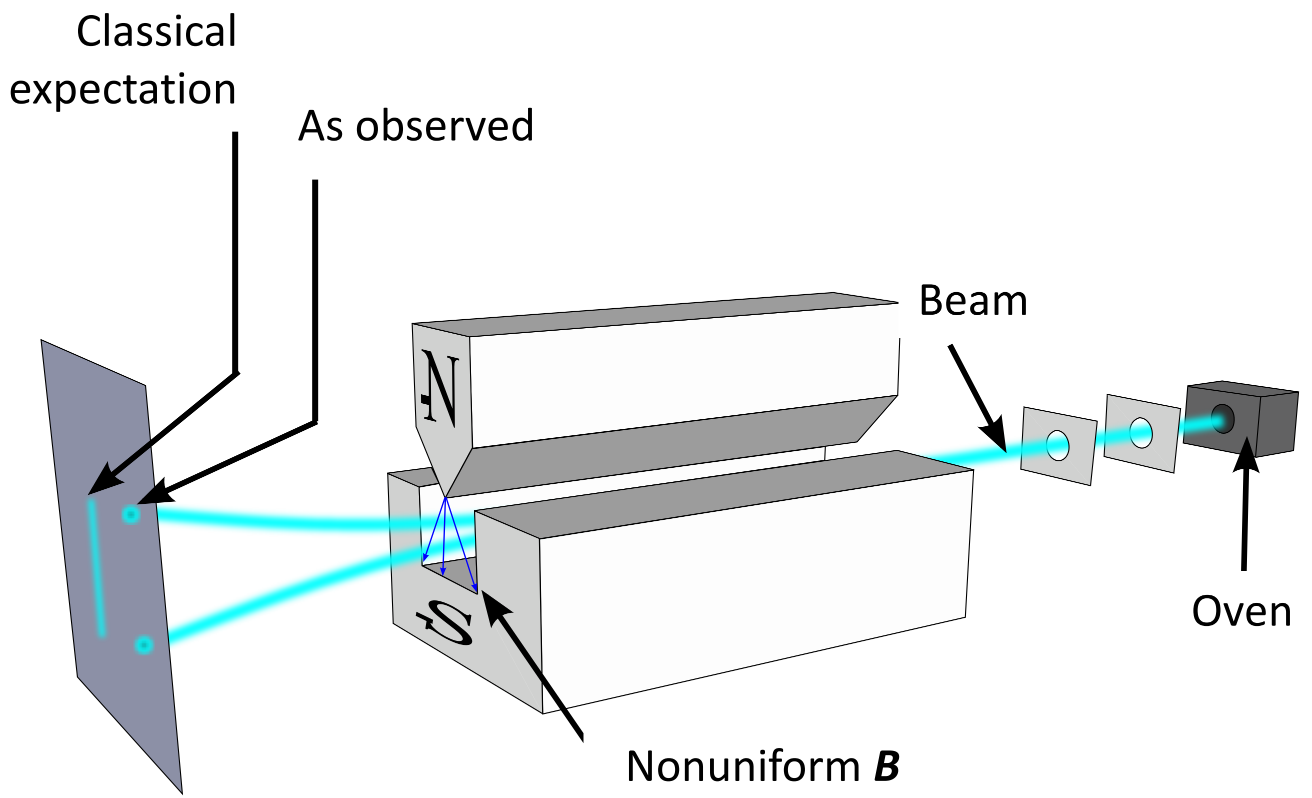
Today in Physics 237: angular momentum and magnetism

- Examples:
  - Use of the spin matrices
  - Spin values other than 1/2.
- Electrons in magnetic fields:
  - Larmor precession
  - The Stern-Gerlach experiment



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Spin-matrix example

G&S problem 4.32:

a. Find the eigenvalues and eigenspinors of $\vec{S}_y$.

b. If you were to measure S_y for a particle in the general state

$$\chi = a\chi_+ + b\chi_- = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} \quad ,$$

(Equation 4.139), what values might you get, and what is the probability of each? Check that the probabilities add up to 1. Note: *a* and *b* need not be real!

c. If you were to measure S_y^2, what values might you get, and with what probabilities?

