From:
Stellar Evolution, ed. R.F. Stein and A.G.W
Cameron (New York:Plenum, 1966),
pp. 3-79

STELLAR EVOLUTION: A SURVEY WITH ANALYTIC MODELS

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INTRODUCTION

To increase our physical insight into the evolution of stars, we shall, in this introductory paper, reproduce the basic features of stellar structure (as found from accurate calculations), by purely analytic considerations. We will, therefore, not attempt accurate calculations of structures and evolutionary tracks. Rather, we shall first discuss the general properties of stellar structure and evolution, and then construct analytic models for the early homogeneous and advanced inhomogeneous stages of evolution.

EQUATIONS OF STELLAR STRUCTURE

The structure of stars is determined by the conditions of mass conservation, momentum conservation, energy conservation, and the mode of energy transport (Schwarzschild, 1958, and Wrubel, 1958). Rotation and magnetic fields will be neglected so that a star will be spherically symmetric.

Hydrostatic Equilibrium

A star changes very slowly during most of its life and so may be considered in hydrostatic equilibrium. Two forces balance to keep a nonrotating star in hydrostatic equilibrium: the gravitational force directed inward and the gas and radiation pressure force directed outward. The equation of hydrostatic equilibrium is

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2} \tag{1.1}$$

The total pressure is the sum of gas and radiation pressure

$$P = P_{gas} + P_{rad}$$

For an ideal gas

$$P_{gas} = \frac{k}{\mu H}\rho T \tag{1.2}$$

where $H = 1.67 \times 10^{-24}$ g is the mass of a proton, $k = 1.38 \times 10^{-16}$ ergs/deg, and μ is the mean molecular weight.

$$P_{rad} = \tfrac{1}{3}aT^4 \tag{1.3}$$

where $a = 7.57 \times 10^{-15}$ ergs/cm^3-deg^4. $M(r)$ is the mass inside a sphere of radius

r. The equation of mass conservation is

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho \tag{1.4a}$$

so that

$$M(r) = \int_0^r 4\pi r^2 \rho \, dr \tag{1.4b}$$

G is the gravitational constant, $G = 6.67 \times 10^{-8}$ dyn-cm^2/g^2.

Energy Conservation

The total energy of an element of material is

$$E = U + \Omega + K \tag{1.5}$$

where U is the internal energy of the gas, Ω is gravitational potential energy, and K is the kinetic energy of large-scale mass motion, which we are neglecting here. The internal energy of a gas plus radiation is

$$U = \frac{1}{\gamma - 1} NkT + \frac{1}{\rho} aT^4$$

where γ is the ratio of specific heats ($\gamma = \frac{5}{3}$ for a monatomic ideal gas). The sources and sinks of energy are (1) energy release or absorption by nuclear reactions, and (2) energy transport into and out of the element of material.

Let $\mathscr{E}$ be the net release of energy per gram per second, and F be the energy flux. The equation of conservation of energy is then

$$\frac{dE}{dt} = \frac{dU}{dt} + \frac{d\Omega}{dt} = \mathscr{E} - \frac{1}{\rho} \operatorname{div} F \text{ ergs/g-sec}$$

The change of gravitational potential energy is

$$d\Omega = -dW = P\,dv = -\frac{P}{\rho^2} d\rho$$

Define the luminosity L_r as the total net energy flux through a spherical shell of radius r, so that

$$L_r = 4\pi r^2 F$$

Then the equation of energy conservation is

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \left[\mathscr{E} + \frac{P}{\rho^2}\frac{d\rho}{dt} - \frac{dU}{dt}\right] \tag{1.6}$$

Energy Transport

Energy is transported by radiation and convection, and by conduction when the electrons are degenerate:

$$\frac{L_r}{4\pi r^2} = F_{\text{rad}} + F_{\text{conv}} \tag{1.7}$$

In the interior of a star, where the radiation is almost isotropic, the force due to the gradient of the radiation pressure is equal to the momentum absorbed from the radiation beam in passing through matter:

$$\frac{dP_R}{dr} = -\frac{\varkappa\rho}{c}\frac{L_r}{4\pi r^2}$$

where $P_R = \frac{1}{3}aT^4$ is the radiation pressure, $(\varkappa\rho)^{-1}$ is the photon mean free path, and c is the velocity of light. Thus, in the interior of the star, the radiative energy flux is

$$F_{\text{rad}} = -\frac{4acT^3}{3\varkappa\rho}\frac{dT}{dr} \tag{1.8}$$

and the temperature gradient necessary to drive the radiation flux is

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\varkappa\rho}{T^3}\frac{L_r}{4\pi r^2} \tag{1.9}$$

Convection occurs in those regions of a star where the temperature gradient is steeper than the adiabatic gradient. The convective flux is (Spiegel, 1965), crudely, the energy fluctuation (excess or deficiency) of an element of gas times its velocity, averaged over horizontal directions,

$$F_{\text{conv}} = \overline{\rho c_p w \theta} \tag{1.10}$$

where w is the radial velocity fluctuation and θ the temperature fluctuations in the matter. The convective flux depends on the superadiabatic gradient

$$\beta = -\left[\frac{dT}{dr} - \left(\frac{dT}{dr}\right)_{\text{ad}}\right] \tag{1.11}$$

Because convection is an extremely efficient energy-transport mechanism in the interior of a star, the superadiabatic gradient is very small and the temperature gradient will be very nearly equal to the adiabatic gradient:

$$\frac{dT}{dr} = \left(\frac{dT}{dr}\right)_{\text{ad}} = \frac{\Gamma - 1}{\Gamma}\frac{T}{P}\frac{dP}{dr} \tag{1.12}$$

where Γ is the effective ratio of specific heats, including ionization, dissociation, and radiation. Near the surface, where the photon mean free path is long, there is a leakage of heat by radiation from the convective elements and the convective temperature gradient is greater than the adiabatic gradient.

Stellar Structure

Order of magnitude estimates of the density, pressure, and temperature of a star can easily be made from the condition of hydrostatic equilibrium. The mean density of a star of mass M and radius R is

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} \tag{1.13}$$

In the equation of hydrostatic equilibrium (1.1), setting

$$\frac{dP}{dr} \approx \frac{P_c - P_0}{R} \approx \frac{P_c}{R}$$

where P_c is the central and P_0 the surface pressure, gives

$$P_c \approx \frac{GM\bar{\rho}}{R} \approx \frac{GM^2}{R^4} \tag{1.14}$$

Let $\beta = P_{gas}/P$ be the ratio of gas pressure to total pressure and assume that the material of the star is a perfect gas. Then the central temperature is obtained from the perfect-gas law (equation 1.2):

$$T_c \approx \frac{\mu\beta H}{k}\frac{P_c}{\bar{\rho}} \approx \frac{\mu\beta H}{k}\frac{GM}{R} \tag{1.15}$$

The mean energy-generation rate is

$$\bar{\mathscr{E}} = \frac{L}{M}$$

For the sun

$$\begin{aligned} L &= 3.89 \times 10^{33}\ \text{ergs/sec} \\ M &= 1.99 \times 10^{33}\ \text{g} \\ R &= 6.95 \times 10^{10}\ \text{cm} \end{aligned} \tag{1.16}$$

Thus, the internal conditions of stars are of the order of magnitude

$$\begin{aligned} \bar{\rho} &= 1.41\left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)^3\ \text{g/cm}^3 \\ P_c &\approx 1.1 \times 10^{16}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R_\odot}{R}\right)^4\ \text{dyn/cm}^2 \\ T_c &= 2.3 \times 10^7\mu\beta\left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)\,^\circ\text{K} \\ \bar{\mathscr{E}} &= 1.9\left(\frac{L}{L_\odot}\right)\left(\frac{M_\odot}{M}\right)\ \text{ergs/g-sec} \end{aligned} \tag{1.17}$$

as functions of the stars' mass, radius, and luminosity given in solar units.

For a more detailed account of the restrictions imposed by hydrostatic equilibrium on stellar structure, see Chandrasekhar (1939).

This section is concluded by calculating the gravitational potential energy of a sphere of uniform density. The gravitational potential energy is

$$\Omega = -\int_0^R \frac{GM(r)}{r}\,dM(r) = \tfrac{1}{2}\int_0^R \Phi\,dM(r) \tag{1.18}$$

where Φ is the gravitational potential.

For a sphere of uniform density, the equation of hydrostatic equilibrium (1.1) is

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{d}{dr}\left(\frac{P}{\rho}\right) = -\frac{d\Phi}{dr}$$

Upon integrating, using the boundary condition that $P/\rho \to 0$ at the surface, we get

$$-\Phi + \Phi_s = \frac{P}{\rho} \quad \text{and} \quad \Phi_s = \frac{GM}{R}$$

then

$$\begin{aligned} \Omega &= -\tfrac{1}{2}\frac{GM}{R}\int_0^R dM(r) - \tfrac{1}{2}\int_0^R \frac{P}{\rho}\,dM(r) \\ &= -\tfrac{1}{2}\frac{GM^2}{R} - \tfrac{1}{2}\int_0^R P4\pi r^2\,dr \\ &= -\tfrac{1}{2}\frac{GM^2}{R} + \frac{4\pi}{6}\int \frac{dP}{dr}r^3\,dr \\ &= -\tfrac{1}{2}\frac{GM^2}{R} + \tfrac{1}{6}\Omega \end{aligned}$$

Thus, the gravitational potential energy of a sphere of uniform density is

$$\Omega = -\tfrac{3}{5}\frac{GM^2}{R} \tag{1.19}$$

The absolute value of the gravitational potential energy in an actual star will be somewhat larger, but of the same order of magnitude.

STELLAR EVOLUTION

A star is a self-gravitating mass of gas in space. The evolutionary trend of internal stellar conditions is determined by hydrostatic equilibrium and the loss of energy by a star due to radiation into space. The life history of a star is the progressive concentration of its mass toward its center, pulled by its own gravitational field, which releases gravitational energy, supplies the energy losses, and heats up the gas. As the gas becomes hotter, thermonuclear reactions among various nuclei become possible. At certain temperatures, the thermonuclear reactions can supply the energy losses, the gas and radiation pressure can support the star, and the gravitational contraction is temporarily halted.

A necessary condition for hydrostatic equilibrium is the virial theorem for a self-gravitating mass (Chandrasekhar, 1939):

$$2K + \Omega = 3(\gamma - 1)U + \Omega = 0 \tag{2.1}$$

Here, K is the total thermal energy of the mass, U is its internal energy, and Ω is its gravitational potential energy. The virial theorem requires that the thermal energy of a star equal half the absolute value of its gravitational potential energy (since Ω is intrinsically negative). As a star contracts and releases gravitational energy, Ω becomes more negative, and the thermal energy must increase. Half of the gravitational energy that is released is stored as thermal energy, increasing the temperature in the interior of the star, and half is radiated away.

The mean relation of temperature to density can be derived from the virial theorem. For a sphere of gas whose internal pressure is given by the perfect gas law with ratio of specific heats $\gamma = \frac{5}{3}$, the virial theorem (2.1) becomes

$$2U + \Omega = 0 \tag{2.2}$$

For a uniform density distribution, the gravitational energy (equation 1.19) is

$$\Omega = -\tfrac{3}{5}\frac{GM^2}{R}$$

and the internal energy is

$$U = \tfrac{3}{2}kT\frac{M}{\mu H} \tag{2.3}$$

where $M/\mu H$ is the number of particles. Thus,

$$T = \tfrac{1}{5}\frac{\mu H}{k}G\frac{M}{R} \tag{2.4}$$

The density (equation 1.13) is

$$\bar{\rho} = \frac{3}{4\pi}\frac{M}{R^3}$$

so that

$$T = \tfrac{1}{5}\left(\frac{4\pi}{3}\right)^{1/3}\frac{G\mu H}{k}M^{2/3}\bar{\rho}^{1/3}$$

Thus, the relation between temperature and density for stars with negligible radiation pressure is

$$T = 4.1 \times 10^6\mu\left(\frac{M}{M_\odot}\right)^{2/3}\bar{\rho}^{1/3}\ ^\circ\mathrm{K} \tag{2.5}$$

For nonuniform density distributions, the same relation holds between the local temperature and density, but with a slightly different numerical coefficient.

