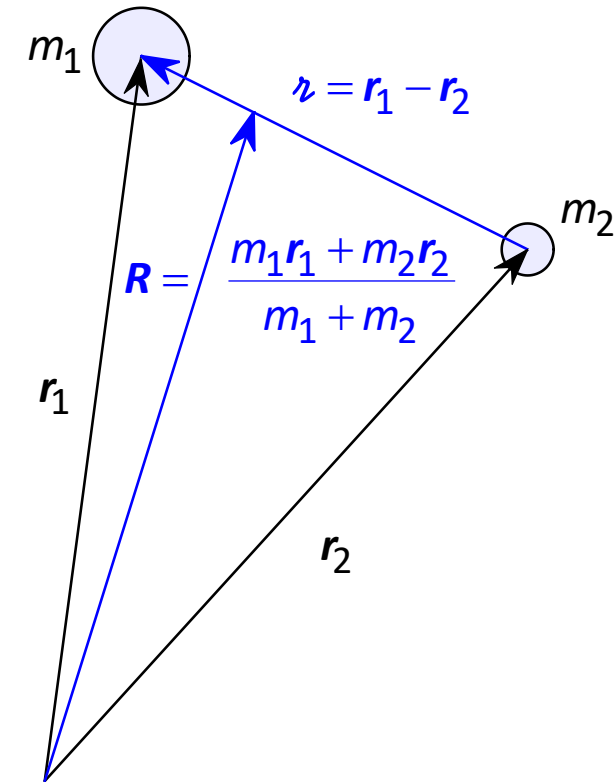


Today in Physics 237: angular momentum addition II

- Projection-operator and Clebsch-Gordan technology
- Two-quantum states
 - Noninteracting quanta and (merely) entangled states
 - Interacting quanta, central potentials, and the center of mass frame.



Angular momentum addition in quantum mechanics

Summary of our latest results:

- We concentrated on the spin part of the wavefunction, as an angular momentum separate from other effects in the system – tacitly noting that the spin operators \hat{S}^2 and \hat{S}_z commute with each other and with \hat{H} .
- We considered a **basis** of spin states in a two quantum system, $|s_1 s_2 m_1 m_2\rangle$, starting with the bases consisting of eigenstates of \hat{S}_j^2 and \hat{S}_{jz} , $|s_j m_j\rangle$, for the individual quanta.
- We constructed from this basis the eigenstates of **total** spin and its z component, $\hat{S}^2 = (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2 = \hat{S}_{1z}^2 + \hat{S}_{2z}^2 + 2\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$ and $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$.
- This new basis of eigenstates, $|s_1 s_2 s m\rangle$, turns out to be related to the other two-quantum basis by the transformation

$$|s_1 s_2 s m\rangle = \sum_{\substack{m_1, m_2, \\ m_1 + m_2 = m}} \langle s_1 s_2 m_1 m_2 | s_1 s_2 s m \rangle |s_1 s_2 m_1 m_2\rangle .$$

Sum over all m_1 and m_2 ,
subject to $m_1 + m_2 = m$

- **This is how vector addition works in quantum mechanics.**

Angular momentum addition in quantum mechanics (continued)

- For a quantum system with **two** angular momenta of any sort, \mathbf{S} or \mathbf{L} , the system's state is given by our two-spin result with only a change of symbols. Take \mathbf{J} to be the generic angular-momentum symbol:

$$|j_1 j_2 j m\rangle = \sum_{\substack{m_1, m_2, \\ m_1 + m_2 = m}} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle, \quad \text{Still sum over all } m_1 \text{ and } m_2, \\ \text{subject to } m_1 + m_2 = m$$

where $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, and the quantum number j lies between $j_1 + j_2$ and $|j_1 - j_2|$, inclusive, just as s had to be either 1 or zero for the two spin-half quanta considered in [Lecture 20](#).

- Frequently encountered: $\mathbf{J} = \mathbf{L} + \mathbf{S}$, the sum of **orbital** angular momentum and spin for a single electron.
- The Clebsch-Gordan (C-G) coefficients $\langle j_1 j_2 j m | j_1 j_2 j m\rangle$ are orthogonal. From above,

$$\langle j'_1 j'_2 j' m' | j_1 j_2 j m\rangle = \delta_{jj'} \delta_{mm'} = \sum_{m_1, m_2, m=m_1+m_2} \langle j'_1 j'_2 j' m' | j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle$$

$$\Rightarrow \sum_{m_1, m_2, m=m_1+m_2} \langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle^2 = 1 \quad ; \quad \text{C-G sum rule}$$

Angular momentum addition in quantum mechanics (continued)

