Today in Physics 237: symmetry and angular-momentum-addition redux

- Selection rules for vector operators

- The Wigner-Eckart theorem

$$\langle n'\ell'm'|\hat{T}_i^k|n\ell m\rangle = \frac{1}{\sqrt{2\ell'+1}}\langle n'\ell'||\hat{T}^k||n\ell\rangle\langle \ell'm'ki|\ell m\rangle$$

Vector-operator selection rules

- For generic vector operator $\hat{\boldsymbol{V}}$, define, in analogy with $\hat{\boldsymbol{L}}$ itself, ladder operators $\hat{V}_\pm = \hat{V}_x \pm i\hat{V}_y$.
- As before with scalar operator $\hat{f}$, we examine the commutators between components of $\hat{\boldsymbol{L}}$ and $\hat{\boldsymbol{V}}$:

$$\left[\hat{L}_i,\hat{V}_j\right] = i\hbar\varepsilon_{ijk}\hat{V}_k$$

Recall $\varepsilon_{ijk} = 1$ if $ijk = xyz$ or yzx or zxy; -1 if $ijk = zyx$ or yxz or xzy; 0 otherwise

$$\left[\hat{L}_z,\hat{V}_z\right] = 0$$

$$\left[\hat{L}_z,\hat{V}_\pm\right] = \left[\hat{L}_z,\hat{V}_x\right] \pm i\left[\hat{L}_z,\hat{V}_y\right] = i\hbar\hat{V}_y \pm i\left(-i\hbar\hat{V}_x\right) = \pm\hbar\left(\hat{V}_x \pm i\hat{V}_y\right) = \pm\hbar\hat{V}_\pm$$

$$\left[\hat{L}_\pm,\hat{V}_\pm\right] = \left[\hat{L}_x,\hat{V}_x\right] \pm i\left[\hat{L}_x,\hat{V}_y\right] \pm i\left[\hat{L}_y,\hat{V}_x\right] - \left[\hat{L}_y,\hat{V}_y\right] = 0 \pm i\left(i\hbar\hat{V}_z\right) \pm i\left(-i\hbar\hat{V}_z\right) + 0 = 0$$

$$\left[\hat{L}_\pm,\hat{V}_z\right] = \left[\hat{L}_x,\hat{V}_z\right] \pm i\left[\hat{L}_y,\hat{V}_z\right] = -i\hbar\hat{V}_y \pm i\left(i\hbar\hat{V}_x\right) = \mp\hbar\left(\hat{V}_x \pm \hat{V}_y\right) = \mp\hbar\hat{V}_\pm$$

$$\left[\hat{L}_\pm,\hat{V}_\mp\right] = \left[\hat{L}_x,\hat{V}_x\right] \mp i\left[\hat{L}_x,\hat{V}_y\right] \pm i\left[\hat{L}_y,\hat{V}_x\right] + \left[\hat{L}_y,\hat{V}_y\right] = 0 \mp i\left(i\hbar\hat{V}_z\right) \pm i\left(-i\hbar\hat{V}_z\right) + 0 = \pm 2\hbar V_z$$

Vector-operator selection rules (continued)

- Also as before, we calculate matrix elements of these commutators between states of definite angular momentum $|n'\ell'm'\rangle$ and $|n\ell m\rangle$, finding selection rules thereby. The first couple:

$$\langle n'\ell'm'|[\hat{L}_z,\hat{V}_z]|n\ell m\rangle = \hbar m'\langle n'\ell'm'|\hat{V}_z|n\ell m\rangle - \hbar m\langle n'\ell'm'|\hat{V}_z|n\ell m\rangle = 0 \quad \Rightarrow \quad m'-m=0$$

$$\langle n'\ell'm'|[\hat{L}_z,\hat{V}_\pm]|n\ell m\rangle = \hbar m'\langle n'\ell'm'|\hat{V}_\pm|n\ell m\rangle - \hbar m\langle n'\ell'm'|\hat{V}_\pm|n\ell m\rangle = \pm\hbar\langle n'\ell'm'|\hat{V}_\pm|n\ell m\rangle \quad \Rightarrow \quad m'-m=\pm 1 \quad ;$$

and the next one, using the recursion relations derived in Lecture 21,

$$\langle n'\ell'm'|[\hat{L}_\pm,\hat{V}_z]|n\ell m\rangle = \langle n'\ell'm'|\hat{L}_\pm\hat{V}_z|n\ell m\rangle - \langle n'\ell'm'|\hat{V}_z\hat{L}_\pm|n\ell m\rangle = \mp\hbar\langle n'\ell'm'|\hat{V}_\pm|n\ell m\rangle \quad \Rightarrow$$

$$\mp\hbar\langle n'\ell'm'|\hat{V}_\pm|n\ell m\rangle = \hbar\sqrt{\ell'(\ell'+1)-m'(m'\mp 1)}\langle n'\ell'm'\mp 1|\hat{V}_z|n\ell m\rangle - \hbar\sqrt{\ell(\ell+1)-m(m\pm 1)}\langle n'\ell'm'|\hat{V}_z|n\ell m\pm 1\rangle \quad ,$$

are supposed to remind you of problem D on assignment 11, in which you started from the "corners" to derive the set of nonzero Clebsch-Gordan coefficients $\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$ for $j=2$.

Vector-operator selection rules (continued)

- And that isn't an accident. If one follows this direction one finds – in G&S problems 6.22, 6.23 and 6.24, on assignment Never – that
  - the matrix elements of any component *i* of a vector operator, $\langle n'\ell'm'|\hat{V}_i|n\ell m\rangle$, are nonzero only when $\ell'-\ell=0 \text{ or } \pm1 \quad \text{and} \quad m'-m=0 \text{ or } \pm1.$
  - These are new selection rules for electric-dipole-permitted quantum transitions, which go with those derived from parity conservation in Lecture 26.
  - And one sees that Clebsch-Gordan coefficients demonstrate the role played by angular-momentum changes in transitions between different state vectors.

The Wigner-Eckart theorem

- It can be proven, more elegantly than is apparent from the last few pages, that the matrix elements of any vector operator can be expressed as

$$\langle n'\ell'm'|\hat{V}_i|n\ell m\rangle = \frac{1}{\sqrt{2\ell'+1}}\langle n'\ell'\|\hat{V}\|n\ell\rangle\langle 1i\ell'm'|1\ell im\rangle \quad ,$$

or more generally for a tensor operator of order *k*,

$$\boxed{\langle n'\ell'm'|\hat{T}_i^k|n\ell m\rangle = \frac{1}{\sqrt{2\ell'+1}}\langle n'\ell'\|\hat{T}^k\|n\ell\rangle\langle \ell'm'ki|\ell m\rangle \quad .}$$

Wigner-Eckart theorem

- One will see this proven in one's first graduate quantum-mechanics course.

- It is a powerful tool: it means that a matrix element between $|n'\ell'm'\rangle$ and $|n\ell m\rangle$, of a tensor operator, factors into
  - **a term containing all the dynamics**, independent of angular details: the reduced matrix element, and
  - **a term containing all the angular details** but independent of the dynamics, which turns out to be nothing more than Clebsch-Gordan coefficients.