The above temperature–density relation does not hold for those stars whose internal pressures are predominantly governed by radiation pressure. The gas and radiation pressures contribute equally to the total pressure when

$$\tfrac{1}{3}aT^4 = \frac{k}{\mu H}\rho T$$

or

$$T = 2.55 \times 10^7\rho^{1/3} \tag{2.6}$$

This condition occurs at $5.5M_\odot$. For heavier stars, radiation pressure is predominant. In such cases, $\gamma = \frac{4}{3}$ and the virial theorem gives $U = -\Omega$. Thus,

$$U = \frac{1}{\rho}aT^4 = \tfrac{3}{5}\frac{GM^2}{R}$$

so

$$T = \left(\frac{9G}{20\pi a}\right)^{1/4}\frac{M^{1/2}}{R} \tag{2.7}$$

Expressing R in terms of the mean density (1.13), we obtain the temperature–density relation

$$T = \left(\frac{4\pi}{3}\right)^{1/12}\left(\frac{3G}{5a}\right)^{1/4}M^{1/6}\rho^{1/3}$$
$$= 1.92 \times 10^7\left(\frac{M}{M_\odot}\right)^{1/6}\rho^{1/3}\ ^\circ\mathrm{K} \tag{2.8}$$

The temperature depends on density as before (to the $\frac{1}{3}$ power), but the effect of mass is less pronounced.

The temperature–density relations (2.5) and (2.8) describe the dependence of the temperature on the density inside a star. They also describe the evolution of stars, which consists of progressive gravitational contraction, increasing the central density and temperature according to

$$T \propto \rho^{1/3}$$

Some simplified evolutionary tracks for internal stellar conditions are shown in Figure 1.

When the central density of a star gets very great, the electrons may become degenerate and the equation of state thus changes. The boundary of degeneracy in terms of density and temperature has the asymptotic forms for low and high density (nonrelativistic and relativistic energies):

$$T = 1.2 \times 10^5\left(\frac{\rho}{\mu_e}\right)^{2/3} \quad \text{Low density}$$
$$T = 1.49 \times 10^7\left(\frac{\rho}{\mu_e}\right)^{1/3} \quad \text{High density} \tag{2.9}$$

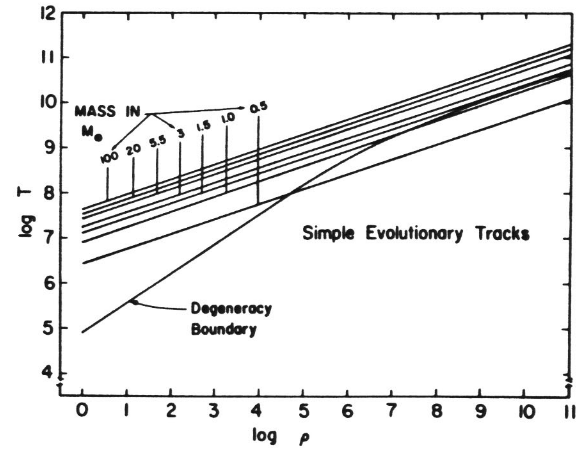

Figure 1. Simple evolutionary tracks for the internal conditions of stars of various masses.

The full boundary curve has been derived by Chandrasekhar (1939). This boundary is also plotted in Figure 1.

Stars of mass less than about $1.3M_\odot$ enter the degenerate region. For these stars, the pressure due to degenerate electrons is so high that further compression is no longer possible. This is essentially the end-point in the evolution of a star of small mass. The star becomes a white dwarf, achieving in this process some maximum temperature which depends specifically on its mass.

The general evolutionary trend of contraction, increasing the central density and temperature, is interrupted periodically by nuclear burning. The energy-generation history of a star is a succession of gravitational contractions which raise the central temperature of the star sufficiently to initiate thermonuclear reactions; the thermonuclear reactions transform a given type of fuel nuclei into heavier nuclei and release energy; the supply of the given fuel nuclei becomes exhausted and the core resumes its gravitational contraction. The order of thermonuclear reactions is determined by the nuclei present and their charges. The larger the nuclear charge, the higher its Coulomb barrier and the higher the kinetic energy (temperature) of the bombarding particles must be to penetrate the barrier and initiate nuclear reactions. A schematic sketch of the energy history of a star is shown in Figure 2. During nuclear burning, the temperature is almost constant. During gravitational contraction, the isotopic composition does not change.

The most abundant element is hydrogen, which also has the lowest charge, 1. It is transformed into He^4, releasing 6×10^{18} erg/g at temperatures above 10^7 °K (Reeves, 1965). Helium is transformed into C^{12} at temperatures above about 10^8 °K, and at slightly higher temperatures carbon reacts with helium to form O^{16}. The

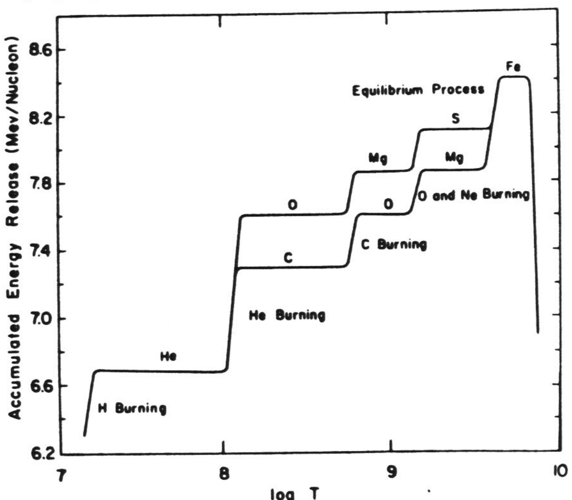

Figure 2. Energy history of a star (schematic diagram). Nuclear burning stages and the resulting composition of the core of the star are shown. Where two curves are drawn, they represent the lower and upper limits of the range of nuclei produced (Reeves, 1963).

amounts of carbon and oxygen produced in the core during helium-burning depend on the central temperature and therefore on the mass of the star. The C and O curves in Figure 2 are the lower and upper limits, respectively. Carbon reacts with itself at temperatures above about 7×10^8 °K; carbon-burning produces nuclei in the range O^{16} to Mg^{26}. The two curves again are the upper and lower limits. Neon photodisintegrates and oxygen reacts with itself at still higher temperatures, about 1.4×10^9 °K. Neon-burning produces predominantly O^{16} and Mg^{24}. Oxygen produces isotopes in the mass range $A = 25$ to 32, with a strong peak at Si^{28}. The two curves show the lower and upper limits.

The full chain of thermonuclear reactions does not occur in all stars. For a star of given mass, there is a maximum central temperature attainable in a non-degenerate core. The exclusion principle requires that the average separation of particles be greater than the electron wavelength:

$$\bar{r} = \left(\frac{m_p}{\rho}\right)^{1/3} > \lambda_e = \frac{\hbar}{(2m_e kT)^{1/2}} \tag{2.10}$$

where $\bar{r}$ is the size of a cube containing one proton and $\lambda_e = \hbar/P$ and $P = (2m_e kT)^{1/2}$. Using expressions (1.13) and (2.4) for $\bar{\rho}$ and T, we must have

$$1 > \frac{\hbar\bar{\rho}^{1/3}}{m_p^{1/3}(2m_e kT)^{1/2}} = 0.0914\mu^{-1/2}\left(\frac{M_\odot}{M}\right)^{1/6}\left(\frac{R_\odot}{R}\right)^{1/2}$$

Thus, the condition for nondegeneracy requires

$$\left(\frac{R}{R_\odot}\right) > 8.36 \times 10^{-3}\mu^{-1}\left(\frac{M_\odot}{M}\right)^{1/3} \tag{2.11}$$

The necessary central temperature for hydrogen-burning is 10^7 °K, so that mass and radius must satisfy the condition, from (2.4),

$$\mu\left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right) \geq \frac{10^7}{4.61 \times 10^6} = 2.16 \tag{2.12}$$

Combining these two requirements, equations (2.11) and (2.12), the minimum mass of a star that can burn hydrogen is

$$\mu^{3/2}\frac{M}{M_\odot} \geq 0.05 \tag{2.13}$$

For helium-burning, the central temperature must be 10^8 °K. The maximum central temperature occurs when the hydrogen-burning shell has burnt its way almost to the surface, so we can treat the core as a homogeneous star. The minimum mass for helium-burning is thus

$$\mu_c^{3/2}\frac{M}{M_\odot} \geq 0.28 \quad \text{or} \quad \frac{M}{M_\odot} \geq 0.18 \tag{2.14}$$

The necessary central temperature for carbon-burning is about 7×10^8 °K, so the minimum mass for carbon-burning is

$$\mu_c^{3/2}\frac{M}{M_\odot} \geq 1.2 \tag{2.15}$$

The necessary central temperature for neon- and oxygen-burning is 1.3×10^9 °K, so the minimum mass for neon- and oxygen-burning is

$$\mu_c^{4/3}\frac{M}{M_\odot} \geq 1.9 \tag{2.16}$$

Oxygen- and neon-burning are the end point of thermonuclear burning stages. Nuclear reactions among larger-mass nuclei (further photodisintegrations and recombinations) would occur in the temperature range of 2 to 4×10^9 °K. However, at these temperatures, the rate of energy dissipation by neutrinos (which are produced in the core and escape directly from the star) is so large that further nuclear reactions are unable to halt the gravitational contraction, but can merely slow it down. These reactions can, however, produce nuclei all the way up to Fe^{56}, and the temperature is high enough to produce statistical equilibrium among the various nuclei.

EARLY STAGES OF EVOLUTION—HOMOGENEOUS STARS

Hydrostatic equilibrium and overall energy conservation determine the evolution of central stellar conditions. For more of the details of evolution, including the star's radius and luminosity, the mode of energy transport from the interior to the surface must also be considered.

The equations of stellar structure—mass conservation, hydrostatic equilibrium, energy conservation, and energy transport—form a system of nonlinear differential equations which must be integrated numerically. It is possible, however, to obtain crude analytic stellar models by separating the condition of hydrostatic equilibrium from the energy transport. In the previous section, the condition of overall hydrostatic equilibrium was expressed by the virial theorem. Now, since a more detailed stellar model is desired, we assume an analytic density distribution, namely, that the density in a star varies linearly from the center to the surface (Cameron, 1963). It is then possible to integrate the equations of mass conservation, hydrostatic equilibrium, and energy generation through the star. Hence, together with the equation of state of an ideal gas, the run of density, mass, pressure, temperature, and luminosity through the star is determined. Also, the central density, pressure and temperature, and the total rate of energy generation are determined as a function of the star's mass and radius. Finally, the different modes of energy transport—radiative transport with Kramer's or electron-scattering opacity and convective transport—are considered. The energy-transport equation can be satisfied at only one typical point of the star because of the approximation made in assuming a given density distribution. This gives a mass–luminosity–radius relation which gives the evolutionary track of the star in the Hertzsprung–Russell diagram.

To summarize: Hydrostatic equilibrium and energy conservation determine the changes in the central stellar conditions, including the mode of energy transport, which gives the changes in the surface conditions—the track in the Hertzsprung–Russell diagram.