We'll assume that χ is normalized, so $\chi^\dagger\chi = \left(a^*\begin{bmatrix}1 & 0\end{bmatrix} + b^*\begin{bmatrix}0 & 1\end{bmatrix}\right)\left(a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}\right) = |a|^2 + |b|^2 = 1 \quad .$

Spin-matrix example (continued)

a.

$$\vec{S}_y = \frac{i}{2}\left(\vec{S}_+ + \vec{S}_-\right) = \frac{\hbar}{2}\vec{\sigma}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \Rightarrow \left|\vec{S}_y - \lambda\vec{I}\right| = \begin{vmatrix} -\lambda & -i\hbar/2 \\ i\hbar/2 & -\lambda \end{vmatrix} = \lambda^2 - \frac{\hbar^2}{4} = 0 \Rightarrow \boxed{\lambda = \pm\frac{\hbar}{2}}.$$

$$\vec{S}_y\begin{bmatrix} a \\ b \end{bmatrix} = \pm\frac{\hbar}{2}\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} -id \\ ic \end{bmatrix} = \pm\frac{\hbar}{2}\begin{bmatrix} c \\ d \end{bmatrix} \Rightarrow \begin{matrix} -id = \pm c \\ ic = \pm d \end{matrix}$$

$$\Rightarrow |c|^2 + |d|^2 = |c|^2 + |c|^2 = 1 \Rightarrow c = \frac{1}{\sqrt{2}} \quad , \quad d = \pm\frac{i}{\sqrt{2}} \Rightarrow \boxed{\chi_{y+} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \quad , \quad \chi_{y-} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}}.$$

b. Call the probability amplitudes e_+ and e_-:

$$e_\pm = \left(\chi_{y\pm}\right)^\dagger \chi = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \mp i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\sqrt{2}}(a \mp ib) \Rightarrow \boxed{\left|e_\pm\right|^2 = \frac{1}{2}\left|a \mp ib\right|^2 \quad \text{, respectively for } S_y = \pm\hbar/2.}$$

Spin-matrix example (continued)

Check that the probabilities add up to one:

$$P = |e_+|^2 + |e_-|^2 = \frac{1}{2}|a-ib|^2 + \frac{1}{2}|a+ib|^2 = \frac{1}{2}[(a+ib)(a-ib)+(a-ib)(a+ib)] = a^2 + b^2 = 1 \quad \checkmark$$

c. Intuitively: since the eigenvalues of $\overleftrightarrow{S}_y$ are $\pm\hbar/2$, those of $\overleftrightarrow{S}_y^2$ must be $\hbar^2/4$, with probability 1. But we are still building intuition:

$$\overleftrightarrow{S}_y^2\chi = \frac{\hbar^2}{4}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\left(a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{\hbar^2}{4}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\left(a\begin{bmatrix}0\\i\end{bmatrix} + b\begin{bmatrix}-i\\0\end{bmatrix}\right) = \frac{\hbar^2}{4}\left(a\begin{bmatrix}0\\1\end{bmatrix} + b\begin{bmatrix}1\\0\end{bmatrix}\right) \quad .$$

So a measurement of $\overleftrightarrow{S}_y^2$ gives either $\hbar^2/4$ with probability $a^2/(a^2+b^2) = a^2$, or $\hbar^2/4$ again with probability $b^2/(a^2+b^2) = b^2$. It's $\hbar^2/4$ either way.

Spin other than 1/2

Of course, not everything in nature has spin 1/2. How about spin 1, for example?

- Spin-1 quanta include gluons, *W* and *Z* bosons, and most photons – that is, the quanta that mediate the strong, weak, and electromagnetic interactions, respectively.

