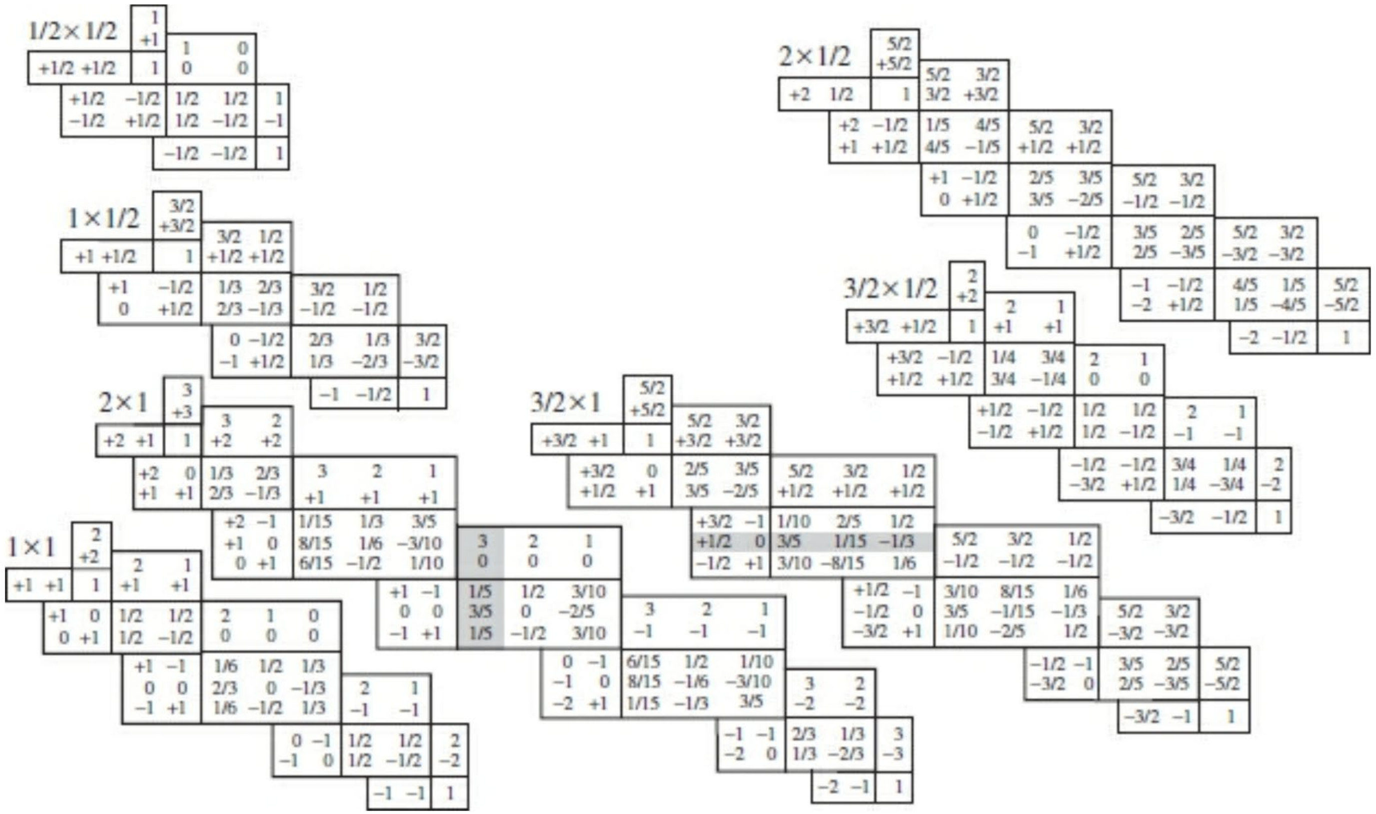
Today in Physics 237: angular momentum addition I

- Simple: addition of spins in a system of two spin-1/2 quanta, intuitively
    - Spin triplets and spin singlets
- Addition of two angular momenta as a basis change
- Clebsch-Gordan coefficients



G&S table 4.8

Angular momentum addition for two spin-1/2 quanta: G&S example 4.5

- Simplest example; like the electrons in H_2, or in neutral atomic He. Or the electron and proton in H, as in G&S.