Linear Stellar Model

Assume that the density in a star varies linearly from the center to the surface:

$$\rho(r) = \rho_c\left(1 - \frac{r}{R}\right) \tag{3.1}$$

where R is the radius of the star. We call this a "linear star model." The equations

of hydrostatic equilibrium and energy generation can now be integrated, but the energy-transport equation can only be satisfied at one point in the star. The mass distribution is [from equation (1.4)]

$$\begin{aligned} M(r) &= \int_0^r 4\pi r^2 \rho(r)\, dr \\ &= \frac{4\pi}{3}\rho_c r^3\left(1 - \frac{3}{4}\frac{r}{R}\right) \end{aligned} \tag{3.2}$$

Hence

$$M(R) = \tfrac{1}{3}\pi\rho_c R^3$$

Thus, the central density is

$$\begin{aligned} \rho_c &= \frac{3M}{\pi R^3} \\ &= 5.64\left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)^3 \text{ g/cm}^3 \end{aligned} \tag{3.3}$$

The pressure is obtained from the equation of hydrostatic equilibrium (1.1)

$$P = P_c - \int_0^r \frac{GM(r)\rho(r)\, dr}{r^2}$$

where P_c is the pressure at the center. Hence

$$P = P_c - \frac{2\pi}{3}G\rho_c^2 r^2\left(1 - \frac{7}{6}\frac{r}{R} + \frac{3}{8}\frac{r^2}{R^2}\right)$$

Applying the boundary condition $P(R) = 0$, we get

$$P = \frac{\pi}{36}G\rho_c^2 R^2\left(5 - \frac{24r^2}{R^2} + \frac{28r^3}{R^3} - \frac{9r^4}{R^4}\right) \tag{3.4a}$$

and the central pressure is

$$P_c = \frac{5\pi}{36}G\rho_c^2 R^2 = 4.44 \times 10^{15}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R_\odot}{R}\right)^4 \text{ dyn/cm}^2 \tag{3.4b}$$

Assume that the radiation pressure is negligible; the temperature is then given by the perfect gas law, equation (1.2),

$$T = \frac{\mu H}{k}\frac{P}{\rho} = \frac{\pi}{36}\frac{G\mu}{kN_0}\rho_c R^2\left(5 + \frac{5r}{R} - \frac{19r^2}{R^2} + \frac{9r^3}{R^3}\right) \tag{3.5a}$$

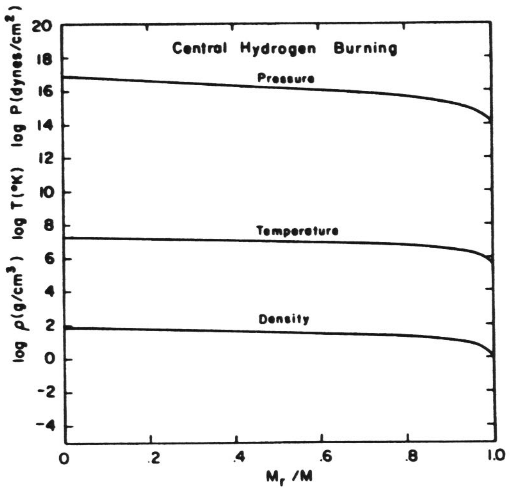

Figure 3.

and the central temperature is

$$T_c = \frac{5\pi}{36}\frac{G\mu H}{k}\rho_c R^2 = 9.62 \times 10^6 \mu\left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right) {}^\circ\mathrm{K} \qquad (3.5b)$$

We now know how the density, temperature, and pressure vary throughout the interior of this linear star model. We have satisfied the condition of hydrostatic equilibrium. The run of pressure, temperature, and density through the star is shown in Figure 3.

We must now consider the condition of energy conservation. The rate of thermonuclear energy generation can be expressed in the form (Reeves, 1965)

$$\mathscr{E} = \mathscr{E}_0 \rho^k \left(\frac{T}{T_0}\right)^n \text{ ergs/g-sec}$$

The total net rate of energy generation is equal to the luminosity of the star (equation 1.6)

$$L = \int_0^R 4\pi\rho(r)\mathscr{E} r^2\, dr$$

For a linear density distribution,

$$L = 4\pi R^3 \mathscr{E}_0 \rho_c^2 \left(\frac{T_c}{T_0}\right)^n I_n$$

$$= \frac{36}{\pi}\mathscr{E}_0\left(\frac{5}{12}\frac{G\mu H}{k}\frac{1}{T_0}\right)^n \frac{M^{1+k+n}}{R^{3k+n}} I_n \qquad (3.6)$$

k+1 = 2 for 2-body reactions (eg. p-p, CNO; not 3α)

where

$$I_n = \int_0^1 x^2(1-x)^{n+k+1}(1+2x-1.8x^2)^n\, dx$$

k+1

is the integral of $[\rho(r)/\rho_c]^2[T(r)/T_c]^n$ over the star and has values of the order 10^{-1} or 10^{-2}. Thus,

$$\frac{L}{L_\odot} = 35.58\mathscr{E}_0 I_n\left[\frac{0.962}{T_{0(7)}}\right]^n \mu^n \left(\frac{M}{M_\odot}\right)^{n+k+1}\left(\frac{R_\odot}{R}\right)^{n+3k} \qquad (3.7)$$

Here, $T_{0(7)}$ is the temperature in units of 10^7 °K. The energy generation and luminosity in a $1M_\odot$ star are shown in Figure 4.

Radiative Energy Transport

Finally, consider the equation that governs the flow of energy through the star. First consider radiative energy transport, equation (1.8):

$$L_r = -4\pi r^2 \frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{dT}{dr}$$

so the temperature gradient necessary to drive the radiative flux through the star is

$$\frac{dT}{dr} = -\frac{3\kappa}{4ac}\frac{\rho}{T^3}\frac{L_r}{4\pi r^2}$$

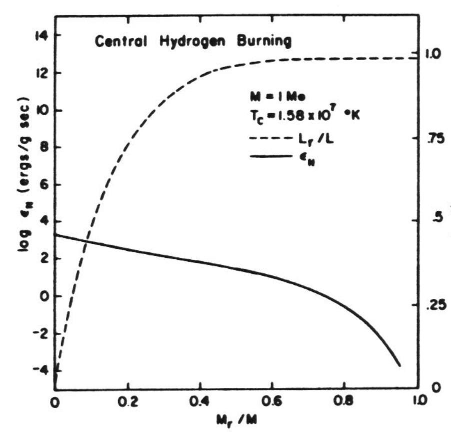

Figure 4.

We consider two types of opacity (Cameron, 1965, and Cox, 1965):

(1) Kramer's opacity

$$\varkappa = \varkappa_0 \rho T^{-3.5} \tag{3.8}$$
$$\varkappa_0 = 4.34 \times 10^{25}\left(\frac{\bar{g}_{bf}}{t}\right)Z(1 + X) + 3.68 \times 10^{22}\bar{g}_{ff}(1 + X)(X + Y)$$

where X, Y, Z are the mass fractions of hydrogen, helium, and all the heavier elements, respectively. The first term is the bound–free and the second the free–free absorption. We take $(t/\bar{g}_{bf}) = 3$ and $\bar{g}_{ff} = 1$. Kramer's opacity is a good approximation at intermediate internal temperatures.

(2) Electron-scattering opacity

$$\varkappa = \varkappa_e = 0.20(1 + X) \tag{3.9}$$

which is dominant at high internal temperatures. We also assume for convenience that all the energy is generated at the center of the star, so that

$$L_r = L = \text{constant}$$

The temperature gradient for Kramer's opacity is

$$\frac{dT}{dr} = \frac{3\varkappa_0}{4ac}\frac{\rho^2}{T^{6.5}}\frac{L}{4\pi r^2} \tag{3.10}$$

Compare this expression for dT/dr with the radial derivative of T from the linear model

$$\frac{dT}{dr} = \frac{\pi}{36}\frac{G\mu\rho_c}{kN_0}R\left(5 - 38\frac{r}{R} + 27\frac{r^2}{R^2}\right) \tag{3.11}$$

For our analytic model, these two expressions for the temperature gradient cannot be equal throughout the star. We determine the luminosity by equating the above two expressions at $r = 0.5R$:

$$L = -4\pi r_{1/2}^2\frac{4ac}{3\varkappa_0}\frac{T_{1/2}^{6.5}}{\rho_{1/2}^2}\left(\frac{dT}{dr}\right)_{1/2}$$

where $r_{1/2} = 0.5R$

$$T_{1/2} = \frac{31}{288}\pi\frac{G\mu\rho_c}{kN_0}R^2$$
$$\rho_{1/2} = 0.5\rho_c$$
$$\left(\frac{dT}{dr}\right)_{1/2} = \frac{-29\pi}{144}\frac{G\mu}{kN_0}\rho_c R$$

Now

$$\rho_c = \frac{3M}{\pi R^3}$$

Thus, when Kramer's opacity dominates,

$$L = \pi^3\frac{29}{81}\left(\frac{31}{96}\right)^{6.5}\frac{ac}{\varkappa_0}\left(\frac{GH}{k}\right)^{7.5}\mu^{7.5}\frac{M^{5.5}}{R^{0.5}} \tag{3.12}$$

or

$$\frac{L}{L_\odot} = \frac{0.988}{(1 + X)[Z + 2.54 \times 10^{-3}(1 - Z)]}\mu^{7.5}\left(\frac{M}{M_\odot}\right)^{5.5}\left(\frac{R}{R_\odot}\right)^{-0.5} \tag{3.13}$$

Solar matter is approximately two-thirds hydrogen and one-third helium by weight. The mean molecular weight for twelve nucleons, of which eight are hydrogen atoms and one a helium atom, is

$$\mu = \frac{\text{Mass}}{\text{Number of particles}} = \frac{8 \times 1 + 1 \times 4}{8 \times 2 + 1 \times 3}$$
$$= \frac{12}{19} = 0.632$$

0.66, using his X&Y.

$$\mu^{7.5} = 0.0320$$

Hence, for solar mass and radius,

$$\frac{L}{L_\odot} = 0.915$$

= 1.2

Thus, the linear star model gives a result which is within 10% of the observed value. The luminosity increases rapidly with the mass of the star and increases slightly with decreasing radius.

When electron scattering is the dominant opacity, the temperature gradient needed to transport the energy flux L is

$$\frac{dT}{dr} = -\frac{3\varkappa_e}{4ac}\frac{\rho}{T^3}\frac{L}{4\pi r^2} \tag{3.14}$$

Determining the luminosity by equating this temperature gradient with the expression for dT/dr obtained in the linear model (3.11) at the midpoint $r = 0.5R$ gives

$$L = -4\pi r_{1/2}^2\frac{4ac}{3\varkappa_e}\frac{T_{1/2}^3}{\rho_{1/2}}\left(\frac{dT}{dr}\right)_{1/2}$$

which is

$$L = \frac{29}{2}\pi^2\left(\frac{31}{288}\right)^3\left(\frac{GH}{k}\right)^4\frac{ac}{\varkappa_e}\mu^4M^3 \tag{3.15}$$

Hence, when electron scattering dominates,

$$\frac{L}{L_\odot} = \frac{179}{1 + X}\mu^4\left(\frac{M}{M_\odot}\right)^3 \tag{3.16}$$

The luminosity is independent of the radius and increases with mass, although less sensitively than for Kramer's opacity.

Equations (3.13) and (3.16) are the radiative mass–luminosity–radius relations for Kramer's and electron-scattering opacity. The effective surface temperature is defined by

$$\text{flux} = \sigma T_{\text{eff}}^4$$

or

$$T_e = \left(\frac{L}{4\pi\sigma R^2}\right)^{\frac{1}{4}} = 5.76 \times 10^3 \left(\frac{L}{L_\odot}\right)^{\frac{1}{4}} \left(\frac{R_\odot}{R}\right)^{\frac{1}{2}} \,{}^\circ\mathrm{K} \tag{3.17}$$

Convective Energy Transport

Convection is an extremely efficient mode of energy transport. Therefore, in a convectively unstable region the entire energy flux can be transferred with only negligible adjustment in the super-adiabatic gradient: $-[(dT/dr) - (dT/dr)_{ad}]$. The energy flux is thus determined by the boundary layer of the convective region (Spiegel, 1965).

If the star has a substantial region with radiative transport, that region will determine the energy flux. If, however, the stellar interior is completely convective, the boundary layer determining the flux is the thin radiative photosphere surrounding the convective zone, where the energy must be transported by radiation since the material is becoming optically thin. The luminosity of the star is then determined by the temperature of the gas at the point from which photons can escape from the star,

$$L = 4\pi R^2 \sigma T_e^4$$

where T_e is the effective surface temperature of the star.

The depth in the star from which photons can escape nearly coincides with the transition point between the radiative and convective regions and occurs at an optical depth of about $\frac{2}{3}$. The radiative temperature gradient drops rapidly as the density decreases, so the temperature is practically constant from this point outward. We thus assume an isothermal photosphere and take the effective temperature as the temperature at the transition point between the convective and radiative regions (Hoyle and Schwarzschild, 1955, and Hayashi, Hoshi, and Sugimoto, 1962).

We assume an opacity law of the form

$$\varkappa = \varkappa_0 P^a T^b$$

Then, since the bottom of the photosphere is at an optical depth $\frac{2}{3}$,

$$\tau = \int_{r_{ph}}^{\infty} \varkappa\rho\, dr = \tfrac{2}{3} = \varkappa_0 T_e^b \int_{r_{ph}}^{\infty} P^a \rho\, dr$$

and from equation (1.1)

$$\rho = -\frac{1}{g}\frac{dP}{dr}$$

so

$$\tfrac{2}{3} = -\varkappa_0 T_e^b \frac{1}{g} \int_{r_{ph}}^{\infty} P^a\, dP = \frac{\varkappa_0 T_e^b P_{ph}^{a+1}}{(a+1)g}$$

where P_{ph} is the pressure at the bottom of the photosphere. Thus, one relation between the temperature and pressure (or density) at the bottom of the photosphere is

$$T_e^b P_{ph}^{a+1} = \tfrac{2}{3}(a+1)\frac{GM}{\varkappa_0 R^2} \tag{3.18}$$

This relation is the boundary condition for the star:

$$P \rightarrow P_{ph} = \tfrac{2}{3}(a+1)\frac{g}{\varkappa_{ph}} \quad \text{as} \quad T \rightarrow T_{eff}$$

This condition is just that the photon mean free path $(\varkappa\rho)^{-1}$ equals the scale height $P/\rho g$ at the boundary so that the radiation can escape from the star at the effective temperature.