Let's find an appropriate basis, and the set of spin operators, for spin 1. (This is G&S problem 4.34, and then some.)

- If $s = 1$, then m can be –1, 0, or +1, so we need another element in the spinors, compared to Lecture 18, page 6:

$$\chi_+ = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \chi_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \text{and} \quad \chi_- = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad .$$

- $\hat{S}_z$ acts on these states according to $\hat{S}_z\chi_+ = \hbar\chi_+$, $\hat{S}_z\chi_0 = 0$, and $\hat{S}_z\chi_- = -\hbar\chi_-$.

Spin other than 1/2 (continued)

- We can find the matrix representation of $\hat{S}_z$ the same way we did for spin ½:

$$\overleftrightarrow{S}_z = \begin{bmatrix} p & q & r \\ t & u & v \\ x & y & z \end{bmatrix} \quad ; \quad \overleftrightarrow{S}_z \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p \\ t \\ x \end{bmatrix} = \hbar \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad p = \hbar, \quad t = x = 0 \quad ,$$

and so forth, leading to

$$\boxed{\overleftrightarrow{S}_z = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}} \quad .$$

- For the spin-1 ladder operators. We still have $\hat{S}_\pm |s,m\rangle = \sqrt{s(s+1) - m(m\pm 1)}\,|s,m\pm 1\rangle$, as you have shown (for $\boldsymbol{L}$) in assignment 9 (G&S problem 4.21). This gives $\hat{S}_\pm |1,m\rangle = \sqrt{2 - m(m\pm 1)}\,|1,m\pm 1\rangle$, or

Spin other than 1/2 (continued)

$$\left.\begin{array}{l}\vec{S}_+\chi_+ = 0 \quad , \quad \vec{S}_+\chi_0 = \hbar\sqrt{2}\chi_+ \quad , \quad \vec{S}_+\chi_- = \hbar\sqrt{2-(-1+1)}\chi_0 = \hbar\sqrt{2}\chi_0 \quad , \\ \vec{S}_-\chi_+ = \hbar\sqrt{2}\chi_0 \quad , \quad \vec{S}_-\chi_0 = \hbar\sqrt{2+(1-1)}\chi_- \quad , \quad \vec{S}_-\chi_- = 0\end{array}\right\} \boxed{\vec{S}_+ = \hbar\sqrt{2}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} \quad , \quad \vec{S}_- = \hbar\sqrt{2}\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}} .$$

- From the ladder-operator matrices we get $\vec{S}_x$ and $\vec{S}_y$:

$$\vec{S}_x = \frac{1}{2}\left(\vec{S}_+ + \vec{S}_-\right) = \boxed{\frac{\hbar}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}} \quad , \quad \vec{S}_y = \frac{1}{2i}\left(\vec{S}_+ - \vec{S}_-\right) = \boxed{\frac{i\hbar}{\sqrt{2}}\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}} .$$

Spin other than 1/2 (continued)

- And $\vec{S}^2 = \vec{S}_x^2 + \vec{S}_y^2 + \vec{S}_z^2$:

$$\vec{S}^2 = \vec{S}_x^2 + \vec{S}_y^2 + \vec{S}_z^2 = \frac{\hbar^2}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \frac{\hbar^2}{2}\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} + \hbar^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{2\hbar^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

This is a good check; if $\vec{S}^2$ isn't a real scalar times the identity matrix, something has gone wrong.