- Each quantum's spin is specified by individual values of *s* and *m.* Considering the quanta to be distinguishable for now, it's clear how the *z* components add to produce *m* for the vector sum, since there are only four arrangements:

$$\left|\uparrow\uparrow\right\rangle = \left|s_1 s_2 m_1 m_2\right\rangle = \left|\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right\rangle \qquad m = m_1 + m_2 = 1$$

$$\left|\uparrow\downarrow\right\rangle = \left|s_1 s_2 m_1 m_2\right\rangle = \left|\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{-1}{2}\right\rangle \qquad m = 0$$

$$\left|\downarrow\uparrow\right\rangle = \left|s_1 s_2 m_1 m_2\right\rangle = \left|\frac{1}{2}\frac{1}{2}\frac{-1}{2}\frac{1}{2}\right\rangle \qquad m = 0$$

$$\left|\downarrow\downarrow\right\rangle = \left|s_1 s_2 m_1 m_2\right\rangle = \left|\frac{1}{2}\frac{1}{2}\frac{-1}{2}\frac{-1}{2}\right\rangle \qquad m = -1$$

These (combined) states can, perforce, be considered orthogonal. Suppose they are also normalized.

Angular momentum addition for two spin-1/2 quanta (continued)

- The first and last clearly correspond to *s* = 1: $\left|\uparrow\uparrow\right\rangle = \left|sm\right\rangle = \left|11\right\rangle, \quad \left|\downarrow\downarrow\right\rangle = \left|sm\right\rangle = \left|1\,-1\right\rangle$. As for the other two…
- If we apply the sum of the lowering operators for the two spins (**presumed independent**), $\hat{S}_- = \hat{S}_{1-} + \hat{S}_{2-}$, to the *s* = 1, *m* = 1 state, we are guaranteed to get *s* = 1, *m* = 0. Use $\hat{S}_-\left|sm\right\rangle = \hbar\sqrt{s(s+1) - m(m-1)}\left|sm\right\rangle$, as you showed in G&S problem 4.21:

$$\hat{S}_-\left|\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right\rangle = \left(\hat{S}_{1-} + \hat{S}_{2-}\right)\left|\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right\rangle = \hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)}\left(\left|\frac{1}{2}\frac{1}{2}\frac{-1}{2}\frac{1}{2}\right\rangle + \left|\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{-1}{2}\right\rangle\right) = \hbar\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) \quad .$$

- Evidently the *s* = 1, *m* = 0 state is a **linear combination of the original *m* = 0 states**. Call $\left|sm\right\rangle = \left|10\right\rangle = A\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right)$; this state is normalized if

$$1 = A^2\left(\left\langle\uparrow\downarrow\right| + \left\langle\downarrow\uparrow\right|\right)\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) = A^2\left(\underbrace{\left\langle\uparrow\downarrow|\uparrow\downarrow\right\rangle}_{1} + \underbrace{\left\langle\uparrow\downarrow|\downarrow\uparrow\right\rangle}_{0} + \underbrace{\left\langle\downarrow\uparrow|\uparrow\downarrow\right\rangle}_{0} + \underbrace{\left\langle\downarrow\uparrow|\downarrow\uparrow\right\rangle}_{1}\right) = 2A^2 \quad \Rightarrow \quad A = \frac{1}{\sqrt{2}} \quad .$$

- Thus the *s* = 1, *m* = 1, 0, -1 states are $\boxed{\left|sm\right\rangle = \left|11\right\rangle = \left|\uparrow\uparrow\right\rangle \quad , \quad \left|10\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) \quad , \quad \text{and} \quad \left|1\,-1\right\rangle = \left|\downarrow\downarrow\right\rangle \quad .}$

Angular momentum addition for two spin-1/2 quanta (continued)

- The other obvious linear combination of m = 0 states, $\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$, is orthogonal to $|10\rangle$:

$$\frac{1}{2}\left(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|\right)\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = \frac{1}{2}\left(\cancelto{1}{\langle\uparrow\downarrow|\uparrow\downarrow\rangle} - \cancelto{0}{\langle\uparrow\downarrow|\downarrow\uparrow\rangle} + \cancelto{0}{\langle\downarrow\uparrow|\uparrow\downarrow\rangle} - \cancelto{1}{\langle\downarrow\uparrow|\downarrow\uparrow\rangle}\right) = 0 \quad .$$