A second condition on T_e and P_{ph} can be obtained from the condition for the boundary of the convective zone, namely,

$$F_C = F_R$$

In the expression for the convective flux (1.10), approximate the velocity w by half the sound velocity

$$c = \left(\frac{\gamma k T}{\mu H}\right)^{\frac{1}{2}}$$

since c is an upper limit to the velocity. Also, approximate $\rho c_p \theta$ by γ times the internal energy

$$U = \tfrac{3}{2}kT\frac{\rho}{\mu H} = \tfrac{3}{2}P$$

Then the convective flux is

$$F_C = \tfrac{1}{2}\rho c_p w\theta \approx \tfrac{1}{2}\gamma\frac{c}{2}U = \frac{3\gamma}{8}\left(\gamma\frac{k}{\mu H}\right)^{\frac{1}{2}} P T^{\frac{1}{2}} \tag{3.19}$$

The radiative flux is

$$F_R = \sigma T_e^4 \tag{3.20}$$

Thus, equating (3.19) and (3.20), the transition point is given by

$$P_{ph} = \frac{8}{3\gamma}\left(\frac{\mu H}{\gamma k}\right)^{\frac{1}{2}} \sigma T_e^{3.5} \tag{3.21}$$

The conditions (3.18) and (3.21) can be combined to determine the effective temperature, which is

$$T_e = \left[\tfrac{2}{3}(1+a)\left(\frac{3\gamma}{8}\right)^{1+a} \frac{G}{\sigma^{1+a}\varkappa_0}\left(\frac{\gamma k}{H}\right)^{(1+a)/2} \mu^{-(1+a)/2}\left(\frac{M}{R^2}\right)\right]^{1/[b+3.5(1+a)]} \tag{3.22}$$

In the outer layers of stars, the opacity is due primarily to H^- and is an increasing function of pressure and temperature, so $a, b > 0$. The H^- opacity is very temperature sensitive, so b is large. Thus, T_{eff} is nearly constant; it increases slightly with increasing mass and decreases slightly with increasing radius.

The approximate power law form for the opacity obtained from the detailed opacity calculations in the region about 3500 °K is as follows:

For population I stars ($X = 0.6$, $Y = 0.38$, $Z = 0.02$)

$$\varkappa = 6.9 \times 10^{-26} P^{0.7} T^{5.3}$$

For population II stars ($X = 0.9$, $Y = 0.099$, $Z = 0.001$)

$$\varkappa = 6.1 \times 10^{-40} P^{0.6} T^{9.4}$$

where X, Y, Z are the mass fractions of hydrogen, helium, and all the heavier elements, respectively. The luminosity is found by inverting equation (3.17):

$$\frac{L}{L_\odot} = \frac{4\pi R^2 \sigma T_e^4}{L_\odot} = \left(\frac{T_e}{5.76 \times 10^3}\right)^4 \left(\frac{R}{R_\odot}\right)^2 \tag{3.23}$$

Then the effective temperature and luminosity are:

For population I

$$T_e = 7.5 \times 10^3 \left(\frac{M}{M_\odot}\right)^{0.089} \left(\frac{R}{R_\odot}\right)^{-0.178}$$

$$\frac{L}{L_\odot} = 2.86 \left(\frac{M}{M_\odot}\right)^{0.356} \left(\frac{R}{R_\odot}\right)^{1.288} \tag{3.24a}$$

For population II

$$T_e = 6.19 \times 10^3 \left(\frac{M}{M_\odot}\right)^{0.0666} \left(\frac{R}{R_\odot}\right)^{-0.133}$$

$$\frac{L}{L_\odot} = 1.34 \left(\frac{M}{M_\odot}\right)^{0.2665} \left(\frac{R}{R_\odot}\right)^{1.466} \tag{3.24b}$$

The effective temperature is less sensitive to the radius for population II than for population I stars because the opacity is more sensitive to temperature. In population II stars, there are fewer metals with low ionization potentials to provide electrons to form H^-. The electrons must now come partly from the ionization of hydrogen which has a high ionization potential, so the electron pressure will be very temperature sensitive.

In stars with high surface density, the relation (3.21) between the pressure and temperature at the bottom of the photosphere is not valid, because in deriving it from the boundary condition $F_C = F_R$ we evaluated the convective flux by assuming that the temperature fluctuation is of the order of magnitude of the temperature itself. This assumption is valid only in stars where convection is inefficient near the surface due to low density and large radiative losses from the convective elements. In stars with high surface density, convection is very efficient and the temperature gradient in the convective region is nearly adiabatic throughout. In this case, the temperature fluctuations are much smaller than the order of magnitude of the temperature itself.

For stars with high surface density, we therefore go to the opposite extreme from the low surface density case and assume that the temperature gradient is adiabatic throughout the convective zone. We may then use the adiabatic relation between pressure and temperature. In the interior

$$P = KT^{\gamma/(\gamma-1)} = KT^{2.5}$$

since (neglecting radiation pressure) $\gamma = \frac{5}{3}$, except in the hydrogen-ionization zone. For a fully convective star

$$K = \text{constant} = \frac{P_c}{T_c^{2.5}}$$

From the linear model (3.4) and (3.5)

$$P_c = \frac{5}{4\pi} \frac{GM^2}{R^4}$$

$$T_c = \frac{5}{12} \frac{G\mu H}{k} \frac{M}{R}$$

thus

$$K = \frac{5}{4\pi}\left(\frac{12}{5}\right)^{2.5} \left(\frac{k}{\mu H}\right)^{2.5} G^{-1.5} M^{-0.5} R^{-1.5}$$

$$= 1.53 \times 10^{-2} \mu^{-2.5} \left(\frac{M}{M_\odot}\right)^{-0.5} \left(\frac{R}{R_\odot}\right)^{-1.5}$$

In particular, the above relation holds at the bottom of the hydrogen-ionization zone.

If we neglect the effect of hydrogen ionization, which reduces γ, then at the boundary between the convective zone and the photosphere

$$P_{ph} = KT_e^{2.5}$$

with the same K as for the interior. This relation, combined with the optical depth condition from equation (3.18), gives the effective temperature

$$T_e = \left[\tfrac{2}{3}(1 + a)\frac{GM}{\varkappa_0 R^2} K^{-(1+a)}\right]^{1/[b+2.5(1+a)]}$$

$$= \left\{\tfrac{2}{3} \frac{1+a}{\varkappa_0} \left[\frac{4\pi}{5}\left(\frac{5}{12}\right)^{2.5} \left(\frac{H}{k}\right)^{2.5}\right]^{1+a} G^{2.5+1.5a} \times \mu^{2.5(1+a)} M^{1.5+0.5a} R^{1.5a-0.5}\right\}^{1/[b+2.5(1+a)]} \tag{3.25}$$

For population I

$$T_e = 2.18 \times 10^3 \left(\frac{M}{M_\odot}\right)^{0.194} \left(\frac{R}{R_\odot}\right)^{0.0576}$$

$$\frac{L}{L_\odot} = 0.02 \left(\frac{M}{M_\odot}\right)^{0.776} \left(\frac{R}{R_\odot}\right)^{2.23}$$

For population II

$$T_e = 2.5 \times 10^3 \left(\frac{M}{M_\odot}\right)^{0.1715} \left(\frac{R}{R_\odot}\right)^{0.0298}$$

$$\frac{L}{L_\odot} = 0.035 \left(\frac{M}{M_\odot}\right)^{0.686} \left(\frac{R}{R_\odot}\right)^{2.119}$$

The hydrogen ionization can, however, be treated exactly and we can relate $K_e = P_{ph}/T_e^{2.5}$ at the top of the hydrogen-ionization zone to

$$K = P_b/T_b^{2.5} = P_c/T_c^{2.5}$$

at its bottom. The effect of the ionization zone is to decrease

$$\frac{d \ln T}{d \ln P} = \frac{\Gamma - 1}{\Gamma}$$

so that the temperature will decrease less than the pressure going outward through the ionization zone. Then $K_e < K$ and T_{eff} will be increased. Since the temperature varies adiabatically through the ionization zone, the entropy is constant across it. The entropy per unit mass is

$$s = \frac{Xk}{H}\left[\tfrac{5}{2}(1 + x + \delta) + \frac{\chi}{kT} + \ln\left(\frac{2\pi H}{h^2}\right)^{3/2} + \delta \ln\left(\frac{8\pi H}{h^2}\right)\right.$$

$$\left. + x \ln\left(\frac{2\pi m_e}{h^2}\right)^{3/2} + (1 + x + \delta) \ln \frac{(kT)^{5/2}(1 + x + \delta)}{P}\right] \tag{3.26}$$

where χ is the ionization energy of hydrogen, $\delta = Y/4X$, and x is the fraction of hydrogen ionized. Evaluating $s =$ constant above and below the hydrogen-ionization zone, that is, for $x = 0$ and $x = 1$, respectively, gives

$$\frac{P_{ph}}{T_e^{2.5}} = K_e = (1 + \delta)\left[\left(\frac{2\pi m_e}{h^2}\right)^{3/2} (ke)^{5/2}\right]^{-1/(1+\delta)} \left(\frac{K}{2 + \delta}\right)^{(2+\delta)/(1+\delta)}$$

Thus, the effective temperature is

$$T_{eff} = \left[\tfrac{2}{3}\frac{1 + a}{\varkappa_0}\left\{\frac{1}{1 + \delta}\left[\left(\frac{2\pi m_e^2}{h^2}\right)^{3/2} (ke)^{5/2}\right]^{1/(1+\delta)} \left[\frac{4\pi(2 + \delta)}{5}(\tfrac{5}{12})^{3/2}\left(\frac{H}{k}\right)^{5/2}\right]^{(2+\delta)/(1+\delta)}\right\}^{1+a}\right.$$

$$\times G^{1 - 1.5(1+a)[(2+\delta)/(1+\delta)]} \mu^{2.5(1+a)[(2+\delta)/(1+\delta)]} M^{1 + 0.5(1+a)[(2+\delta)/(1+\delta)]}$$

$$\left. \times R^{1.5(1+a)[(2+\delta)/(1+\delta)] - 2}\right]^{1/[b + 2.5(1+a)]} \tag{3.27}$$

Again, in the high surface density as in the low surface density case, the effective temperature is very insensitive to mass and radius.