Electrons in magnetic fields

- Classically, rotating charged objects are magnetic dipoles, as they are comprised of current loops.
- Also classically, the dipole moment $\boldsymbol{\mu}$ would depend upon the 3-D distribution of those currents.
- Quantum-mechanically we represent that 3-D distribution **constitutively** by defining the **gyromagnetic ratio**, γ:

$$\hat{\boldsymbol{\mu}} = \gamma \hat{\mathbf{S}} \quad .$$

- As usual, quantum mechanics has surprises for those who lean on classical analogy: the electron gyromagnetic ratio is **almost** twice what they would expect:

$$\gamma_e = \frac{\mu_e}{\left(\frac{\hbar}{2}\right)} = g_e \frac{-e}{2m_e c} \left(g_e \frac{-e}{2m_e} \text{ in SI units} \right) = -g_e \frac{\mu_B}{\hbar} \quad , \qquad \text{expect } g_e = -1$$

where $\mu_B = e\hbar/2m_c c \left(= e\hbar/2m_c \text{ in SI units}\right)$ is the **Bohr magneton**, and the electron's **g-factor** is, experimentally and understood theoretically,

$$g_e = -2.00231930436092 \pm .00000000000036 \quad .$$

Electrons in magnetic fields

- The Hamiltonian for a spinning quantum at rest in a magnetic field **B** is $\hat{H} = -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{B} = -\gamma \boldsymbol{B} \cdot \hat{\boldsymbol{S}}$, where $\hat{\boldsymbol{S}}$ is the vector spin operator appropriate for the quantum: for spin 1/2, in terms of the Pauli spin matrices (Lecture 18, p. 9) ,

$$\vec{\boldsymbol{S}} = \frac{\hbar}{2}\vec{\boldsymbol{\sigma}} \quad , \quad \text{where} \quad \vec{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad \vec{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad , \quad \text{and} \quad \vec{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad ,$$

- We know all this works for electrons due to experimental observations of **Larmor precession**, and to the **Stern-Gerlach** experiment, which are well worth understanding in detail.

Larmor precession

- Put an electron in a uniform $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$: $\vec{H} = -\gamma B_0 \vec{S}_z = -\frac{\gamma B_0 \hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- The eigenvectors of H would therefore be the same as those of $\vec{S}_z$, i.e. $\chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- And as the corresponding spin eigenvalues are $\pm\hbar/2$, the energy eigenvalues are $E_\pm = \mp\gamma B_0 \hbar/2$.
- The time-dependent Schrödinger equation reads as $i\hbar(\partial\chi/\partial t) = \hat{H}\chi$, which has the usual solutions

$$\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar} = \begin{bmatrix} a e^{i\gamma B_0 t/\hbar} \\ b e^{-i\gamma B_0 t/\hbar} \end{bmatrix} .$$

- Take, as initial conditions, $\chi(0) = \begin{bmatrix} a \\ b \end{bmatrix}$, $|a|^2 + |b|^2 = 1$.

Larmor precession (continued)

- Parameterize the solutions in traditional fashion: $a = \cos(\alpha/2), \quad b = \sin(\alpha/2); \;\chi(t) = \begin{bmatrix} \cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/\hbar} \\ \sin\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t/\hbar} \end{bmatrix}$.

- The illuminating part of this is the expectation values of components of spin:

$$\langle S_x \rangle = \chi(t)^\dagger \vec{S}_x \chi(t) = \begin{bmatrix} \cos\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t/\hbar} & \sin\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/\hbar} \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/\hbar} \\ \sin\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t/\hbar} \end{bmatrix}$$