- And because $|j_1 j_2 j m\rangle$ is a basis just like $|j_1 j_2 m_1 m_2\rangle$, we can also write the sum over the states as

$$|j_1 j_2 m_1 m_2\rangle = \sum_{\substack{m_1, m_2, \\ m=m_1+m_2}} |j_1 j_2 j m\rangle \langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle ,$$

whence
$$\langle j_1 j_2 m'_1 m'_2 | j_1 j_2 m_1 m_2\rangle = \delta_{m_1 m'_1} \delta_{m_2 m'_2} = \sum_{\substack{m_1, m_2, \\ m=m_1+m_2}} \langle j_1 j_2 m'_1 m'_2 | j_1 j_2 j m\rangle \langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle$$

$$\Rightarrow \sum_{\substack{m_1, m_2, \\ m=m_1+m_2}} |\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle|^2 = 1 .$$

- This is the same as the sum rule above, if the Clebsch-Gordan coefficients are real. (They are, as we're about to see.)

Calculation of Clebsch-Gordan coefficients

- \mathbf{J} , being an angular momentum, corresponds to operators which behave the same as those for \mathbf{L} and \mathbf{S} :

$$\hat{j}^2 |j_1 j_2 j m\rangle = \hbar^2 j(j+1) |j_1 j_2 j m\rangle \quad , \quad \hat{j}_z = \hbar m |j_1 j_2 j m\rangle \quad ,$$

$$\hat{j}_\pm |j_1 j_2 j m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j_1 j_2 j m \pm 1\rangle \quad , \quad \langle j_1 j_2 j m | (\hat{j}_\pm)^\dagger = \sqrt{j(j+1) - m(m \pm 1)} \langle j_1 j_2 j m \pm 1 | \quad .$$

- Its ladder operators can be used to obtain ratios of Clebsch-Gordan coefficients.

- Consider the matrix element $\langle j_1 j_2 m_1 m_2 | \hat{j}_\pm | j_1 j_2 j m \rangle$. First apply the operator to the ket:

$$\langle j_1 j_2 m_1 m_2 | \hat{j}_\pm | j_1 j_2 j m \rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j, m \pm 1 \rangle \quad .$$

- Then start over, with $\hat{j}_\pm = \hat{j}_{1\pm} + \hat{j}_{2\pm}$, and apply it to the bra. Because $\hat{j}_\pm = \hat{j}_x \pm i\hat{j}_y$, $(\hat{j}_\pm)^\dagger = \hat{j}_x \mp i\hat{j}_y$, so as \hat{j}_\pm acts leftward here on the bra – rather than $(\hat{j}_\pm)^\dagger$ – the \pm flips:

$$\begin{aligned} \langle j_1 j_2 m_1 m_2 | \hat{j}_\pm | j_1 j_2 j m \rangle &= \hbar \sqrt{j_1(j_1+1) - m_1(m_1 \mp 1)} \langle j_1 j_2, m_1 \mp 1, m_2 | j_1 j_2 j, m \rangle \\ &\quad + \hbar \sqrt{j_2(j_2+1) - m_2(m_2 \mp 1)} \langle j_1 j_2 m_1, m_2 \mp 1 | j_1 j_2 j, m \rangle \quad . \end{aligned}$$

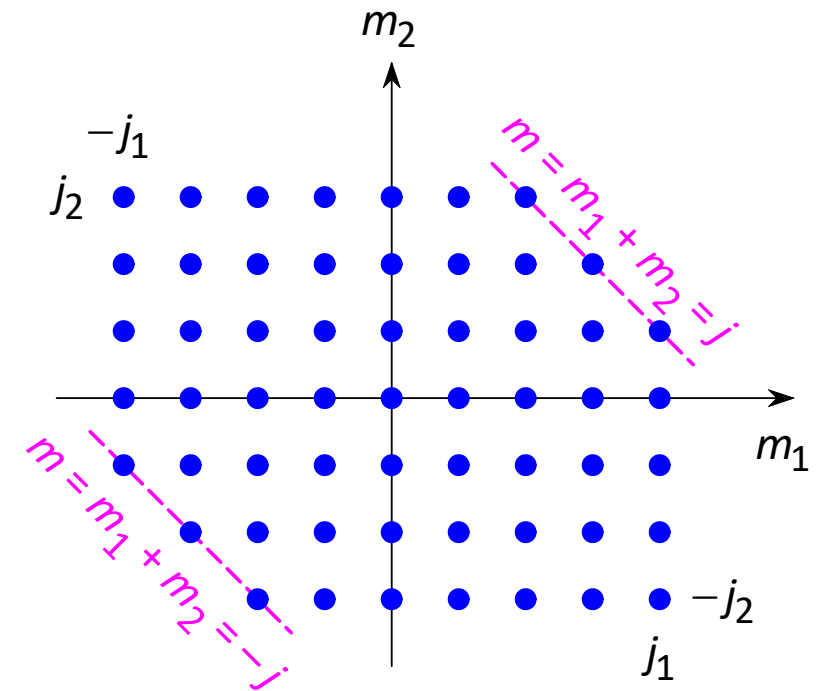
Calculation of Clebsch-Gordan coefficients (continued)