- It still has m = 0, it cannot have s = 1, so it must be $|sm\rangle = |00\rangle$. Summary:

$$\left.\begin{aligned} |sm\rangle &= |11\rangle = |\uparrow\uparrow\rangle \\ &= |10\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ &= |1\,{-1}\rangle = |\downarrow\downarrow\rangle \end{aligned}\right\} \textbf{ Spin triplet} \qquad |sm\rangle = |00\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \Big\} \textbf{ Spin singlet}$$

- So far, we have *asserted* that s = 1 and 0 respectively for the triplet and singlet states. We can prove it by demonstrating the triplet and singlet states to be eigenstates of $\hat{S}^2$ with eigenvalues respectively $2\hbar^2$ and zero.

Angular momentum addition for two spin-1/2 quanta (continued)

- To do this, recall the action of the spin operators on the eigenstates of $\hat{S}^2$ and $\hat{S}_z$ for spin ½, shown in Lecture 18:

$$\overset{\leftrightarrow}{S}_x\chi_+ = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}0\\ 1\end{bmatrix} \quad\Leftrightarrow\quad \hat{S}_x\left|\uparrow\right\rangle = \frac{\hbar}{2}\left|\downarrow\right\rangle$$

$$\overset{\leftrightarrow}{S}_x\chi_- = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}1\\ 0\end{bmatrix} \quad\Leftrightarrow\quad \hat{S}_x\left|\downarrow\right\rangle = \frac{\hbar}{2}\left|\uparrow\right\rangle$$

$$\overset{\leftrightarrow}{S}_y\chi_+ = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{i\hbar}{2}\begin{bmatrix}0\\ 1\end{bmatrix} \quad\Leftrightarrow\quad \hat{S}_y\left|\uparrow\right\rangle = \frac{i\hbar}{2}\left|\downarrow\right\rangle$$

$$\overset{\leftrightarrow}{S}_y\chi_- = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = -\frac{i\hbar}{2}\begin{bmatrix}1\\ 0\end{bmatrix} \quad\Leftrightarrow\quad \hat{S}_y\left|\downarrow\right\rangle = -\frac{i\hbar}{2}\left|\uparrow\right\rangle$$

$$\overset{\leftrightarrow}{S}_z\chi_+ = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}1\\ 0\end{bmatrix} \quad\Leftrightarrow\quad \hat{S}_z\left|\uparrow\right\rangle = \frac{\hbar}{2}\left|\uparrow\right\rangle$$

$$\overset{\leftrightarrow}{S}_z\chi_- = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = -\frac{\hbar}{2}\begin{bmatrix}0\\ 1\end{bmatrix} \quad\Leftrightarrow\quad \hat{S}_z\left|\downarrow\right\rangle = -\frac{\hbar}{2}\left|\downarrow\right\rangle$$

Angular momentum addition for two spin-1/2 quanta (continued)

- Now write $\hat{S}^2 = \left(\hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2\right)^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2$, and consider first the operation of the dot product on $\left|\uparrow\downarrow\right\rangle$:

$$\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 \left|\uparrow\downarrow\right\rangle = \left(\hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z}\right)\left|\uparrow\downarrow\right\rangle = \frac{\hbar^2}{4}\left|\downarrow\uparrow\right\rangle - \left(\frac{i\hbar}{2}\right)^2\left|\downarrow\uparrow\right\rangle - \frac{\hbar^2}{4}\left|\uparrow\downarrow\right\rangle = \frac{\hbar^2}{4}\left(2\left|\downarrow\uparrow\right\rangle - \left|\uparrow\downarrow\right\rangle\right) \quad .$$

- Similarly, for $\left|\downarrow\uparrow\right\rangle$,

$$\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 \left|\downarrow\uparrow\right\rangle = \frac{\hbar^2}{4}\left|\uparrow\downarrow\right\rangle - \left(\frac{i\hbar}{2}\right)^2\left|\uparrow\downarrow\right\rangle - \frac{\hbar^2}{4}\left|\downarrow\uparrow\right\rangle = \frac{\hbar^2}{4}\left(2\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right) \quad .$$