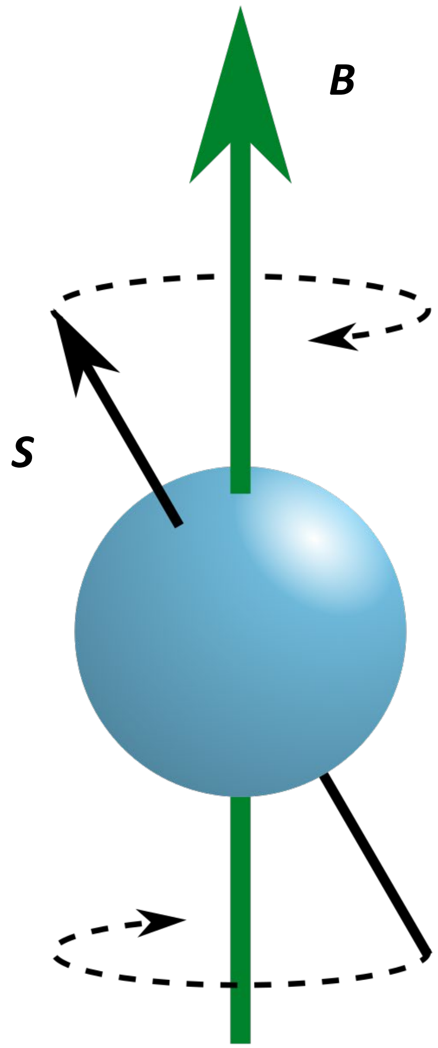
$$= \frac{\hbar}{2} \cos\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t/\hbar} \sin\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t/\hbar} + \sin\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/\hbar} \cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/\hbar}$$

Use $\sin 2x = 2 \sin x \cos x$, and the Euler formula:

$$= \frac{\hbar}{2} \sin\alpha \cos\left(\frac{\gamma B_0 t}{\hbar}\right) \quad ;$$

Larmor precession (continued)