- Combine the last two results to produce the **upper and lower** Clebsch-Gordan **recursion relations**:

$$\sqrt{j(j+1) - m(m \pm 1)} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j, m \pm 1 \rangle =$$

$$\sqrt{j_1(j_1 + 1) - m_1(m_1 \mp 1)} \langle j_1 j_2, m_1 \mp 1, m_2 | j_1 j_2 j, m \rangle + \sqrt{j_2(j_2 + 1) - m_2(m_2 \mp 1)} \langle j_1 j_2 m_1, m_2 \mp 1 | j_1 j_2 j, m \rangle .$$

- The recursion relations and the sum rule suffice to calculate all the C-G coefficients.
 - They also show that if **one C-G coefficient is real, they all are real**.
- Procedure for calculating C-G coefficients with these relations:
 - For each j from $|j_1 - j_2|$ to $j_1 + j_2$, identify the m_1 and m_2 for which $-j_1 \leq m_1 \leq j_1$, $-j_2 \leq m_2 \leq j_2$, and $j \leq m = m_1 + m_2 \leq j$. The coefficients are zero otherwise: the z component of a vector cannot exceed its magnitude.

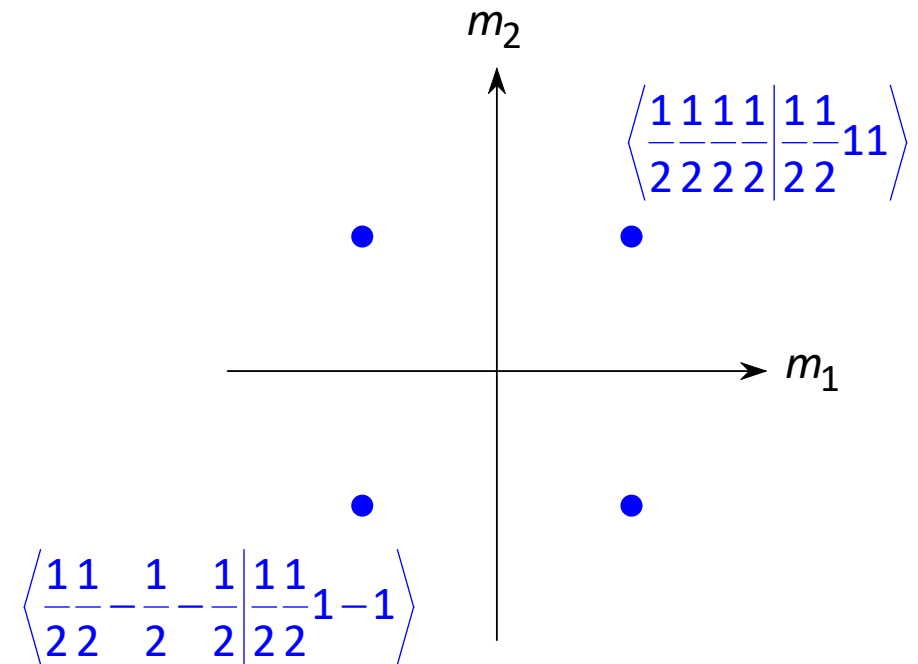


Calculation of Clebsch-Gordan coefficients (continued)

- From now on, we will illustrate the procedure using two spin-1/2 quanta, since we know what the answer is from [Lecture 20](#).
- Start at a corner along the left or right edge of the array of points in the $m_1 - m_2$ plane.
 - One will often know, without calculation, what the value of the “corner” points are.
 - As in [Lecture 20](#): we knew that

$$\left\langle \begin{array}{c|c} 1 & 1 \\ \hline 2 & 2 \end{array} \begin{array}{c|c} 1 & 1 \\ \hline 2 & 2 \end{array} \begin{array}{c} 1 \\ 1 \end{array} \right\rangle = 1 \quad \text{and} \quad \left\langle \begin{array}{c|c|c} 1 & 1 & 1 \\ \hline 2 & 2 & 2 \end{array} \begin{array}{c} 1 \\ 1 \\ -1 \end{array} \right\rangle = 1$$

(see page 3 of that lecture).