- Thus

$$\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 \left|10\right\rangle = \frac{\hbar^2}{4\sqrt{2}}\left(2\left|\downarrow\uparrow\right\rangle - \left|\uparrow\downarrow\right\rangle + 2\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right) = \frac{\hbar^2}{4\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) = \frac{\hbar^2}{4}\left|10\right\rangle \quad .$$

- Similarly,

$$\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 \left|00\right\rangle = \frac{\hbar^2}{4\sqrt{2}}\left(2\left|\downarrow\uparrow\right\rangle - \left|\uparrow\downarrow\right\rangle - 2\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) = -\frac{3\hbar^2}{4\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right) = -\frac{3\hbar^2}{4}\left|00\right\rangle \quad .$$

Angular momentum addition for two spin-1/2 quanta (continued)

- Finally, using the $\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2$ results along with $\hat{S}^2 \left|sm\right\rangle = s(s+1)\left|sm\right\rangle$,

$$\hat{S}^2\left|11\right\rangle = \left(\hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2\right)\left|\uparrow\uparrow\right\rangle = \left[\left(\frac{\hbar}{2}\right)\left(\frac{3\hbar}{2}\right) + \left(\frac{\hbar}{2}\right)\left(\frac{3\hbar}{2}\right) + 2\left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right)\right]\left|\uparrow\uparrow\right\rangle = 2\hbar^2\left|11\right\rangle$$

$$\hat{S}^2\left|10\right\rangle = \frac{1}{\sqrt{2}}\left(\hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2\right)\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) = \frac{1}{\sqrt{2}}\left[\left(\frac{\hbar}{2}\right)\left(\frac{3\hbar}{2}\right) + \left(\frac{\hbar}{2}\right)\left(\frac{3\hbar}{2}\right) + 2\frac{\hbar^2}{4}\right]\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) = 2\hbar^2\left|10\right\rangle$$

$$\hat{S}^2\left|1-1\right\rangle = \left(\hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2\right)\left|\downarrow\downarrow\right\rangle = \left[\left(\frac{\hbar}{2}\right)\left(\frac{3\hbar}{2}\right) + \left(\frac{\hbar}{2}\right)\left(\frac{3\hbar}{2}\right) + 2\left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right)\right]\left|\downarrow\downarrow\right\rangle = 2\hbar^2\left|1-1\right\rangle$$

and $$\hat{S}^2\left|00\right\rangle = \frac{1}{\sqrt{2}}\left(\hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2\right)\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right) = \frac{1}{\sqrt{2}}\left[\left(\frac{\hbar}{2}\right)\left(\frac{3\hbar}{2}\right) + \left(\frac{\hbar}{2}\right)\left(\frac{3\hbar}{2}\right) - 2\frac{3\hbar^2}{4}\right]\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) = 0$$, q.e.d.

- Note that the results are independent of which electron is #1 and which is #2.

Angular momentum addition for any two spins

- So the operator $\hat{P}_1 \equiv \frac{\hbar}{2}\left(\frac{3}{4} + \hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2\right)$ annihilates a singlet state and multiplies a triplet state by 1. Acting on any linear combination of triplets and singlets, it **projects** the triplet part of the state.
- Similarly, $\hat{P}_0 \equiv 1 - \hat{P}_1 = \frac{\hbar}{2}\left(\frac{3}{4} - \hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2\right)$ is a **projection operator** for the singlet state.
- This suggests using our projection-operator language (Lecture 11) to express the combination of two spins into a single state: $\hat{P}_1 = \left|1m\right\rangle\left\langle 1m\right|$ and $\hat{P}_0 = \left|00\right\rangle\left\langle 00\right|$, where $m = m_1 + m_2$.
- In that language, the projection operators transform between the $\left|sm\right\rangle$ and $\left|s_1 s_2 m_1 m_2\right\rangle$ bases:

$$\left|\frac{1}{2}\frac{1}{2}m_1 m_2\right\rangle = \sum_{m=m_1+m_2} \left|s,m\right\rangle\left\langle s,m\left|\frac{1}{2}\frac{1}{2}m_1 m_2\right.\right\rangle \quad .$$

- For two quanta with other spins, we can use the same scheme. For s_1 and s_2, the allowed values of s are

$$s = s_1 + s_2,\, s_1 + s_2 - 1,\, s_1 + s_2 - 2, \ldots, \left|s_1 - s_2\right| \quad .$$