- Similarly, $\langle S_y \rangle = -\frac{\hbar}{2}\sin\alpha\sin\left(\frac{\gamma B_0 t}{\hbar}\right)$ and $\langle S_z \rangle = \frac{\hbar}{2}\cos\alpha$.
- So the electron's spin ***S* precesses** about the direction of the magnetic field.
- The rate at which S precesses is called the **Larmor frequency**: $\omega_L = \gamma B_0$, and, apart from the unexpected value of the electron's g-factor, is the same as the classical result signified by the use of the blue sphere in the diagram at right.



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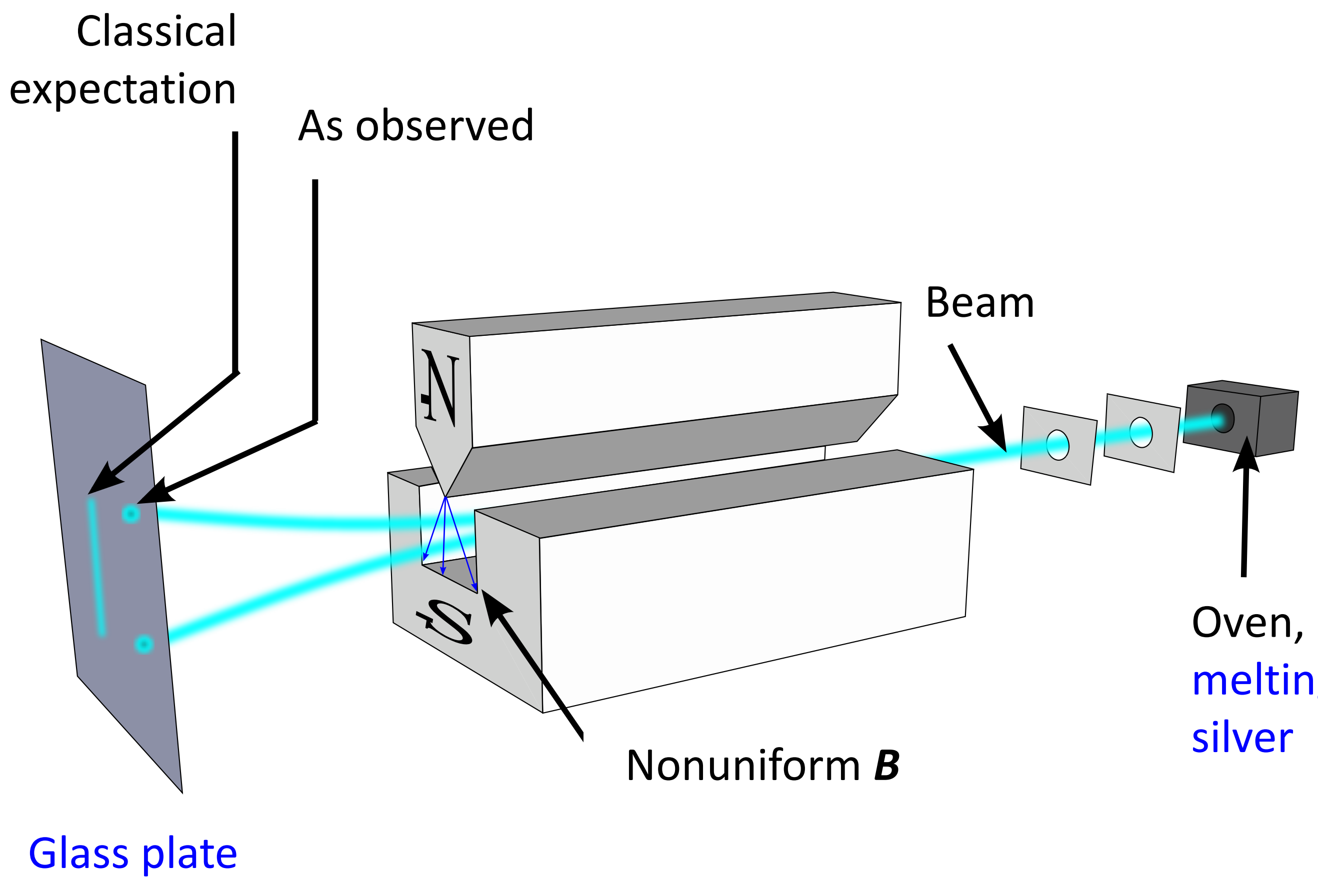
The Stern-Gerlach experiment