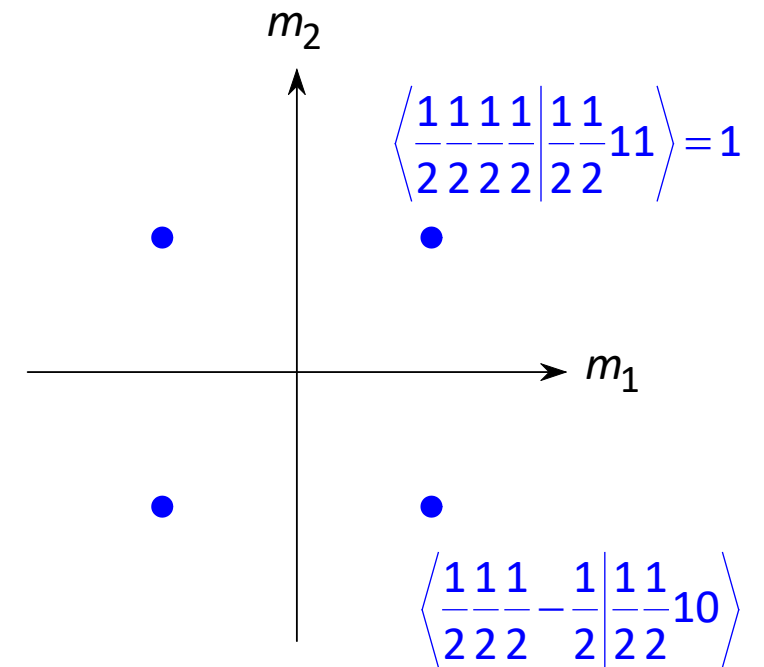


$$j_1 = \frac{1}{2}, \quad j_2 = \frac{1}{2}, \quad j = 1 \quad .$$

Calculation of Clebsch-Gordan coefficients (continued)

- Then apply a recursion relation in which a known point appears in the formula, and in which one of the terms is zero by virtue of needing an m larger than the corresponding j .
- Like the two points labelled here:

$$\begin{aligned} & \sqrt{j(j+1) - m(m-1)} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j, m-1 \rangle = \\ & \sqrt{j_1(j_1+1) - m_1(m_1+1)} \langle j_1 j_2, m_1+1, m_2 | j_1 j_2 j, m \rangle \\ & + \sqrt{j_2(j_2+1) - m_2(m_2+1)} \langle j_1 j_2 m_1, m_2+1 | j_1 j_2 j, m \rangle \end{aligned}$$



$$j_1 = \frac{1}{2}, \quad j_2 = \frac{1}{2}, \quad j = 1 \quad .$$

Calculation of Clebsch-Gordan coefficients (continued)

$$\sqrt{2-1(0)} \left\langle \frac{111}{222} - \frac{1}{2} \left| \frac{11}{22} 1,0 \right\rangle =$$

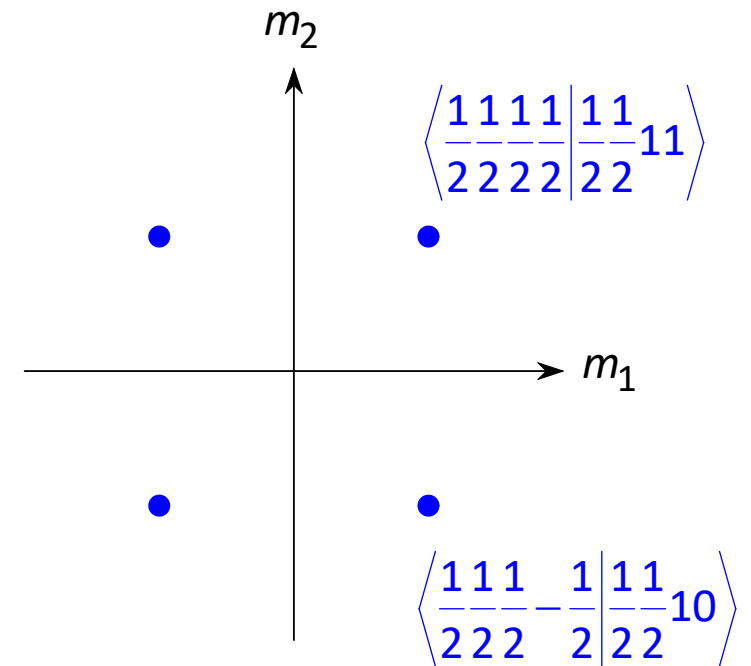
$$\sqrt{\frac{13}{22} - \frac{13}{22}} \left\langle \frac{11}{22}, \frac{3}{2}, -\frac{1}{2} \left| \frac{11}{22} 1,1 \right\rangle + \sqrt{\frac{13}{22} + \frac{11}{22}} \left\langle \frac{11}{22}, \frac{1}{2}, +\frac{1}{2} \left| \frac{11}{22} 1,1 \right\rangle \right.$$

0

$$\sqrt{2} \left\langle \frac{111}{222} - \frac{1}{2} \left| \frac{11}{22} 1,0 \right\rangle = \sqrt{1} \cdot 1 \Rightarrow \left\langle \frac{111}{222} - \frac{1}{2} \left| \frac{11}{22} 1,0 \right\rangle = \frac{1}{\sqrt{2}}$$

