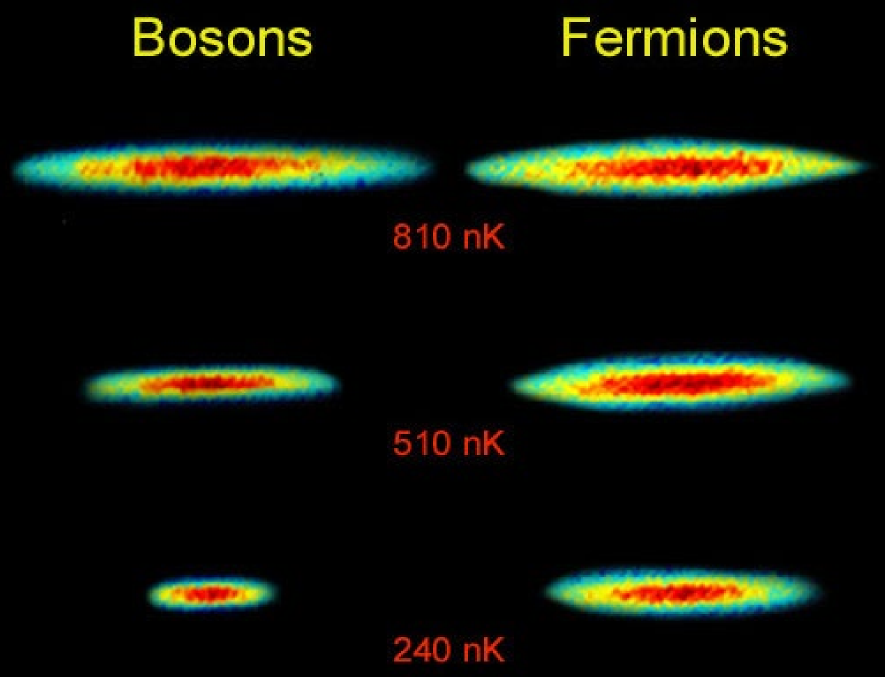
Today in Physics 237: spin and statistics

- Bosons and Fermions
  - The Pauli exclusion principle
  - Symmetric and antisymmetric wavefunctions
- The exchange "force"
  - An outcome of symmetry and antisymmetry
  - Toward explaining differences of behavior between bosons and fermions, like those at right



Clusters of ultracold atoms in a magnetooptical atom trap (Truscott & Hulet 2010): ^{7}Li (total spin 0, bosonic, left), and ^{6}Li (total spin ½, fermionic, right). Note that the ^{6}Li cluster doesn't get smaller as temperature decreases below ~300 nK, but the ^{7}Li condenses further as the temperature drops.

Identical quanta, spin and statistics

Back to two noninteracting, entangled quanta, say, $\psi(\boldsymbol{r}_1,\boldsymbol{r}_2)=\psi_a(\boldsymbol{r}_1)\psi_b(\boldsymbol{r}_2)$ or $|10\rangle=\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\right)$.

- We have noted that neither the form of the state vectors, nor the results of operations on them, depend upon which quantum is which.

- But the fact is that we could not, **even in principle,** tell or specify which is which. The quanta we have discussed are **identical**. All electrons are absolutely identical to one another; all protons, neutrons, photons, same. All their antiquanta too. There is no classical analogue to this feature of fundamental quanta.

- This is not a metaphysical statement, but an experimental one, based on the behavior of quanta of definite spin:

  - Entangled quanta with **half-integer spin**, called **fermions**, **exclude** one another in the sense that no two of them which share an "entangled" volume of space can have the same quantum numbers.

    - This is the well-known **Pauli exclusion principle**.

  - Those with whole-integer spin, called **bosons**, can share volume and quantum numbers, and **condense** spontaneously if given the opportunity.

Identical quanta, spin and statistics (continued)

- There are two ways to construct a wavefunction which does not distinguish between identical quanta *a* and *b*:

$$\psi_\pm(\mathbf{r}_1,\mathbf{r}_2) = A\left(\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \pm \psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)\right) \quad \text{, so that}$$

$$\psi_+(\mathbf{r}_1,\mathbf{r}_2) = \psi_+(\mathbf{r}_2,\mathbf{r}_1) \qquad \text{Bose statistics}$$

$$\text{or} \quad \psi_-(\mathbf{r}_1,\mathbf{r}_2) = -\psi_-(\mathbf{r}_2,\mathbf{r}_1) \qquad \text{Fermi statistics} \quad ;$$

"Statistics" is used here in a peculiar technical sense.

we say that wavefunctions are **symmetric** under exchange of identical bosons, and **antisymmetric** under exchange of identical fermions.

  - In these terms, the Pauli principle is characterized by having the wavefunction vanish if the quanta are in the same state and volume: if $\psi_b = \psi_a$, then $A\left(\psi_a(\mathbf{r}_1)\psi_a(\mathbf{r}_2) - \psi_a(\mathbf{r}_1)\psi_a(\mathbf{r}_2)\right) = 0$.

- For theoretical purposes: this experimental behavior is most often taken as an axiom in nonrelativistic quantum mechanics, but one can make the spin-statistics connection appear non-axiomatically in relativistic quantum mechanics, as Dirac did in the 1930s.

The exchange force

Here "exchange" is used as an adjective, and the "force" isn't a force, strictly speaking.

- Consider a 1-D two-quantum system, in which the quanta occupy orthonormal states ψ_a and ψ_b.