This is the clearest signal of angular momentum quantization.

- In nonuniform $\boldsymbol{B}$, a magnetic dipole experiences a net force $\boldsymbol{F} = \nabla(\boldsymbol{\mu}\cdot\boldsymbol{B})$, as well as the torque $\boldsymbol{\tau} = \boldsymbol{\mu}\times\boldsymbol{B}$ which gave rise to the potential energy and Hamiltonian used above.
- Suppose there is a beam of quanta with spin S travelling in the y direction, which encounters a nonuniform magnetic field

$$\boldsymbol{B}(x,y,z) = -\alpha x\hat{\boldsymbol{x}} + (B_0 + \alpha z)\hat{\boldsymbol{z}} \quad .$$



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The Stern-Gerlach experiment (continued)

- Each quantum experiences a force given by

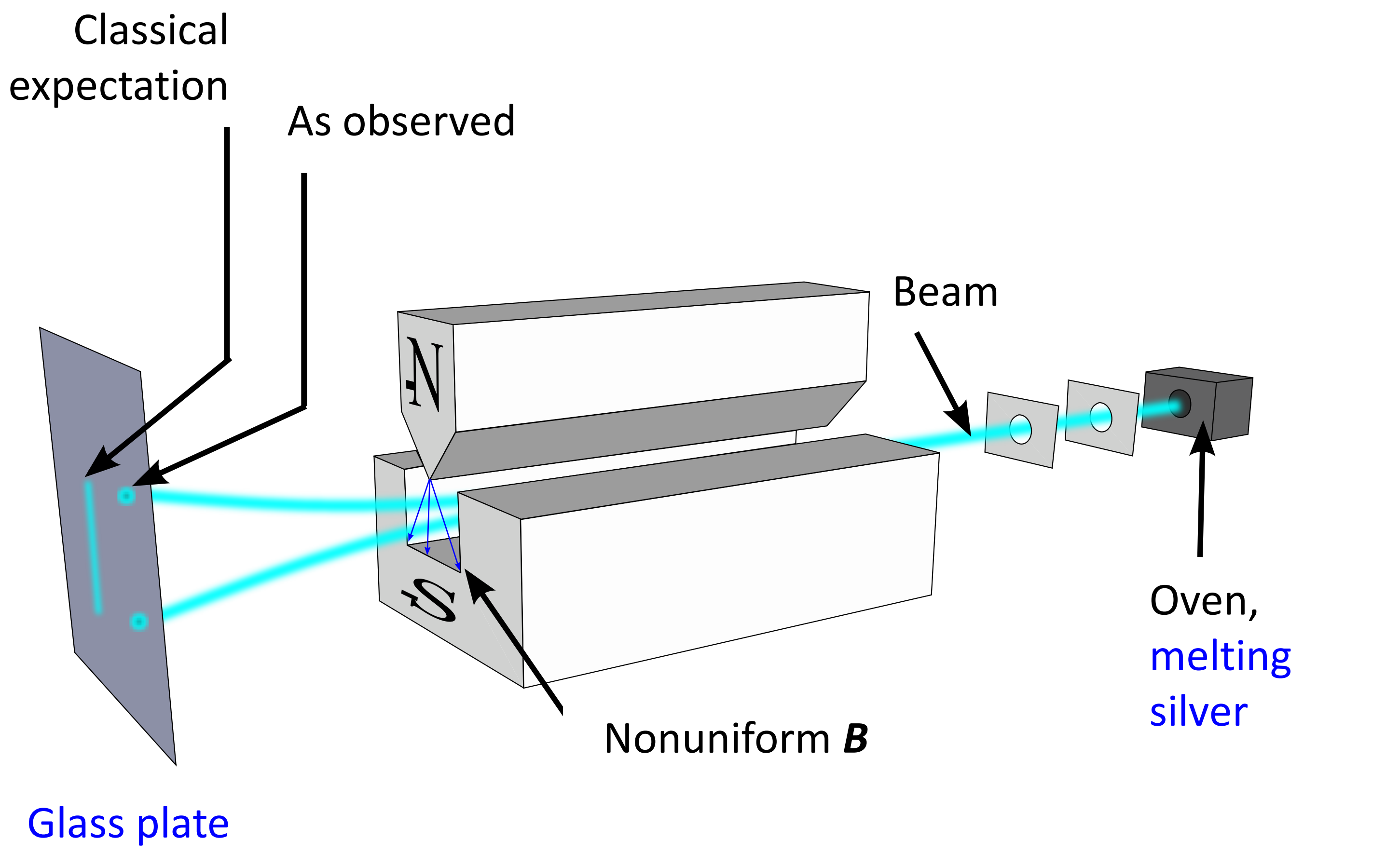
$$\boldsymbol{F} = \nabla(\boldsymbol{\mu}\cdot\boldsymbol{B})$$

$$= \nabla\left[-\gamma\alpha x S_x + \gamma\left(B_0 + \alpha z\right)S_z\right]$$

$$= -\gamma\alpha S_x\hat{\boldsymbol{x}} + \gamma\alpha S_z\hat{\boldsymbol{z}} \quad .$$

- The x component oscillates rapidly about zero with time (Larmor precession), and thus averages out over time $\gg 2\pi/\gamma B_0$, leaving

