ,$$

and, as you will find for L in G&S problem 4.21 on this week's assignment,

$$\hat{S}_\pm|sm\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}|s(m\pm 1)\rangle \quad .$$

where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ and $m = -s, -s+1, \dots, s$.

Spin 1/2

- If a quantum has $|\mathbf{S}| = \hbar/2$, we can represent its states using a basis of only two: up and down, names after the only two orientations of one Cartesian component of \mathbf{S} – as usual, the z component, but also as usual, it would work for any choice:

$$|sm\rangle = \left| \frac{1}{2} + \frac{1}{2} \right\rangle, \quad \left| \frac{1}{2} - \frac{1}{2} \right\rangle, \quad \text{or} \quad |\uparrow\rangle, \quad |\downarrow\rangle.$$

Represented as 2×1 column vectors, a.k.a. **spinors**, $\chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- That these are eigenstates of \hat{S}^2 and \hat{S}_z may seem obvious: $\hat{S}^2 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \frac{3\hbar^2}{4} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$, $\hat{S}_z \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = -\frac{\hbar}{2} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$, etc.
- To use the column vector representation as in [Lecture 12](#), we need matrix representations of our operators \hat{S}^2 and \hat{S}_z :

$$\vec{S}^2 \text{ or } \vec{S}_z = \begin{bmatrix} c & d \\ e & f \end{bmatrix}, \quad \text{such that} \quad \vec{S}^2 \chi_{\pm} = \frac{3\hbar^2}{4} \chi_{\pm} \quad \text{and} \quad \vec{S}_z \chi_{\pm} = \pm \frac{\hbar}{2} \chi_{\pm} :$$

Spin 1/2 (continued)

- That is,

$$\vec{S}^2 \chi_+ = \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ e \end{bmatrix} = \frac{3\hbar^2}{4} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c = \frac{3\hbar^2}{4}, \quad e = 0; \quad \vec{S}^2 \chi_- = \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ f \end{bmatrix} = \frac{3\hbar^2}{4} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow d = 0, \quad f = \frac{3\hbar^2}{4};$$

$$\Rightarrow \vec{S}^2 = \frac{3\hbar^2}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\vec{S}_z \chi_+ = \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ e \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c = \frac{\hbar}{2}, \quad e = 0; \quad \vec{S}_z \chi_- = \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ f \end{bmatrix} = -\frac{\hbar}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow d = 0, \quad f = -\frac{\hbar}{2};$$

$$\Rightarrow \vec{S}_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Spin 1/2 (continued)

- And again with the ladder operators:

$$\vec{S}_+\chi_- = \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ f \end{bmatrix} = \hbar \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow d = \hbar, f = 0; \quad \vec{S}_-\chi_+ = \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ e \end{bmatrix} = \hbar \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow c = 0, e = \hbar;$$

$$\vec{S}_+\chi_+ = \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ e \end{bmatrix} = 0 \Rightarrow c = 0, e = 0; \quad \vec{S}_-\chi_- = \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ f \end{bmatrix} = 0 \Rightarrow d = 0, f = 0;$$

$$\Rightarrow \vec{S}_+ = \hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \vec{S}_- = \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

- From the ladder operators we get expressions for \vec{S}_x and \vec{S}_y :

Spin 1/2 (continued)

$$\vec{S}_x = \frac{1}{2}(\vec{S}_+ + \vec{S}_-) = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \vec{S}_y = \frac{i}{2}(\vec{S}_+ - \vec{S}_-) = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

- The trio corresponding to the Cartesian components of \mathbf{S} ,

$$\vec{S} = \frac{\hbar}{2} \vec{\sigma}, \quad \text{where } \vec{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \vec{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \text{and } \vec{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

are called the **Pauli spin matrices**.

- Note that \vec{S}^2 and all three components of \vec{S} are Hermitian (i.e. self-adjoint; equal to the complex conjugate of their transpose). Not the ladder operators, though.