Clebsch-Gordan coefficients

- So $$\left|s_1 s_2 m_1 m_2\right\rangle = \sum_{\substack{s=s_1'+s_2',\\ m=m_1+m_2}} \left|s,m\right\rangle\left\langle s,m\middle|s_1' s_2' m_1 m_2\right\rangle \quad .$$ Sum over all m and s, subject to $s = s_1' + s_2'$ and $m = m_1 + m_2$

  The $\left\langle s,m\middle|s_1' s_2' m_1 m_2\right\rangle$ are called **Clebsch-Gordan (C-G) coefficients**. G&S use $C^{s_1' s_2' s}_{m_1 m_2 m}$ instead of $\left\langle s,m\middle|s_1' s_2' m_1 m_2\right\rangle$.

- To simplify the sum, consider the matrix element $\left\langle s,m\middle|\hat{S}_1^2\middle|s_1' s_2' m_1 m_2\right\rangle$. We see by operating $\hat{S}_1^2$ to the right, then left:

$$\left\langle s,m\middle|\hat{S}_1^2\middle|s_1' s_2' m_1 m_2\right\rangle = \hbar^2 s_1'\left(s_1'+1\right)\left\langle s,m\middle|s_1' s_2' m_1 m_2\right\rangle = \hbar^2 s_1\left(s_1+1\right)\left\langle s,m\middle|s_1' s_2' m_1 m_2\right\rangle \quad ,$$

  that $\left\langle s,m\middle|s_1' s_2' m_1 m_2\right\rangle$ is zero unless $s_1' = s_1$ and $s_2' = s_2$.

- Also, doing the same thing with $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$,

$$\left\langle s,m\middle|\hat{S}_z\middle|s_1' s_2' m_1 m_2\right\rangle = \left(m_1+m_2\right)\left\langle s,m\middle|s_1' s_2' m_1 m_2\right\rangle = m\left\langle s,m\middle|s_1' s_2' m_1 m_2\right\rangle \quad ,$$

  so $\left\langle s,m\middle|s_1' s_2' m_1 m_2\right\rangle$ is also zero unless $m = m_1 + m_2$, as we also saw for spin ½ above.

Clebsch-Gordan coefficients (continued)

- Thus the only nonvanishing coefficients are of the form $\langle s_1 s_2 s, m = m_1 + m_2 | s_1 s_2 m_1 m_2 \rangle$, and the sum becomes

$$|s_1 s_2 m_1 m_2\rangle = \sum_{m=m_1+m_2} \langle s_1 s_2 s m | s_1 s_2 m_1 m_2 \rangle | s_1 s_2 s m \rangle \quad ,$$

Sum over all m, subject to $m = m_1 + m_2$

or, in reverse,

$$|s_1 s_2 s m\rangle = \sum_{\substack{m_1, m_2, \\ m_1+m_2=m}} \langle s_1 s_2 m_1 m_2 | s_1 s_2 s m \rangle | s_1 s_2 m_1 m_2 \rangle \quad .$$

Sum over all m_1 and m_2, subject to $m_1 + m_2 = m$

- For example, the C-G coefficients for addition of spins for two spin-1/2 quanta can be read off from pages 3-4, and the terms in the second sum above can be identified:

$$\left\langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2} \middle| \tfrac{1}{2}\tfrac{1}{2} 1 1 \right\rangle = 1 \quad , \quad \left\langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2} \middle| \tfrac{1}{2}\tfrac{1}{2} 1 0 \right\rangle = \frac{1}{\sqrt{2}} \quad , \quad \left\langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2} \middle| \tfrac{1}{2}\tfrac{1}{2} 1 0 \right\rangle = \frac{1}{\sqrt{2}} \quad , \quad \left\langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-1}{2} \middle| \tfrac{1}{2}\tfrac{1}{2} 1 -1 \right\rangle = 1 \quad ,$$

$$\left\langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2} \middle| \tfrac{1}{2}\tfrac{1}{2} 0 0 \right\rangle = \frac{1}{\sqrt{2}} \quad , \quad \left\langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2} \middle| \tfrac{1}{2}\tfrac{1}{2} 0 0 \right\rangle = -\frac{1}{\sqrt{2}} \quad .$$