For population I

$$T_e = 5.65 \times 10^3 \left(\frac{M}{M_\odot}\right)^{0.27} \left(\frac{R}{R_\odot}\right)^{0.288}$$

$$\frac{L}{L_\odot} = 0.93 \left(\frac{M}{M_\odot}\right)^{1.08} \left(\frac{R}{R_\odot}\right)^{3.15} \tag{3.28a}$$

For population II

$$T_e = 4.63 \times 10^3 \left(\frac{M}{M_\odot}\right)^{0.1925} \left(\frac{R}{R_\odot}\right)^{0.204}$$

$$\frac{L}{L_\odot} = 0.42 \left(\frac{M}{M_\odot}\right)^{0.77} \left(\frac{R}{R_\odot}\right)^{2.816} \tag{3.28b}$$

Evolutionary Tracks

The evolutionary tracks of stars in the Hertzsprung–Russell diagram depend on the mode of energy transport, which determines the mass–luminosity–radius relation. For fully convective stars, the luminosity is determined by the surface condition. Since the opacity is very temperature sensitive, the effective temperature is nearly constant, independent of the radius, and the track in the Hertzsprung–Russell diagram is a nearly vertical line. The mass–luminosity–radius relations for a fully convective star are given by equations (3.24) and (3.28), so the track in the H–R diagram will be:

For population I ($X = 0.6$, $Y = 0.38$, $Z = 0.02$)

$$\log\left(\frac{L}{L_\odot}\right) = -7.236 \log\left(\frac{T_e}{8.65 \times 10^3}\right) + \log\left(\frac{M}{M_\odot}\right) \quad \text{(Low surface density)}$$

$$\log\left(\frac{L}{L_\odot}\right) = 10.94 \log\left(\frac{T_e}{5.69 \times 10^3}\right) - 1.874 \log\left(\frac{M}{M_\odot}\right) \quad \text{(High surface density)} \tag{3.29a}$$

For population I ($X = 0.6$, $Y = 0.38$, $Z = 0.02$)

$$\log\left(\frac{L}{L_\odot}\right) = -11 \log\left(\frac{T_e}{6.19 \times 10^3}\right) + \log\left(\frac{M}{M_\odot}\right) \quad \text{(Low surface density)}$$

$$\log\left(\frac{L}{L_\odot}\right) = 13.8 \log\left(\frac{T_e}{4.93 \times 10^3}\right) - 1.887 \log\left(\frac{M}{M_\odot}\right) \quad \text{(High surface density)} \tag{3.29b}$$

For stars with radiative energy transport, the flux [equation (1.8)] is

$$L \propto r^2 \frac{T^3}{\varkappa\rho}\frac{dT}{dr} \propto \frac{R^2}{\bar{\varkappa}}\frac{T_c}{R}$$

since T^3/ρ is approximately constant. The opacity in the interior of a star is nearly constant, so the luminosity is approximately proportional to RT_c, which is constant, independent of the radius. Thus the track in the Hertzsprung–Russell diagram is a nearly horizontal line. As the temperature rises, Kramer's opacity

$$\varkappa \propto \rho T^{-3.5} \propto T^{-0.5}$$

decreases, and the luminosity increases slightly. The mass–luminosity–radius relation for a radiative star with Kramer's opacity is given by equation (3.13) and the path in the H–R diagram is

$$\log\left(\frac{L}{L_\odot}\right) = 0.8 \log\left(\frac{T_e}{5.83 \times 10^3}\right) + 4.4 \log\left(\frac{M}{M_\odot}\right) + 6 \log \mu$$
$$-0.8 \log[(1 + X)\{Z + 2.54 \times 10^{-3}(1 - Z)\}] \qquad (3.30)$$

If the central temperature becomes very high and the central density is low, the dominant opacity is due to electron scattering. Electron-scattering opacity is independent of temperature and density, so the luminosity is constant. The mass–luminosity–radius relation is given by equation (3.16):

$$\frac{L}{L_\odot} = \frac{179}{1 + X}\mu^4\left(\frac{M}{M_\odot}\right)^3$$
$$T_e = 2.14 \times 10^4(1 \times X)^{-1/4}\mu\left(\frac{M}{M_\odot}\right)^{3/4}\left(\frac{R}{R_\odot}\right)^{-1/2} \qquad (3.31)$$

The changes in the stellar radius depend on the sources of energy and the internal structure of the star.

PRE-MAIN-SEQUENCE CONTRACTION PHASE

The linear stellar model is now applied to the pre-main-sequence contraction stage of evolution. A star is formed from a condensation of the interstellar gas that is dense enough to become opaque to its own radiation. Then, as the gas contracts, its temperature will rise. As the temperature rises, the gas, composed predominantly of hydrogen and helium, is ionized. Much energy is necessary to ionize the gas, which means that the temperature cannot rise much above 10^4 °K until the hydrogen is ionized. The ionization of the hydrogen and helium leads to gravitational instability, since the energy released by the contraction does not increase the kinetic energy per particle (the temperature) but goes into the ionization energy of the atoms. Hence, the contraction of the gas does not raise the pressure sufficiently to permit the gas to remain in hydrostatic equilibrium; the ratio of specific heats γ falls below $\frac{4}{3}$, and the collapse must continue.

A stable star is not formed until its major constituents (hydrogen and helium) are ionized throughout most of the gas fragment. The energy necessary for ionization comes from gravitational potential-energy release (Truran, 1964). Let I be the total dissociation plus ionization energy per gram of stellar material. The gravitational potential energy is $\Omega \approx -GM^2/R$. From the virial theorem, half the gravitational energy released goes into thermal energy. Thus,

$$IM = \frac{1}{2}\frac{GM^2}{R}$$

so

$$R_{max} = \frac{1}{2}\frac{GM}{I}$$

The dissociation plus ionization energy is

$$I = N_0\left(\frac{X}{2}D_H + XE_H + \frac{Y}{4}E_{He}\right)$$

where $N_0 = 6.025 \times 10^{23}$ is the number of hydrogen atoms per gram; D_H is the dissociation energy of hydrogen molecules, $D_H = 4.476\,\text{eV} = 7.16 \times 10^{-22}$ ergs; E_H is the ionization energy of hydrogen, $E_H = 13.595\,\text{eV} = 21.75 \times 10^{-12}$ ergs; E_{He} is the ionization energy of helium, $E_{He} = 24.581 + 54.403\,\text{eV} = 78.984\,\text{eV} = 1.26 \times 10^{-10}$ ergs. Since $Y = 1 - X$

$$I = 1.9 \times 10^{13}(1 - 0.3X)\,\text{ergs/g}$$

The maximum radius of a stable star is thus

$$\left(\frac{R}{R_\odot}\right)_{max} = \frac{50.3}{1 - 0.3X}\left(\frac{M}{M_\odot}\right) \qquad (3.32)$$

Such a marginally stable star is the starting point of stellar evolution.

When a star becomes stable, its internal temperatures are of the order of 10^5 °K, and the opacity is so high that the radiative transport of energy is impeded. Further, there are extensive ionization zones which increase the specific heat and reduce γ to less than $\frac{5}{3}$ throughout large regions of the star. Thus the adiabatic gradient

$$\left(\frac{dT}{dr}\right)_{ad} = -\frac{\gamma - 1}{\gamma}\frac{\mu H}{k}g$$

will be small and the star will be unstable to convection throughout most of its interior. Its luminosity will then be determined by the surface conditions. For a given star, the rate of contraction is limited by the rate at which energy can be radiated away

$$L = -\frac{1}{2}\frac{d\Omega}{dt} \approx -\frac{1}{2}\frac{GM^2}{R}\left(\frac{1}{R}\frac{dR}{dt}\right)$$

so

$$-\frac{1}{R}\frac{dR}{dt} = \alpha\frac{LR}{GM^2}$$

where $\alpha \approx 1$. The equilibrium stellar structure is that with the highest contraction rate, i.e., with the highest luminosity, so the condition for a fully convective star is that the convective luminosity exceed the radiative luminosity. Initially, stars are fully convective. The effective temperature and luminosity of a marginally stable star, as given by the fully convective linear model for low surface density (3.24), are

$$\left.\begin{aligned} T_e &= 3.6 \times 10^3\left(\frac{M}{M_\odot}\right)^{-0.089} \\ \frac{L}{L_\odot} &= 5.71 \times 10^2\left(\frac{M}{M_\odot}\right)^{1.644} \end{aligned}\right\} \text{population I}$$

$$\left.\begin{aligned} T_e &= 3.52 \times 10^3\left(\frac{M}{M_\odot}\right)^{-0.0667} \\ \frac{L}{L_\odot} &= 5.91 \times 10^2\left(\frac{M}{M_\odot}\right)^{1.733} \end{aligned}\right\} \text{population II}$$

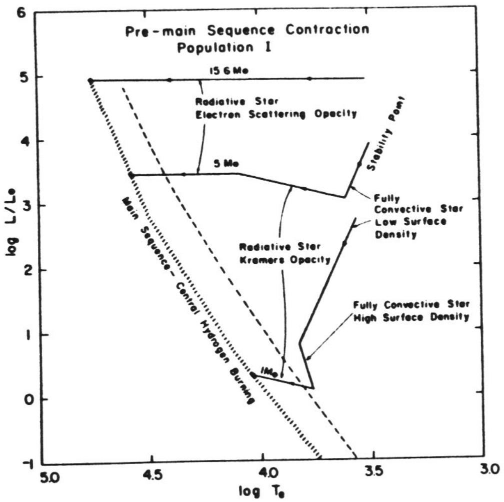

Figure 5. Hertzsprung–Russell diagram of the pre-main-sequence contraction evolutionary tracks and initial main sequence. The tracks are labeled with the type of energy transport determining the direction of that portion of the track. Dotted curve is observed main sequence (Hayashi, Hoshi, and Sugimoto, 1962, and Schwarzschild, 1958).

These relations give the starting point of a star's evolutionary track in the Hertzsprung–Russell diagram.

As a star contracts, when fully convective, the effective temperature is nearly constant. The tracks in the H–R diagram for fully convective stars follow equation (3.29). These tracks are shown in Figure 5. The tracks for population II stars are similar to, but slightly steeper than, those for population I stars.

As a star contracts, its central temperature increases according to (equation 3.5):

$$T_c \approx \frac{\mu H}{k}\frac{GM}{R}$$

The rising temperature increases the emission of radiation and reduces the opacity. A central core which is in radiative equilibrium will develop. When about half the star is in radiative equilibrium, it leaves the fully convective path. The luminosity of a radiative star is nearly constant, and the effective temperature varies as $T_e \sim L^{\frac{1}{4}}R^{-\frac{1}{2}}$. Thus, as the star contracts, the effective temperature rises and the star moves to the left in the H–R diagram at nearly constant luminosity. The track in the H–R diagram for a radiative star follows equation (3.30) for Kramer's opacity and equation (3.31) for electron-scattering opacity. Typical radiative tracks in the H–R diagram are shown in Figure 5.

Time Scale of Contraction

The luminosity of a star is the rate of change of total energy

$$L = \frac{\Delta E}{\Delta t} = -\frac{1}{2}\frac{\Delta\Omega}{\Delta t}$$

The gravitational energy is [from equation (1.19)]

$$-\Omega \approx \frac{GM^2}{R}$$

so

$$L = \frac{1}{2}\frac{GM^2}{R\Delta t}$$

Thus, the time scale of the contraction phase is

$$\Delta t = \frac{1}{2}\frac{GM^2}{LR} = 1.59 \times 10^7 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R_\odot}{R}\right)\left(\frac{L}{L_\odot}\right)^{-1} \text{years} \qquad (3.33)$$

Pre-main-sequence contraction times are listed in Table I.

CENTRAL HYDROGEN-BURNING

As a star contracts, its central temperature rises until it is high enough for hydrogen thermonuclear reactions to produce the energy radiated away from the star. At this point, the contraction stops and the star spends most of its lifetime

Table I. Evolutionary Time Scales (Years)

Mass ($M_\odot$)	Population	Pre-main-sequence contraction	Central hydrogen-burning	Hydrogen shell-burning	Central helium-burning
0.7	I	7×10^7	5×10^{10}	8×10^8	1×10^8
	II	8×10^7	5×10^{10}	5×10^9	6×10^7
1	I	2×10^7	9×10^9	4×10^8	5×10^7
	II	3×10^7	9×10^9	2×10^9	4×10^7
2	I	2×10^6	5×10^8	6×10^7	2×10^7
	II	3×10^6	7×10^8	2×10^8	2×10^7
5	I	1×10^5	3×10^7	4×10^6	1×10^7
	II	4×10^5	8×10^7	1×10^7	2×10^7
7	I	6×10^4	1×10^7	2×10^6	7×10^6
	II	2×10^5	4×10^7	7×10^6	8×10^6
10	I	3×10^4	8×10^6	1×10^6	3×10^6
	II	1×10^5	2×10^7	3×10^6	3×10^6
15.6	I	1×10^4	3×10^6	5×10^5	1×10^6
	II	6×10^4	9×10^6	1×10^6	1×10^6

burning hydrogen into helium. The locus of luminosity versus effective surface temperature of such stars (burning hydrogen in their cores and still of nearly homogeneous composition) defines the main sequence in the Hertzsprung–Russell diagram.

The luminosity of a star is determined mainly by the thermal conductivity (radiative) of the stellar material. The central temperature is determined by the adjustment of the nuclear-energy generation to maintain mechanical and thermal equilibrium throughout the star. Nuclear-energy generation processes are very temperature sensitive, and thus nuclear-energy sources play the role of thermostats. The radius of the star depends on the temperature and the mass distribution.

