Statistics & Error Propagation Tutorial

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Every measurement you make has an uncertainty. Understanding that uncertainty is what separates a *result* from a *guess*. The knowledge we have of the physical world is obtained by doing experiments and making measurements, and it is impossible to measure any physical quantity exactly. A result without an uncertainty is incomplete; the uncertainty is not a sign of failure — it is an essential part of the measurement itself.

As a concrete example, consider measuring the length of a pendulum to determine g. Even with a good ruler and careful technique, factors like reading angle, thermal expansion of the string, and the precision of the ruler itself introduce unavoidable uncertainty. The result of any measurement therefore has two essential components:

1. a numerical value (in specified units) giving the best estimate of the quantity, and
2. an uncertainty describing how confident we are in that estimate.

For example: $L = 95.3 \pm 0.1$ cm.

Significant Figures

The *significant figures* of a measured or calculated quantity are the meaningful digits in it. Follow these conventions:

1. Any non-zero digit is significant. Thus 549 has three significant figures.
2. Zeros *between* non-zero digits are significant. Thus 4023 has four.
3. Leading zeros are not significant. $0.000034 = 3.4 \times 10^{-5}$ has two.
4. For numbers with decimal points, trailing zeros are significant. 2.00 has three significant figures; 0.050 has two.
5. For integers without decimal points, trailing zeros may or may not be significant. Write 400. (with decimal) or 4×10^2 to make the intention clear.
6. Defined and counted quantities (e.g. the number of apples, the speed of light in vacuum) have infinitely many significant figures.

Rounding rules

The last significant figure of a result should be in the same decimal place as its uncertainty. Round the uncertainty to one or two significant figures first, then match the result to it.

Correct:

$$g = 9.82 \pm 0.02 \text{ m/s}^2, \qquad T = 10.0 \pm 1.5 \text{ s}, \qquad N = 4 \pm 1$$

Incorrect:

$$g = 9.82 \pm 0.02385 \text{ m/s}^2, \qquad T = 10.0 \pm 2 \text{ s}, \qquad N = 4 \pm 0.5$$

After **addition or subtraction**, the result is significant only to the place of the largest last significant figure among the inputs:

$$89.332 + 1.1 = 90.432 \longrightarrow 90.4$$

After **multiplication or division**, keep as many significant figures as the input with the fewest:

$$2.80 \times 4.5039 = 12.61092 \longrightarrow 12.6$$

The Nature of Experimental Error

Error does not mean blunder. Misreading a scale, using the wrong setting on an instrument, or knocking over your lab partner's apparatus are *mistakes*, not errors — they can and should be caught and discarded.

Error also does not mean the difference between your result and an accepted value. Handbook values are themselves measurements with their own uncertainties.

Error refers to the irreducible uncertainty that remains even after every precaution has been taken. If you repeat a measurement carefully, you will not get exactly the same number each time. That spread is what we mean by error, and it must be characterized and reported.

Systematic Errors

Systematic errors shift all measurements in the same direction. They arise from incorrect instrument calibration, consistently improper technique, or unaccounted-for physical effects (e.g. friction in an Atwood machine, or air resistance in a free-fall measurement). Large systematic errors must be identified and eliminated; small ones will always remain. When several independent experiments using different techniques agree, systematic errors are less likely to be responsible for any discrepancy.

Random Errors

Random errors fluctuate unpredictably from one measurement to the next. They arise from limited instrument sensitivity, environmental noise, or the inherently probabilistic nature of some physical processes (radioactive decay, for example). Unlike systematic errors, random errors cannot be eliminated — but their effect on the *mean* of many measurements decreases as $1/\sqrt{N}$.

Many experimental results carry two separate uncertainties, written as $x \pm \delta_{\text{stat}} \pm \delta_{\text{sys}}$, distinguishing statistical (random) from systematic contributions. When only one uncertainty is quoted, it represents the combination of both sources added in quadrature (see Section 4).

Poisson Statistics

A common source of random error in physics is a counting process governed by Poisson statistics. Radioactive decay, photon counting, and particle detection all follow this distribution. If the expected number of counts is n, the statistical uncertainty is

$$\Delta n = \sqrt{n}.$$

When n is unknown, the measured count n_{meas} is used as the estimate.

Example. In Lab 1, you analyze the distribution of spam emails to decide whether spammers prefer daytime hours. Out of 1595 emails, 784 arrive during the day and 811 at night. Each bin has uncertainty $\sqrt{n}$, so the daytime count is 784 ± 28 and the nighttime count is 811 ± 28. The difference (27) is comparable to the uncertainty — there is no statistically significant preference.

Describing a Distribution of Measurements

Suppose an experiment is repeated N times, yielding values $x_1, x_2, \ldots, x_N$.

Mean

The best estimate of the true value is the arithmetic mean,

$$\bar{x} = \frac{1}{N}\sum_{k=1}^{N} x_k.$$

Averaging reduces the effect of random errors because positive and negative fluctuations tend to cancel. Repeating an experiment is the only reliable way to gain confidence in its accuracy.

Standard Deviation

The *standard deviation* s describes the typical spread of individual measurements around the mean:

$$s = \sqrt{\frac{\sum_{k=1}^{N}(x_k - \bar{x})^2}{N-1}}.$$

We divide by $N-1$ (not N) because we are estimating the true mean with $\bar{x}$ rather than using the true mean itself; dividing by N would systematically underestimate the spread. For a Gaussian (normal) distribution, about 68% of individual measurements fall within $\bar{x} \pm s$.