- The upper-left point can either be worked out similarly, or with the sum rule. Using the latter, summing along the $m=0$ diagonal containing the upper left and lower right points,

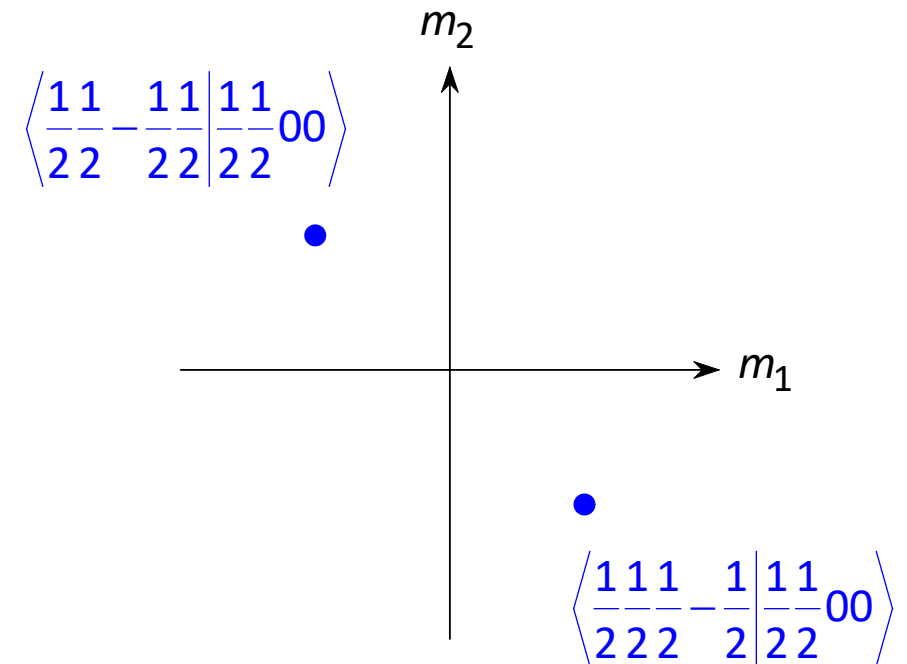
$$\left\langle \frac{11}{22} - \frac{1}{2} \left| \frac{11}{22} 1,0 \right\rangle^2 = 1 - \left\langle \frac{111}{222} - \frac{1}{2} \left| \frac{11}{22} 1,0 \right\rangle^2 = \frac{1}{2} \Rightarrow \left\langle \frac{11}{22} - \frac{1}{2} \left| \frac{11}{22} 1,0 \right\rangle = \frac{1}{\sqrt{2}} . \quad j_1 = \frac{1}{2}, \quad j_2 = \frac{1}{2}, \quad j = 1 .$$



Calculation of Clebsch-Gordan coefficients (continued)

- Then move on to the next value of j : $j = 0$, in the case of spin $\frac{1}{2}$. Only two points left on the graph.
- Since $j = m = 0$, the left-hand side is zero for both upper and lower recursion relations. Take the upper, and use $m_1 = m_2 = 1/2$:

$$\begin{aligned}
 & \sqrt{0(1) - 0(+1)} \left\langle \frac{1111}{2222} \middle| \frac{11}{22} 01 \right\rangle = 0 \\
 & = \sqrt{\frac{1(3)}{2(2)} - \frac{1(1)}{2(2)}} \left\langle \frac{11}{22}, -\frac{1}{2}, \frac{1}{2} \middle| \frac{11}{22} 00 \right\rangle \\
 & \quad + \sqrt{\frac{1(3)}{2(2)} - \frac{1(1)}{2(2)}} \left\langle \frac{111}{222}, -\frac{1}{2} \middle| \frac{11}{22} 00 \right\rangle \\
 \Rightarrow & \left\langle \frac{11}{22}, -\frac{1}{2}, \frac{1}{2} \middle| \frac{11}{22} 00 \right\rangle = - \left\langle \frac{111}{222}, -\frac{1}{2} \middle| \frac{11}{22} 00 \right\rangle \equiv z .
 \end{aligned}$$