The basic features of the structure of homogeneous stars can be determined by dimensional analysis. The dependence of the central temperature and density on chemical composition, mass, and radius is determined by the condition of hydrostatic equilibrium and the equation of state [from equations (1.15) and (1.13)]

$$T_c \propto \mu\beta \frac{M}{R} \qquad \rho_c \propto \frac{M}{R^3} \tag{3.34}$$

The luminosity and radius are then determined by the energy balance. The equation for radiative energy transport (1.8) is

$$L = -4\pi r^2 \frac{16\sigma}{3} \frac{T^3}{\varkappa\rho} \frac{dT}{dr}$$

Assuming an opacity law of the form

$$\varkappa = \varkappa_0 \rho^a T^b$$

the luminosity is

$$L \propto \varkappa_0^{-1} (\mu\beta)^{4-b} M^{3-a-b} R^{3a+b} \tag{3.35}$$

The rate of nuclear-energy generation is [equation (1.6)]

$$L = 4\pi \int \mathscr{E} \rho r^2 \, dr$$

Assuming the rate of nuclear-energy generation per gram has the form

$$\mathscr{E} = \mathscr{E}_0 \rho^k T^n$$

the total rate of energy generation is

$$L \propto \mathscr{E}_0 (\mu\beta)^n M^{1+k+n} R^{-3k-n} \tag{3.36}$$

When the rate of energy generation equals the rate of energy loss (luminosity), then the dependence of the radius, luminosity, and effective temperature on the mass and chemical composition is (Hayashi, Hoshi, and Sugimoto, 1962)

$$R \propto (\mathscr{E}_0 \varkappa_0)^{1/l} (\mu\beta)^{(n+b-4)/l} M^{(k+n+a+b-2)/l}$$
$$L \propto \varkappa_0^{-(n+3k)/l} \mathscr{E}_0^{(3a+b)/l} (\mu\beta)^{[n(4+3a)+3k(4-b)]/l} M^{[n(3+2a)+k(9-2b)+3a+b]/l} \tag{3.37}$$
$$T_e^4 \propto \varkappa_0^{-(n+3k-2)/l} \mathscr{E}_0^{(3a+b-2)/l} (\mu\beta)^{[n(2+3a)+3k(4-b)-2b+8]/l} M^{[n(1+2a)+k(7-2b)+a-b+4]/l}$$

where $l = n + 3k + 3a + b$ and $b \leq 0$ in the interior. The central temperature and density are

$$T_c \propto (\mathscr{E}_0 \varkappa_0)^{-1/l} (\mu\beta)^{(4+3k+3a)/l} M^{2(k+a+1)/l}$$
$$\rho_c \propto (\mathscr{E}_0 \varkappa_0)^{-3/l} (\mu\beta)^{-3(n+b-4)/l} M^{-2(n+b-3)/l} \tag{3.38}$$

Thus the radius, luminosity, effective temperature, and central temperature increase with mass, and the central density increases with mass for the *p-p* chain, $n = 4$, but decreases with mass for the CNO cycle, $n \approx 18$.

The main sequence is the locus of points in the luminosity–effective temperature diagram

$$\log\left(\frac{L}{L_\odot}\right) = 4\frac{n(3+2a) + k(9-2b) + 3a + b}{n(1+2a) + k(7-2b) + a - b + 4} \log T_e + \text{constant}$$
$$= 4\frac{5n + 15.5}{3n + 15.5} \log T_e + \text{constant} \qquad \text{(Kramer's)} \tag{3.39}$$
$$= 4\frac{3n + 9}{n + 11} \log T_e + \text{constant} \qquad \text{(electron scattering)}$$

The central temperature of a contracting star is

$$T_c = 9.62 \times 10^7 \mu \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)$$

Hydrogen burning starts at about $T_c = 8 \times 10^6$ °K. Thus, a star will start generating energy by nuclear reactions when its radius is

$$\frac{R}{R_\odot} = 1.2\mu \left(\frac{M}{M_\odot}\right) \tag{3.40}$$

Stars of small mass, $M \leq 2M_\odot$, burn hydrogen by the p-p chain at a temperature around 1.5×10^7 °K. The rate of energy generation is approximately

$$\mathscr{E} = \mathscr{E}_0 \rho \left(\frac{T}{1.5 \times 10^7}\right)^4 \text{ ergs/g-sec}$$
$$\mathscr{E}_0 = X_H^2$$

where X_H is the mass fraction of hydrogen. Massive stars, $M \gtrsim 2M_\odot$, burn hydrogen by the CNO cycle at a temperature of about 2×10^7 °K. The rate of energy generation is approximately

$$\mathscr{E} = \mathscr{E}_0 \rho \left(\frac{T}{2 \times 10^7}\right)^{18} \text{ ergs/g-sec}$$
$$\mathscr{E}_0 = 451 X_H X_{CNO}$$

where X_{CNO} is the mass fraction of C + N + O. The energy-generation rates for the linear model are [from equation (3.7)]

$$\frac{L}{L_\odot} = 4.98 \times 10^{-3} \mu^4 \left(\frac{M}{M_\odot}\right)^6 \left(\frac{R_\odot}{R}\right)^7 \qquad \text{(p-p chain)}$$
$$\frac{L}{L_\odot} = 1.157 \times 10^{-4} \mu^{14} \left(\frac{M}{M_\odot}\right)^{16} \left(\frac{R_\odot}{R}\right)^{17} \qquad \text{(CNO cycle)}$$

The properties of population I stars on the main sequence—burning hydrogen in their cores—as given by the linear model, are:

For the p-p chain and Kramer's opacity

$$\begin{aligned}
\frac{R}{R_\odot} &= 0.312\mu^{-0.538}\left(\frac{M}{M_\odot}\right)^{0.0769} \\
\frac{L}{L_\odot} &= 49.1\,\mu^{7.77}\left(\frac{M}{M_\odot}\right)^{5.46} \\
T_e &= 2.18 \times 10^4\,\mu^{2.21}\left(\frac{M}{M_\odot}\right)^{1.058} \\
\log\left(\frac{L}{L_\odot}\right) &= 5.16 \log T_e - 0.74 \log \mu - 20.7 \\
T_c &= 3.05 \times 10^7\,\mu^{1.54}\left(\frac{M}{M_\odot}\right)^{0.923} \\
\rho_c &= 186\,\mu^{1.615}\left(\frac{M}{M_\odot}\right)^{0.769}
\end{aligned} \tag{3.41}$$

For the CNO cycle and Kramer's opacity

$$\begin{aligned}
\frac{R}{R_\odot} &= 0.451\,\mu^{0.395}\left(\frac{M}{M_\odot}\right)^{0.697} \\
\frac{L}{L_\odot} &= 43.5\,\mu^{7.3}\left(\frac{M}{M_\odot}\right)^{5.18} \\
T_e &= 2.34 \times 10^4\,\mu^{1.63}\left(\frac{M}{M_\odot}\right)^{0.871} \\
\log\left(\frac{L}{L_\odot}\right) &= 5.948 \log T_e - 2.39 \log \mu - 24.36 \\
T_c &= 1.98 \times 10^7\,\mu^{0.606}\left(\frac{M}{M_\odot}\right)^{0.364} \\
\rho_c &= 65.8\,\mu^{-0.455}\left(\frac{M}{M_\odot}\right)^{-0.909}
\end{aligned} \tag{3.42}$$

Stars switch over from the p-p chain to the CNO cycle at a central temperature of about 2×10^7 °K, which occurs at a mass of about $M = 2M_\odot$. For the CNO cycle

and electron-scattering opacity,

$$\begin{aligned}
\frac{R}{R_\odot} &= 0.454\,\mu^{0.588}\left(\frac{M}{M_\odot}\right)^{0.765} \\
\frac{L}{L_\odot} &= 112\,\mu^{4}\left(\frac{M}{M_\odot}\right)^{3} \\
T_e &= 2.77 \times 10^4\,\mu^{0.706}\left(\frac{M}{M_\odot}\right)^{0.368} \\
\log\left(\frac{L}{L_\odot}\right) &= 8.16 \log T_e - 1.76 \log \mu - 34.15 \\
T_c &= 2.12 \times 10^7\,\mu^{0.412}\left(\frac{M}{M_\odot}\right)^{0.235} \\
\rho_c &= 60.3\,\mu^{-1.765}\left(\frac{M}{M_\odot}\right)^{-1.294}
\end{aligned} \tag{3.43}$$

Stars switch over from Kramer's to electron scattering as the dominant opacity in the deep interior, for mass $M > 3M_\odot$ for population I and $M > 2M_\odot$ for population II.

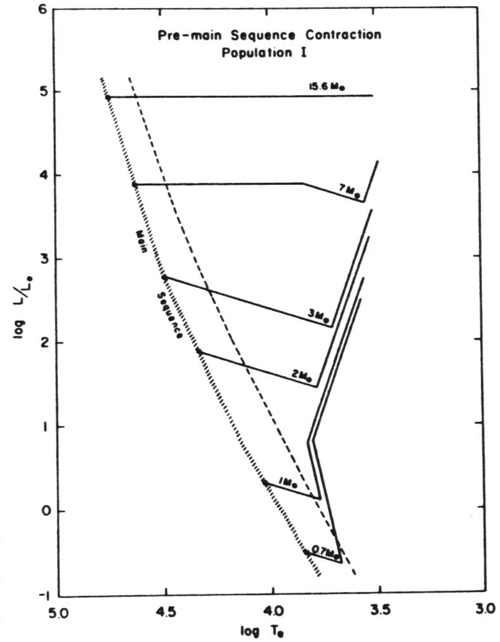

Figure 6. Evolutionary tracks of stars in H–R diagram during pre-main-sequence contraction. The main sequence is also shown. Dotted curve is observed main sequence.

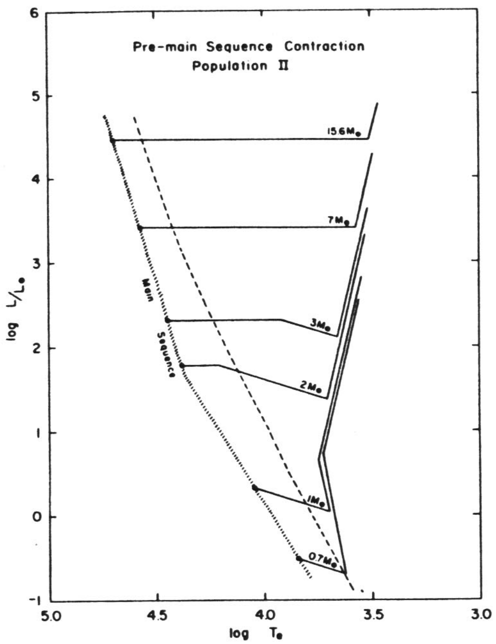

Figure 7. Evolutionary tracks of stars in H–R diagram during pre-main-sequence contraction. The main sequence is also shown. Dotted curve is observed main sequence.

The evolutionary tracks for different mass stars are shown in Figures 5 through 8. During the pre-main-sequence contraction, the stars contract to release gravitational potential energy to supply the radiative energy losses from the surface of the star. The radius of the star decreases. The direction of the track is determined by the mode of energy transport: convection with low surface density, convection with high surface density, radiation with electron-scattering opacity, or radiation with Kramer's opacity. Stars, when they first become stable, are fully convective, except for very massive stars $M > 12M_\odot$ (population I) and $M > 17M_\odot$ (population II). Inclusion of radiation pressure will, however, modify this result by increasing the convective instability. Stars become radiative when the radiative luminosity is greater than the fully convective luminosity.

The main-sequence is the region of the H–R diagram where central hydrogen-burning occurs. Here the central temperature is high enough for the hydrogen thermonuclear reactions to supply the energy radiated away. There are three sections of the main sequence with different slopes, depending on the mode of energy generation and the type of opacity. Because the linear model is not sufficiently centrally condensed, the radius must shrink in order to raise the central temperature to high enough values to generate the luminosity, so the main sequence is shifted to higher effective temperature and slightly higher luminosity than that obtained from accurate calculations.

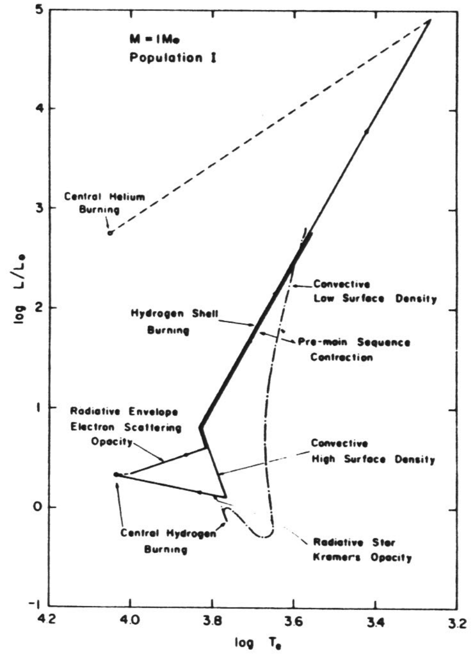

Figure 8. Evolutionary track in H–R diagram of star at one solar mass. Solid and dotted curves are from analytic models. Dashed-dot curve is results of calculations using Henyey method by Ezer and Cameron (1965).