- If the quanta are distinguishable (non-identical) then the system is described by $\psi(x_1,x_2)=\psi_a(x_1)\psi_b(x_2)$;
  if they are identical, then the system is described by

$$\psi(x_1,x_2)=A\left[\psi_a(x_1)\psi_b(x_2)\pm\psi_b(x_1)\psi_a(x_2)\right] \quad ;$$

$$1=\int|\psi|^2\,dx=|A|^2\left(\int|\psi_a|^2\,dx+\int\psi_a^*\psi_b dx+\int\psi_b^*\psi_a dx+\int|\psi|^2\,dx\right)=2A^2\Rightarrow A=\frac{1}{\sqrt{2}} \quad .$$

- It's illuminating to calculate $\left\langle(x_1-x_2)^2\right\rangle=\left\langle x_1{}^2\right\rangle+\left\langle x_2{}^2\right\rangle-2\langle x_1x_2\rangle$ in these three cases. First, distinguishable quanta:

$$\left\langle x_1^2\right\rangle=\int x_1^2\left|\psi_a(x_1)\right|^2dx_1\int\left|\psi_b(x_2)\right|^2dx_2=\left\langle x^2\right\rangle_a \quad , \quad \left\langle x_2^2\right\rangle=\int\left|\psi_a(x_1)\right|^2dx_1\int x_2^2\left|\psi_b(x_2)\right|^2dx_2=\left\langle x^2\right\rangle_b \quad ,$$

The exchange force (continued)

$$\langle x_1 x_2 \rangle = \int x_1 |\psi_a(x_1)|^2 dx_1 \int x_2 |\psi_b(x_2)|^2 dx_2 = \langle x \rangle_a \langle x \rangle_b \quad \Rightarrow \quad \left\langle (x_1 - x_2)^2 \right\rangle_d = \left\langle x_1{}^2 \right\rangle_a + \left\langle x_2{}^2 \right\rangle_b - 2\langle x \rangle_a \langle x \rangle_b \quad .$$

- We haven't specified enough of the details to get very precise answers, but this is precise enough. Consider now the identical-quanta case, first the squares:

$$2\left\langle x_1^2 \right\rangle = \int x_1^2 |\psi_a(x_1)|^2 dx_1 \int |\psi_b(x_2)|^2 dx_2 + \int x_1^2 |\psi_b(x_1)|^2 dx_1 \int |\psi_a(x_2)|^2 dx_2$$

$$\pm \int x_1^2 \psi_a(x_1)^* \psi_b(x_1) dx_1 \int \psi_b(x_2)^* \psi_a(x_2) dx_2 \pm \int x_1^2 \psi_b(x_1)^* \psi_a(x_1) dx_1 \int \psi_a(x_2)^* \psi_b(x_2) dx_2$$

Orthonormality

$$\left\langle x_1^2 \right\rangle = \frac{1}{2}\left(\left\langle x^2 \right\rangle_a + \left\langle x^2 \right\rangle_b \pm 0 \pm 0 \right) = \frac{1}{2}\left(\left\langle x^2 \right\rangle_a + \left\langle x^2 \right\rangle_b \right) \quad .$$

Similarly,

$$\left\langle x_2^2 \right\rangle = \frac{1}{2}\left(\left\langle x^2 \right\rangle_b + \left\langle x^2 \right\rangle_a \right) = \left\langle x_1^2 \right\rangle \quad .$$

The exchange force (continued)

- And the cross term:

$$2\langle x_1 x_2\rangle = \int x_1 |\psi_a(x_1)|^2 dx_1 \int x_2 |\psi_b(x_2)|^2 dx_2 + \int x_1 |\psi_b(x_1)|^2 dx_1 \int x_2 |\psi_a(x_2)|^2 dx_2$$

$$\pm \int x_1 \psi_a(x_1)^* \psi_b(x_1) dx_1 \int x_2 \psi_b(x_2)^* \psi_a(x_2) dx_2 \pm \int x_1 \psi_b(x_1)^* \psi_a(x_1) dx_1 \int x_2 \psi_a(x_2)^* \psi_b(x_2) dx_2$$

Complex conjugates of each other

$$\langle x_1 x_2\rangle = \frac{1}{2}\left(\langle x\rangle_a \langle x\rangle_b + \langle x\rangle_b \langle x\rangle_a \pm \langle x\rangle_{ab} \langle x\rangle_{ba} \pm \langle x\rangle_{ba} \langle x\rangle_{ab}\right) = \langle x\rangle_a \langle x\rangle_b \pm \left|\langle x\rangle_{ab}\right|^2 \quad .$$

- Thus the expectation value of the inter-quantum distance is

$$\left\langle (x_1 - x_2)^2 \right\rangle_{\pm} = \left\langle x^2 \right\rangle_a + \left\langle x^2 \right\rangle_b - 2\langle x\rangle_a \langle x\rangle_b \mp 2\left|\langle x\rangle_{ab}\right|^2 \quad .$$

The exchange force (continued)

- Rename the inter-quantum distance $x_1 - x_2 = \Delta x$;

$$\left\langle (\Delta x)^2 \right\rangle_{\pm} = \left\langle x^2 \right\rangle_a + \left\langle x^2 \right\rangle_b - 2\langle x \rangle_a \langle x \rangle_b \mp 2\left|\langle x \rangle_{ab}\right|^2 = \left\langle (\Delta x)^2 \right\rangle_d \mp 2\left|\langle x \rangle_{ab}\right|^2 \quad .$$

- Compared to the case of the quanta considered distinguishable,
  - identical bosons (+ on LHS) are closer together (– on RHS), and
  - identical fermions (vice versa) are further apart.
- This is effect is called the **exchange force between identical quanta**: it is as though identical bosons are attracted to each other, and identical fermions repelled, by this “force”…
  - … which is a consequence of the symmetrization on page 3, rather than a real force,
  - and apparently harmonizes with Bose-Einstein condensation and the Pauli exclusion principle.