$$j_1 = \frac{1}{2}, \quad j_2 = \frac{1}{2}, \quad j = 0 .$$

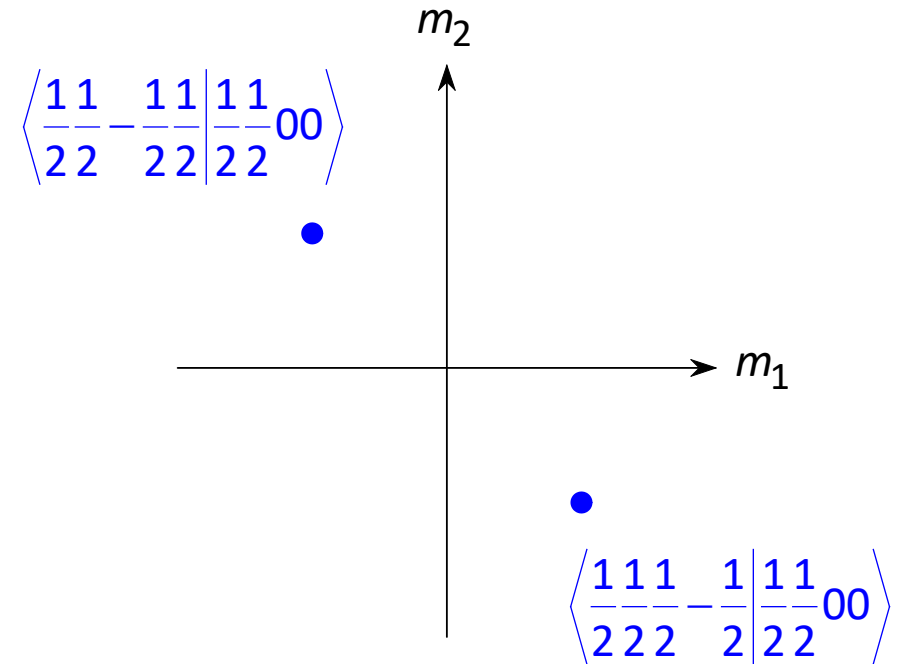
Calculation of Clebsch-Gordan coefficients (continued)

- Now use the sum rule:

$$1 = z^2 + (-z)^2 = 2z^2 \Rightarrow z = \frac{1}{\sqrt{2}} ;$$

$$\left\langle \frac{11}{22}, -\frac{1}{2}, \frac{1}{2} \middle| \frac{11}{22}, 00 \right\rangle = \frac{1}{\sqrt{2}} ,$$

$$\left\langle \frac{111}{222}, -\frac{1}{2} \middle| \frac{11}{22}, 00 \right\rangle = -\frac{1}{\sqrt{2}} .$$



- Same result as before: [Lecture 20](#), p. 10.

$$j_1 = \frac{1}{2}, \quad j_2 = \frac{1}{2}, \quad j = 0 .$$

Calculation of Clebsch-Gordan coefficients (continued)

- A general form for the C-G coefficients was first derived by Giulio Racah ([Racah 1942](#), equation 15):

$$\begin{aligned} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle &= \sqrt{(2j+1)/(j_1 + j_2 + j + 1)!} \sqrt{(j_1 + j_2 - j)! (j_2 + j - j_1)! (j + j_1 - j_2)!} \\ &\quad \times \sqrt{(j_1 + m_1)! (j_1 - m_1)! (j_2 + m_2)! (j_2 - m_2)! (j + m)! (j - m)!} \\ &\quad \times \sum_k (-1)^k / \left[(j_1 + j_2 - j - k)! (j + j_1 - m_2 - k)! (j - j_2 + m_1 + k)! (j_1 - m_1 - k)! (j_2 + m_2 - k)! k! \right] , \end{aligned}$$

where the sum is over all values of k consistent with factorial notation, factorials of negative numbers being undefined. Not very easy to use.

- The earliest computers were used to turn this formula into extensive tables of C-G coefficients. See, e.g., those by Richard Cohen ([1949](#), Ph.D. dissertation, Caltech), which he produced in the form of exact integer ratios instead of floating-point numbers like most contemporary products.

One system, two independent quanta

- The Hamiltonian and the Schrödinger equation for two quanta, lying at positions \mathbf{r}_1 and \mathbf{r}_2 , with momenta $\mathbf{p}_1 = -i\hbar\nabla_1$ and $\mathbf{p}_2 = -i\hbar\nabla_2$, and potential energy $V(\mathbf{r}_1, \mathbf{r}_2, t)$, are

$$\hat{H} = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\mathbf{r}_1, \mathbf{r}_2, t) \quad \text{and} \quad i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \hat{H}\Psi(\mathbf{r}_1, \mathbf{r}_2, t) \quad .$$

- The probability that the quanta lie in within infinitesimal volumes $d\tau_1 = d^3r_1$ and $d\tau_2 = d^3r_2$ at \mathbf{r}_1 and \mathbf{r}_2 is

$$p d\tau_1 d\tau_2 = |\Psi(\mathbf{r}_1, \mathbf{r}_2, t)|^2 d\tau_1 d\tau_2 \quad , \quad \text{where} \quad \int |\Psi(\mathbf{r}_1, \mathbf{r}_2, t)|^2 d\tau_1 d\tau_2 = 1 \quad .$$