The equation for the consumption of nuclear fuel is

$$\frac{dX}{dt} = -\frac{\mathcal{E}}{E} \qquad \text{(radiative zone)}$$

$$\frac{dX}{dt} = -\frac{1}{E(M_2 - M_1)}\int_{M_1}^{M_2} \mathcal{E}\, dM(r) \qquad \text{(convective zone)}$$

where X is the concentration of fuel nuclei and E is the energy released per gram of fuel consumed. This equation can be solved for the time scale of central nuclear-burning,

$$\Delta t \approx \frac{M_c}{L} E \Delta X \tag{3.44}$$

where $L/M_c \approx \bar{\mathcal{E}}$, the mean rate of energy generation; E is the energy release per gram; and $\Delta X \approx 1$. Lifetimes of stars near the main sequence are given in Table I.

Convective Core

A star which is generating energy at its center by a very temperature-sensitive process will have a convective core. The energy-generation region is very small, so the luminosity increases very rapidly with radius. The flux $F = L/4\pi r^2$ will then be extremely large, since the radius is very small, which forces the radiative temperature gradient to become superadiabatic in order to carry the flux. This causes instability to convection.

The boundary condition for the convective core is

$$\left(\frac{dT}{dr}\right)_{\rm rad} = \left(\frac{dT}{dr}\right)_{\rm ad} \tag{3.45}$$

$$\left(\frac{dT}{dr}\right)_{\rm ad} = \frac{1}{(N+1)_{\rm ad}} \frac{T}{P}\frac{dP}{dr} = -\frac{1}{(N+1)_{\rm ad}} \frac{T}{P}\frac{GM(r)\rho}{r^2} \tag{3.46}$$

and

$$(N+1)_{\rm ad} = \frac{32 - 24\beta - 3\beta^2}{8 - 6\beta} \tag{3.47}$$

where

$$\beta = \frac{P_g}{P}$$

$$\left(\frac{dT}{dr}\right)_{\rm rad} = -\frac{3}{16\sigma}\frac{\varkappa\sigma}{T^3}\frac{L}{4\pi r^2} = \frac{1}{(N+1)_{\rm rad}}\frac{T}{P}\frac{dP}{dr} \tag{3.48}$$

where

$$(N+1)_{\rm rad} = \frac{16\pi c G(1-\beta)M(r)}{\varkappa L_r} \tag{3.49}$$

where

$$(1-\beta) = \frac{P_{\rm rad}}{P} = \tfrac{1}{3}\frac{aT^4}{P}$$

Thus, the condition for convective instability is

$$(N+1)_{\rm rad} \leq (N+1)_{\rm ad} \tag{3.50}$$

Expressed in another form, the boundary of the convective core will be at

$$\begin{aligned} q = \frac{M(r)}{M} &= (N+1)_{\rm ad}\frac{3}{16\pi acG}\frac{\varkappa P}{T^4}\frac{L_r}{M} \\ &= (N+1)_{\rm ad}\frac{1}{16\pi cG}\frac{\varkappa L_r}{(1-\beta)M} \end{aligned} \tag{3.51}$$

For a convective core to exist, the effective polytropic index must be $N_{\rm ad}$ and decreasing inward at some point in the star (Naur and Osterbrock, 1953), i.e., at the core boundary

$$\frac{d\ln(N+1)_{\rm rad}}{d\ln r} \geq 0$$

Assuming $\varkappa = \varkappa_0\rho^a T^{-b}$, then

$$\begin{aligned} \frac{d\ln(N+1)_{\rm rad}}{d\ln r} &= 4\frac{d\ln T}{d\ln r} + \frac{d\ln M(r)}{d\ln r} - \frac{d\ln P}{d\ln r} - \frac{d\ln\varkappa}{d\ln r} - \frac{d\ln L_r}{d\ln r} \\ &= \left[\frac{4+b+a}{N+1} - (1+a)\right]\frac{d\ln P}{d\ln r} + \frac{d\ln M(r)}{d\ln r} - \frac{d\ln L_r}{d\ln r} \\ &= -\left[\frac{4+b+a}{N+1} - (1+a)\right]V + U - W \end{aligned} \tag{3.52}$$

where

$$U = \frac{d\ln M(r)}{d\ln r} = \frac{4\pi r^3\rho}{M_r}$$

$$V = -\frac{d\ln P}{d\ln r} = \frac{GM(r)\rho}{rP}$$

$$W = \frac{d\ln L_r}{d\ln r}$$

Expand $M(r)$, P, L_r, and T about their central values:

$$M(r) = \tfrac{4}{3}\pi\rho_c r^3$$

$$P = P_c - \tfrac{2}{3}\pi G\rho_c^2 r^2$$

$$L_r = \tfrac{4}{3}\pi\rho_c\mathscr{E}_c r^3$$

$$T = T_c - \frac{1}{N+1}\frac{T}{P}\Delta P = T_c - \frac{1}{N+1}\tfrac{2}{3}\pi G\frac{T_c}{P_c}\rho_c^2 r^2$$

At the center, $U_c = 3$, $V_c = 0$, and $W_c = 3$, so $d\ln(N+1)_{\rm rad}/d\ln r = 0$ at the center. Thus, the condition for a convective core

$$D = \frac{d\ln(N+1)_{\rm rad}}{d\ln r} = -\left(\frac{4+b+a}{N+1} - a - 1\right)V + U - W \geq 0$$

becomes

$$\frac{dD}{dV} \geq 0$$

since $D_c = 0$ and V increases outward.

To evaluate dU/dV and dW/dV at the center, we must develop ρ and $M(r)$ to higher order, since in lowest order $dU = dW = 0$.

$$\begin{aligned} \rho = \frac{\mu H}{k}\frac{P}{T} &= \rho_c\left(1 - \frac{N}{N+1}\tfrac{2}{3}\pi G\frac{\rho_c^2}{P_c}r^2\right) \\ &= \rho_c\left(1 - \frac{N}{N+1}Cr^2\right) \end{aligned}$$

where $C = \frac{2}{3}\pi G(\rho_c^2/P_c)$, so

$$M(r) = 4\pi\int_0^r \rho r^2\,dr = \frac{4\pi}{3}\rho_c r^3\left(1 - \frac{3}{5}\frac{N}{N+1}Cr^2\right)$$

Then

$$U = 3\left(1 - \frac{2}{5}\frac{N}{N+1}Cr^2\right)$$

$$V = 2Cr^2$$

thus

$$\frac{dU}{dV} = -\frac{3}{5}\frac{N}{N+1}$$

Now consider W.

$$L_r = 4\pi\mathcal{E}_0\int_0^r \rho^{1+d}T^{\nu}r^2\,dr$$

assuming an energy-generation rate of the form $\mathcal{E} = \mathcal{E}_0\rho^d T^\nu$, and

$$\rho = \rho_c\left(1 - \frac{N}{N+1}Cr^2\right) \quad \text{and} \quad T = T_c\left(1 - \frac{N}{N+1}Cr^2\right)$$

Thus

$$L_r = 4\pi\mathcal{E}_0\rho_c^{1+d}T_c^\nu\int_0^r\left[1 - \frac{N(1+d)}{N+1}Cr^2\right]\left(1 - \frac{\nu}{N+1}Cr^2\right)r^2\,dr$$

$$= \frac{4\pi}{3}\mathcal{E}_0\rho_c^{1+d}T_c^\nu r^3\left[1 - \frac{3}{5}\frac{\nu + N(1+d)}{N+1}Cr^2\right]$$

$$= \frac{4\pi}{3}\mathcal{E}_c\rho_c r^3\left[1 - \frac{3}{5}\frac{\nu + N(1+d)}{N+1}Cr^2\right]$$

Then

$$W = 3 + \frac{d}{d\ln r}\left[\ln\left\{1 - \frac{3}{5}\frac{\nu + N(1+d)}{N+1}Cr^2\right\}\right]$$

$$= 3 - \frac{6}{5}\frac{\nu + N(1+d)}{N+1}Cr^2$$

so

$$dW = -\frac{12}{5}\frac{\nu + N(d+1)}{N+1}Cr\,dr$$

and

$$dV = 4Cr\,dr$$

Thus

$$\frac{dW}{dV} = -\frac{3}{5}\frac{\nu + N(1+d)}{N+1}$$

The criterion for the existence of a convective core is thus

$$\frac{dD}{dV} = \frac{1}{5(N+1)}(3\nu + 3Nd + 5Na + 5N - 5b - 15) \geq 0 \tag{3.53}$$

where

$$N + 1 = \frac{32 - 24\beta - 3\beta^2}{8 - 6\beta}$$

$$\mathcal{E} = \mathcal{E}_0\rho^d T^\nu$$

$$\kappa = \kappa_0\rho^a T^{-b}$$

Nuclear energy-generation processes have $d \geq 1$, so the condition for a convective core, assuming $\beta = 1$, is

$\nu \geq 4.3$ (for Kramer's opacity)

$\nu \geq 1$ (for electron-scattering opacity)

Thus, all central nuclear burning processes, except the equilibrium p-p chain in stars where Kramer's opacity dominates at the center, cause convective cores. The rate of gravitational energy release is $\mathcal{E}_{gr} = -T(\partial s/\partial t) \propto T$, so the condition for a convective core is

$N \geq 2.4$ or $\beta \leq 0.75$ (for electron-scattering opacity)

$N \geq 2.95$ or $\beta \leq 0.32$ (for Kramer's opacity)

Thus, a convective core can exist from gravitational contraction alone in a massive star. For $\beta = 0$, no temperature or density dependence of $\mathcal{E}$ is needed in order to have a convective core.

Assuming the existence of a convective core and $L_r = L$ at the core boundary, its size is given by

$$q_1 = \frac{(N+1)_{ad}}{1-\beta}\frac{\kappa L}{16\pi cGM} \tag{3.54}$$

For the linear model with electron-scattering opacity [equation (3.15)], but including radiation pressure

$$q_1 = \frac{(N+1)_{ad}}{1-\beta}\frac{29\pi}{32}\left(\frac{31}{288}\right)^3\left(\frac{GH}{k}\right)^4\frac{a}{G}M_\odot^2\left(\frac{1+X_c}{1+X_e}\right)(\beta\mu_e)^4\left(\frac{M}{M_\odot}\right)^2$$

$$= 6.8\times 10^{-4}\frac{(N+1)_{ad}}{1-\beta}\left(\frac{1+X_c}{1+X_e}\right)(\beta\mu_e)^4\left(\frac{M}{M_\odot}\right)^2$$

Assuming β is constant through the star,

$$\frac{1}{1-\beta} = \frac{3P}{aT^4} = \frac{3k}{\beta a\mu H}\frac{\rho}{T^3} = \frac{3k}{\beta a\mu H}\frac{\rho_c}{T_c^3}$$

then the size of the convective core is

$$q_1 = 0.14(N+1)_{ad}\left(\frac{1+X_c}{1+X_e}\right)\left(\frac{\mu_e}{\mu_c}\right)^4 \tag{3.55}$$

Note that the size of the convective core depends on the mass of the star only through the radiation pressure.

ADVANCED STAGES OF EVOLUTION—INHOMOGENEOUS STARS

A star spends most of its life burning hydrogen into helium in its core. The advanced stages of evolution comprise the star's life after central hydrogen-burning. When the hydrogen in the core is completely transformed into helium, the core of the star contracts and heats up. The rising temperature enables hydrogen thermonuclear reactions to occur in a hydrogen-burning shell source surrounding the core. A star in this stage is composed of a helium core, a hydrogen-burning shell source, and a hydrogen envelope. If the star is massive enough, the core continues to contract and heat up, until, at about 10^8 °K, helium-burning thermonuclear reactions occur in the core. Depending on its mass, a star may thereafter proceed to carbon, neon, and oxygen burning.

As a star evolves, each nuclear burning process starts first in the core, exhausts its fuel there, and then burns outward in a shell source as the star heats up. Thus, a star that has passed through several nuclear burning stages will be composed of concentric shells of the products of the different processes, with a hydrogen envelope on the outside and a core of the products of the last nuclear burning stage through which the star has passed. Figure 9 illustrates the shell structure of a star that has passed through all the nuclear burning stages.