Standard Error of the Mean

We are usually more interested in the uncertainty on the *mean* than on a single measurement. The standard error of the mean is

$$\sigma_{\bar{x}} = \frac{s}{\sqrt{N}} = \sqrt{\frac{\sum_{k=1}^{N}(x_k - \bar{x})^2}{N(N-1)}}.$$

The result of a series of N measurements is reported as $\bar{x} \pm \sigma_{\bar{x}}$. There is a 68% probability that the true mean lies within this interval, and a 32% chance it falls outside — meaning roughly 1 in 3 well-conducted experiments will disagree with the true value by more than one standard error.

Example. In Lab 2 (Atwood Machine), you measure the fall time t five times for a fixed drop distance. Suppose you record: $t = 1.31,\ 1.28,\ 1.34,\ 1.30,\ 1.32$ s.

$$\bar{t} = \frac{1.31 + 1.28 + 1.34 + 1.30 + 1.32}{5} = 1.31 \text{ s}$$

$$s = \sqrt{\frac{(0.00)^2 + (0.03)^2 + (0.03)^2 + (0.01)^2 + (0.01)^2}{4}} = \sqrt{\frac{0.0020}{4}} \approx 0.022 \text{ s}$$

$$\sigma_{\bar{t}} = \frac{0.022}{\sqrt{5}} \approx 0.010 \text{ s} \qquad \Rightarrow \qquad t = 1.31 \pm 0.01 \text{ s}$$

For the drop distance s you will estimate Δs directly from the precision of the ruler (typically 2–3 mm).

Propagation of Errors

When a result Z is calculated from measured quantities A and B (each with uncertainties ΔA and ΔB), the uncertainty in Z is found by combining the partial derivatives of Z with respect to each variable in quadrature:

$$\Delta Z = \sqrt{\left(\frac{\partial Z}{\partial A}\,\Delta A\right)^2 + \left(\frac{\partial Z}{\partial B}\,\Delta B\right)^2}.$$

This assumes A and B are *independent*. Adding errors in quadrature (rather than linearly) reflects the fact that random errors do not always combine at their worst.

Common Cases

Table 1 summarises the propagation rules for the most common functional forms.

	Relation	Error formula
1	$Z = A + B$ or $Z = A - B$	$\Delta Z = \sqrt{(\Delta A)^2 + (\Delta B)^2}$
2	$Z = AB$ or $Z = A/B$	$\frac{\Delta Z}{Z} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}$
3	$Z = A^n$	$\frac{\Delta Z}{Z} = \lvert n\rvert\,\frac{\Delta A}{A}$
4	$Z = \ln A$	$\Delta Z = \frac{\Delta A}{A}$
5	$Z = e^A$	$\frac{\Delta Z}{Z} = \Delta A$
6	$Z = \frac{A+B}{2}$	$\Delta Z = \frac{1}{2}\sqrt{(\Delta A)^2 + (\Delta B)^2}$

Table 1: Propagation rules for independent variables A and B.

Addition and Subtraction

For $Z = A + B$ with $A = 100 \pm 3$ and $B = 6 \pm 4$:

$$\Delta Z = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5 \qquad \Rightarrow \qquad Z = 106 \pm 5$$

Multiplication and Division

For $Z = AB$ with $A = 100 \pm 0.3$ and $B = 6 \pm 0.4$:

$$\frac{\Delta Z}{Z} = \sqrt{\left(\frac{0.3}{100}\right)^2 + \left(\frac{0.4}{6}\right)^2} = \sqrt{9 \times 10^{-6} + 0.0044} \approx 0.067 \qquad \Rightarrow \qquad \Delta Z \approx 40$$

so $Z = 600 \pm 40$.

Powers — A Common Pitfall

When the same measured quantity appears more than once, the variables are not independent and the general quadrature rule does not apply. For $Z = A^2$, differentiating directly:

$$\Delta Z = 2A\,\Delta A \qquad \Rightarrow \qquad (10 \pm 1)^2 = 100 \pm 20.$$

Misapplying the multiplication rule (treating $A^2 = A \times A$ as two independent measurements) would give 100 ± 14 — an underestimate.

Example: Measuring g from an Atwood Machine

In Lab 2, you derive g from measurements of mass m_1, m_2, drop distance s, and fall time t. The relevant expression is

$$g = \frac{2s(m_1 + m_2)}{(m_1 - m_2)\,t^2}.$$

Applying propagation, the fractional uncertainty is

$$\frac{\Delta g}{g} = \sqrt{\left(\frac{\Delta s}{s}\right)^2 + \left(\frac{\Delta(m_1 + m_2)}{m_1 + m_2}\right)^2 + \left(\frac{\Delta(m_1 - m_2)}{m_1 - m_2}\right)^2 + \left(\frac{2\,\Delta t}{t}\right)^2}.$$

Notice the factor of 2 on the time term because t appears squared; notice also that $\Delta(m_1 \pm m_2) = \sqrt{(\Delta m_1)^2 + (\Delta m_2)^2}$ by Rule 1. A small uncertainty in Δm can dominate if $m_1 - m_2$ is small — this is why the Atwood machine is most sensitive when the two masses are nearly equal.

References

1. J. R. Taylor, *An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements*, 2nd ed. University Science Books, 1997.
2. P. V. Bork, H. Grote, D. Notz, M. Regler, *Data Analysis Techniques in High Energy Physics Experiments*, Cambridge University Press, 1993.