$$\overline{\boldsymbol{F}} = \gamma\alpha S_z\hat{\boldsymbol{z}} \quad .$$



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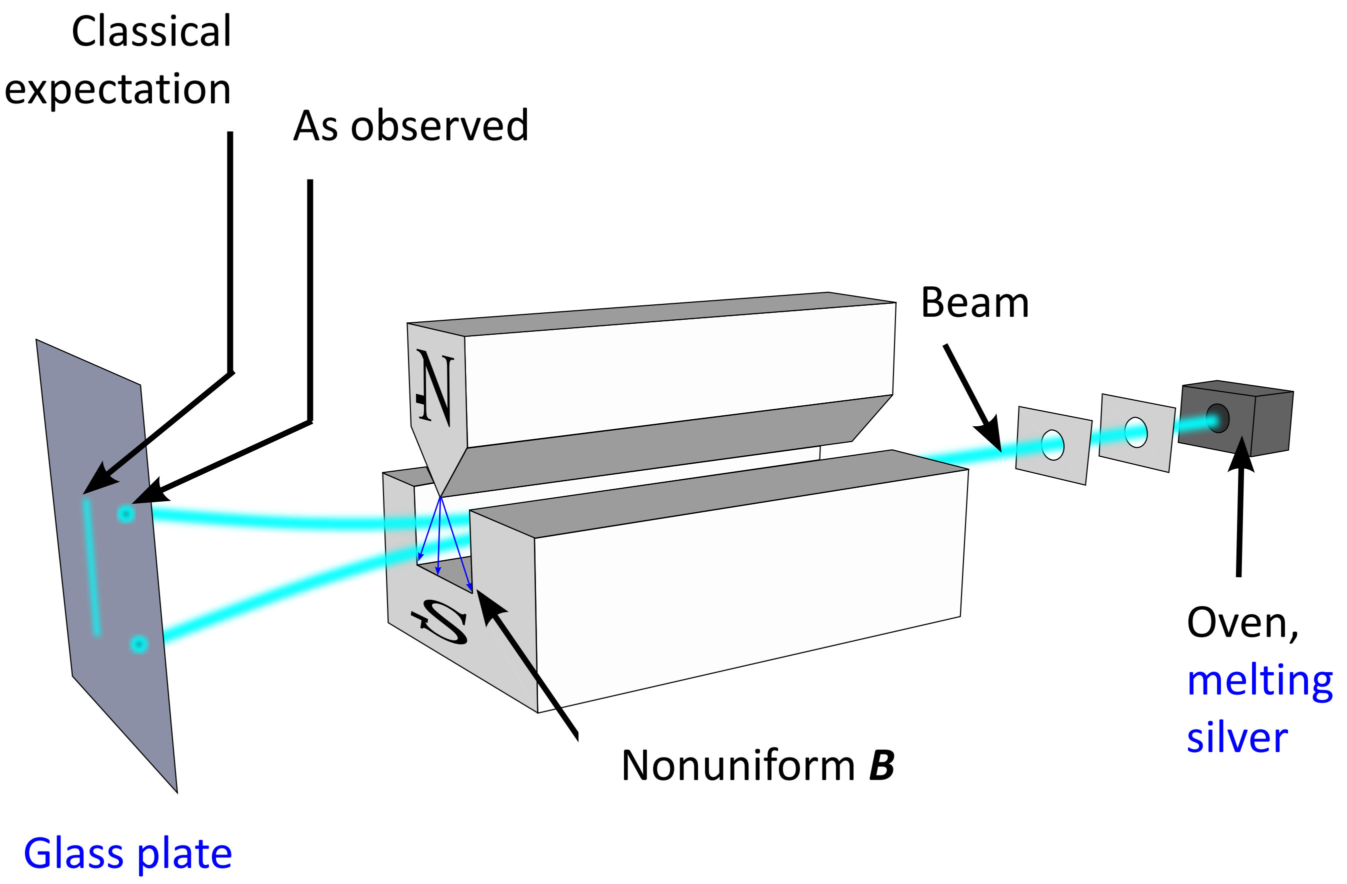
The Stern-Gerlach experiment (continued)

- Classically, we would expect this to result in a continuous smear of particles deposited on the glass plate at the end of the apparatus, as, **classically**, the spin isn’t quantized.

- Instead one sees two dots of particles deposited.

  - Spin is quantized!

  - The number of dots, and the magnitude of the deflection is as expected for the quanta, e.g. spin 1/2 (two dots) and deflection corresponding to $\hbar/2$ for the neutral silver atoms used by Stern and Gerlach.



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