- If the potential energy V is time-independent, the separation solution is as usual

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \psi(\mathbf{r}_1, \mathbf{r}_2) e^{-iEt/\hbar} \quad , \quad \text{where} \quad \left[-\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\mathbf{r}_1, \mathbf{r}_2) \right] \psi(\mathbf{r}_1, \mathbf{r}_2) = E\psi(\mathbf{r}_1, \mathbf{r}_2) \quad .$$

One system, two independent quanta (continued)

- If the quanta are subject to an externally-applied force, but **do not interact with each other**, then the potential energy breaks into two parts, $V(\mathbf{r}_1, \mathbf{r}_2) = V_1(\mathbf{r}_1, \mathbf{r}_2) + V_2(\mathbf{r}_1, \mathbf{r}_2)$.
- Then the time-independent Schrödinger equation separates into a part for each quantum:

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2)$$

$$\psi_b(\mathbf{r}_2) \left(-\frac{\hbar^2}{2m_1} \nabla_1^2 + V_1 \right) \psi_a(\mathbf{r}_1) + \psi_a(\mathbf{r}_1) \left(-\frac{\hbar^2}{2m_2} \nabla_2^2 + V_2 \right) \psi_b(\mathbf{r}_2) = E \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2)$$

$$\frac{1}{\psi_a} \left(-\frac{\hbar^2}{2m_1} \nabla_1^2 + V_1 \right) \psi_a + \frac{1}{\psi_b} \left(-\frac{\hbar^2}{2m_2} \nabla_2^2 + V_2 \right) \psi_b = E = E_a + E_b$$

$$\left(-\frac{\hbar^2}{2m_1} \nabla_1^2 + V_1 \right) \psi_a = E_a \psi_a \quad , \quad \left(-\frac{\hbar^2}{2m_2} \nabla_2^2 + V_2 \right) \psi_b = E_b \psi_b \quad , \quad \Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2)e^{-(E_a + E_b)t/\hbar} \quad .$$

One system, two independent quanta (continued)

- In this case the wavefunction $\Psi(\mathbf{r}_1, \mathbf{r}_2, t)$ is a product of the two single-quantum states. The two quanta are **independent**, one in state a and one in state b .
 - Like the two-spin-1/2 states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$.
- But of course any linear combination of ψ_a and ψ_b also solves the time-independent Schrödinger equation. In this case the time-dependent solution cannot be expressed as a product of independent single-quantum states. We say such noninteracting two-quantum systems are **entangled**.
 - Like the two-spin-1/2 states $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ and $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.
 - Entanglement doesn't mean the quanta exert forces on each other.

One system, two interacting quanta

- Now suppose they do exert forces on each other. The simplest such case is the **central** potential energy, which only depends upon the distance between the quanta: $V(\mathbf{r}_1, \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|)$.
- The electrostatic potential energy is a good example. For two electrons, $V(\mathbf{r}_1, \mathbf{r}_2) = e^2/|\mathbf{r}_1 - \mathbf{r}_2|$.
- Change variables to the **displacement** $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and the **center-of-mass position** $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$:

$$(m_1 + m_2)\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = m_1\mathbf{r}_1 + m_2(\mathbf{r}_1 - \mathbf{r}) = (m_1 + m_2)\mathbf{r}_1 - m_2\mathbf{r} \quad \Rightarrow \quad \mathbf{r}_1 = \mathbf{R} + \frac{m_2}{m_1 + m_2}\mathbf{r}$$

$$\Rightarrow \quad \mathbf{r}_1 = \mathbf{R} + \frac{\mu}{m_1}\mathbf{r} \quad , \quad \mathbf{r}_2 = \mathbf{r} - \mathbf{r}_1 = \mathbf{R} - \frac{m_1}{m_1 + m_2}\mathbf{r} = \mathbf{R} - \frac{\mu}{m_2}\mathbf{r} \quad ,$$

where $\mu = m_1m_2/(m_1 + m_2)$ is the **reduced mass** of the pair of quanta.

- Now endow \mathbf{R} and \mathbf{r} with Cartesian components, $\mathbf{R} = [X \quad Y \quad Z]$, $\mathbf{r} = [x \quad y \quad z]$:

One system, two interacting quanta (continued)

$$\nabla_{1x} = \frac{\partial}{\partial x_1} = \frac{\partial X}{\partial x_1} \frac{\partial}{\partial X} + \frac{\partial x}{\partial x_1} \frac{\partial}{\partial x} = \frac{m_1}{m_1 + m_2} \frac{\partial}{\partial X} + \frac{\partial(x_1 - x_2)}{\partial x_1} \frac{\partial}{\partial x} = \frac{\mu}{m_2} \frac{\partial}{\partial X} + \frac{\partial}{\partial x} = \frac{\mu}{m_2} (\nabla_R)_X + (\nabla_r)_x \quad \text{and}$$

$$\nabla_{2x} = \frac{\partial}{\partial x_2} = \frac{\partial X}{\partial x_2} \frac{\partial}{\partial X} + \frac{\partial x}{\partial x_2} \frac{\partial}{\partial x} = \frac{m_2}{m_1 + m_2} \frac{\partial}{\partial X} + \frac{\partial(x_1 - x_2)}{\partial x_2} \frac{\partial}{\partial x} = \frac{\mu}{m_1} \frac{\partial}{\partial X} - \frac{\partial}{\partial x} = \frac{\mu}{m_1} (\nabla_R)_X - (\nabla_r)_x .$$

- Similarly for the other components of \mathbf{R} and \mathbf{r} , so $\nabla_1 = \frac{\mu}{m_2} \nabla_R + \nabla_r$ and $\nabla_2 = \frac{\mu}{m_1} \nabla_R - \nabla_r$.
- The Laplacians, in turn, are

$$\nabla_1^2 = \nabla_1 \cdot \nabla_1 = \left(\frac{\mu}{m_2} \right)^2 \nabla_R^2 + 2 \frac{\mu}{m_2} \nabla_R \cdot \nabla_r + \nabla_r^2 \quad \text{and}$$

$$\nabla_2^2 = \nabla_2 \cdot \nabla_2 = \left(\frac{\mu}{m_1} \right)^2 \nabla_R^2 - 2 \frac{\mu}{m_1} \nabla_R \cdot \nabla_r + \nabla_r^2 .$$

One system, two interacting quanta (continued)

- Thus the kinetic-energy terms in the Schrödinger equation become

$$\begin{aligned}
 -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 &= -\frac{\hbar^2}{2m_1}\left(\frac{\mu}{m_2}\right)^2\nabla_R^2 - \frac{\hbar^2}{2m_1}2\frac{\mu}{m_2}\nabla_R\cdot\nabla_r - \frac{\hbar^2}{2m_1}\nabla_r^2 - \frac{\hbar^2}{2m_2}\left(\frac{\mu}{m_1}\right)^2\nabla_R^2 + \frac{\hbar^2}{2m_2}2\frac{\mu}{m_1}\nabla_R\cdot\nabla_r - \frac{\hbar^2}{2m_2}\nabla_r^2 \\
 &= -\frac{\hbar^2}{2}\frac{m_1}{(m_1+m_2)^2}\nabla_R^2 - \frac{\hbar^2}{m_1+m_2}\nabla_R\cdot\nabla_r - \frac{\hbar^2}{2m_1}\nabla_r^2 - \frac{\hbar^2}{2}\frac{m_2}{(m_1+m_2)^2}\nabla_R^2 + \frac{\hbar^2}{m_1+m_2}\nabla_R\cdot\nabla_r - \frac{\hbar^2}{2m_2}\nabla_r^2 \\
 &= -\frac{\hbar^2}{2}\left(\frac{1}{m_1+m_2}\nabla_R^2 + \left[\frac{1}{m_1} + \frac{1}{m_2}\right]\nabla_r^2\right) = -\frac{\hbar^2}{2}\left(\frac{1}{m_1+m_2}\nabla_R^2 + \frac{m_1+m_2}{m_1m_2}\nabla_r^2\right) = -\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2 - \frac{\hbar^2}{2\mu}\nabla_r^2,
 \end{aligned}$$

whence

$$-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(r) = E\psi.$$

One system, two interacting quanta (continued)

- Finally, separate: take $\psi = \psi_R \psi_r$, substitute in, divide through by the product, and obtain

$$-\frac{1}{\psi_R} \frac{\hbar^2}{2(m_1 + m_2)} \nabla_R^2 \psi_R - \frac{1}{\psi_r} \frac{\hbar^2}{2\mu} \nabla_r^2 \psi_r + V(r) = E = E_R + E_r$$
$$\Rightarrow \boxed{-\frac{\hbar^2}{2(m_1 + m_2)} \nabla_R^2 \psi_R = E_R \psi_R \quad , \quad -\frac{\hbar^2}{2\mu} \nabla_r^2 \psi_r + V(r) \psi_r = E_r \psi_r \quad ;}$$

the first a free-quantum state, the second a (potentially) bound state.

- We have already solved this system once, albeit in approximation: **for the hydrogen atom**, $m_p = 1836m_e \gg m_e$, so

$$m_1 + m_2 = m_e + m_p \cong m_p \quad , \quad \mu = \frac{m_e m_p}{m_e + m_p} \cong \frac{m_e m_p}{m_p} = m_e \quad .$$

But we chose not to solve the whole-atom free-quantum part, as we had discussed free quanta and wavepackets before.