We first consider some general properties of stars in advanced stages of evolution. The evolutionary trend of stars is toward greater central condensation. Stars contract and increase their central density and temperature. This contraction is occasionally interrupted (but the evolutionary trend is not altered) by nuclear burning in the core of the star.

The increasing central density as a star evolves, together with the existence of nuclear burning shell sources, causes the development of large radii and extended envelopes. The large radii are caused by increasing central condensation, that is, increasing central density but decreasing envelope density. The degree of central condensation is measured by

$$U = \frac{d\ln M(r)}{d\ln r} = \frac{4\pi r^3\rho}{M(r)} = 3\frac{\rho(r)}{\bar{\rho}_r}$$

where $\bar{\rho}_r$ is the mean density interior to r. Since

$$d\ln r = \frac{1}{U}\,d\ln q$$

where $q = M(r)/M$, the radius is

$$\ln R = \int_{q_1}^{1}\frac{1}{U}\,d\ln q + \ln R_1 \tag{4.1}$$

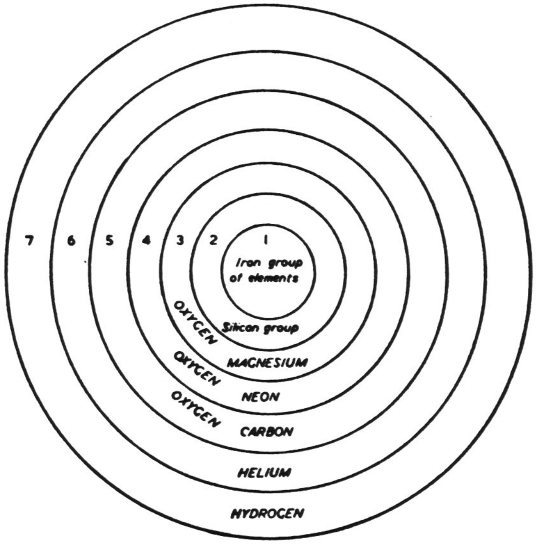

Figure 9. Schematic shell structure of a massive star at the end of nuclear burning. The star is assumed to have passed through all the nuclear burning stages and is approaching equilibrium among the nuclei in the core.

where q_1 and R_1 refer to the core–envelope interface. Now, from equation (1.15),

$$T_c \approx \frac{G\mu_c H}{k}\frac{M_1}{R_1}$$

so

$$R_1 \approx \frac{GHM_1}{k}\frac{\mu_c}{T_c} \tag{4.2}$$

where M_1 is the mass of the core. Thus, the stellar radius is

$$\ln R = \ln\left(\frac{\mu_c}{T_c}\right) + \int_{q_1}^{1}\frac{1}{U}\,d\ln q + \ln\left(\frac{GHM_1}{k}\right) \tag{4.3}$$

The larger the central condensation, the smaller the U near the shell source and the larger the stellar radius.

The central condensation develops as follows: As the central density increases, the pressure gradient $dP/dr = -\rho g$ increases. When a shell source contributes significantly to the energy output of a star, the core luminosity is less than the total luminosity, so the core tends toward an isothermal condition, that is, the temperature increases by less than $T \propto \rho^{1/3}$. Thus, the density gradient in the core increases, $U_1 = 3\rho_1/\bar{\rho}_c$ decreases, and the stellar radius increases. The composition difference between the hydrogen envelope and the helium core also increases the central condensation. From hydrostatic and thermal equilibrium, the physical variables r, $M(r)$, P, and T must be continuous throughout a star. A discontinuity in pressure would entail an infinite acceleration, and a discontinuity in temperature would entail an infinite energy flux. At a composition discontinuity, then, the density will be discontinuous, but ρ/μ and U/μ will be continuous. The composition discontinuity therefore causes a decrease in ρ_1 and so U_1 by a factor of μ_c/μ_e, and also contributes to increasing the stellar radius. All stars in advanced stages of evolution (until their nuclear fuel has been exhausted) have extended envelopes.

Although stellar radii tend to increase during the advanced stages of evolution, their actual magnitude depends on the detailed structure of the star. There is a general empirical rule for determining the variation of a star's radius: *The direction of expansion or contraction in a star is reversed at every nuclear burning shell source and unaffected by any inactive shell.* The reversal of expansion or contraction at a nuclear burning shell source is due to the thermostatic nature of an active nuclear-energy source. A star adjusts itself to maintain a constant temperature in the nuclear-energy source, which causes the radii of the nuclear burning shell sources to tend to remain nearly constant. The mechanism is similar to that which keeps a main-sequence star in equilibrium. If the radii and mass fractions of the shell sources remain constant, the contraction of a zone between two shells, for instance, means that the density at the inner shell of the zone increases but that the density at the outer shell must decrease, since the mass and volume of the zone remain constant. Thus, the density at the inner shell of the next outer zone is decreasing and that zone is expanding (see Figure 10).

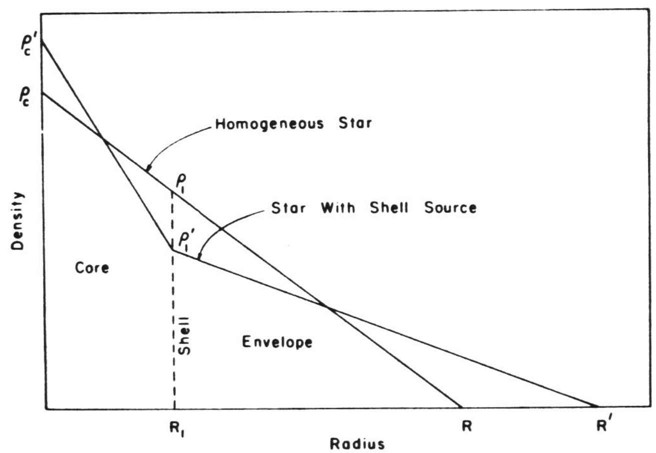

Figure 10.

Consider the zone between two shells of radii $R_0 < R_1$.

$$\underset{R_0}{\vdash}\xrightarrow{\quad r \quad}\underset{R_1}{\dashv}$$

Let m be the mass of this zone and assume $R_1 \gg R_0$. The mean density of the zone is

$$\bar{\rho} = \frac{M_1 - M_0}{\frac{4}{3}\pi(R_1^3 - R_0^3)} \approx \frac{3m}{4\pi R_1^3} \tag{4.4}$$

Thus

$$\frac{\Delta R_1}{R_1} = -\frac{1}{3}\frac{\Delta\bar{\rho}}{\bar{\rho}} \tag{4.5}$$

We also assume the radiation pressure is negligible, so $\beta \approx 1$.

Consider what happens when the radius of the inner shell changes. Suppose R_0 changes by ΔR_0. If the shell at R_0 is not nuclear burning, its properties vary in a manner that preserves hydrostatic equilibrium, that is approximately homologously. Then, by equations (1.13) and (1.15),

$$T \propto \frac{1}{R} \qquad \rho \propto \frac{1}{R^3}$$

so

$$\begin{aligned} \frac{\Delta T_0}{T_0} &= -\frac{\Delta R_0}{R_0} \\ \frac{\Delta\rho_0}{\rho_0} &= -3\frac{\Delta R_0}{R_0} \\ \frac{\Delta P_0}{P_0} &= -4\frac{\Delta R_0}{R_0} \end{aligned} \tag{4.6}$$

Then

$$\frac{\Delta R_1}{R_1} = -\frac{1}{3}\frac{\Delta\rho_0}{\rho_0} = \frac{\Delta R_0}{R_0}$$

Thus, when the inner shell is not nuclear burning, the outer shell's radius changes in the same way as the inner shell's radius, and the shell has no effect on the expansion or contraction.

If a shell is nuclear burning, however, its structure initially changes homologously, but, due to the change in the rate of energy generation, there is an additional, nonhomologous change in the structure. The change in the rate of energy generation is

$$\begin{aligned} \mathscr{E}_N &= \mathscr{E}_0(\rho + \Delta\rho)(T + \Delta T)^n = \mathscr{E}_{N_0}\left(1 + \frac{\Delta\rho}{\rho} + n\frac{\Delta T}{T}\right) \\ &= \mathscr{E}_{N_0}\left(1 - (n + 3)\frac{\Delta R_0}{R_0}\right) \end{aligned}$$

so that

$$\frac{\Delta\mathscr{E}_N}{\mathscr{E}_N} = -(n + 3)\left(\frac{\Delta R_0}{R_0}\right)_1$$

During central hydrogen-burning, the luminosity of a star increases because of the increasing mean molecular weight as hydrogen is depleted in the core. Assuming that the homology relations for homogeneous stars are still valid, then in stars of small mass where Kramer's opacity is dominant from (3.13) and (3.35)

$$L \propto (\mu\beta)^{7.5} \tag{4.9}$$

while in massive stars where electron scattering is dominant from (3.16) and (3.35)

$$L \propto (\mu\beta)^{4} \tag{4.10}$$

The molecular weight in the core increases by about a factor of 2 as hydrogen is consumed.

The energy-generation rate has the form

$$\mathscr{E} = \mathscr{E}_0 X_H X_2 \rho \left(\frac{T}{T_0}\right)^n$$

where X_2 is X_H for the p-p chain and is X_{CNO} for the CNO cycle. The temperature exponent is $n \approx 4$ for the p-p chain and $n \approx 18$ for the CNO cycle. As the hydrogen concentration in the core decreases, the central temperature must rise in order to maintain the rate of energy generation. The radius of the star will therefore tend to shrink [see equation (4.2)],

$$R \propto M/T_c \tag{4.11}$$

The tendency of the radius to decrease due to the increasing central temperature is counteracted by the tendency of the radius to increase due to the growing composition inhomogeneity which decreases

$$U_{1+} = \frac{\mu_{1+}}{\mu_{1-}} U_{1-} \tag{4.12}$$

at the bottom of the envelope. Here, plus and minus refer to the exterior and interior sides of the composition discontinuity (core–envelope interface).

The p-p chain is less sensitive to temperature and more sensitive to hydrogen concentration than the CNO cycle. The central temperature will thus increase much more in small-mass than in large-mass stars. During central hydrogen-burning in small-mass stars, the rapidly increasing central temperature nearly balances the growing composition inhomogeneity and the radius stays nearly constant. In massive stars, the central temperature rises only slightly and the radius increases because of the composition inhomogeneity. The evolutionary track of a star in the H–R diagram during central hydrogen-burning is toward higher luminosity. For low-mass stars, where the radius is approximately constant, the track is nearly parallel to the main sequence. For massive stars, where the radius increases, the track turns off the main sequence to lower effective temperatures.

HYDROGEN SHELL BURNING

As hydrogen is exhausted in the core of a star, the central temperature increases in order to maintain the rate of energy generation. The temperature farther out in the star is then increased and the rate of hydrogen-burning outside the core (where the hydrogen has not been exhausted) is therefore increased. Thus, a shell burning source is ignited.

When hydrogen becomes nearly exhausted in small-mass stars generating energy by the p-p chain, the central temperature has already increased and raised the temperature in the surrounding regions of higher hydrogen concentration sufficiently to produce hydrogen thermonuclear reactions there. When hydrogen becomes nearly exhausted in massive stars, the central temperature has not yet increased much due to the high temperature sensitivity of the CNO cycle. The energy requirements of the star must still be met by the core, so the central temperature must now increase greatly. This causes the radius of the star to contract, and its track in the H–R diagram swings to higher effective temperatures. Eventually, the decrease in X_c outruns the increase in T_c^n and the rate of nuclear-energy generation in the core decreases. The core then starts to contract rapidly and release gravitational energy to supplement the decreasing rate of central nuclear-energy generation. The gravitational contraction raises the central temperature $T_c \propto \rho_c^{1/3}$ and the shell temperature, and ignites the shell source. The more massive the star, the larger the size of the initial convective core and the farther out from the center the hydrogen-rich regions lie. Therefore, the temperature in the hydrogen-rich shell will be lower, the ignition of the shell source will be delayed, and gravitational energy release will supplant nuclear energy generation as the star's primary energy source. Eventually, the contraction will raise the temperature enough to ignite the shell source. To summarize, as hydrogen is exhausted in the core of a star, the temperature increases, nuclear-energy generation in the core decreases, and a hydrogen-burning shell source surrounding the core is ignited.

We now consider in greater detail the increase in central condensation that occurs when a shell source is set up.

It is convenient to discuss the structure of the star in terms of the nondimensional variables U, V, and $N + 1$:

$$U \equiv \frac{d \ln M(r)}{d \ln r} = \frac{4\pi r^3 \rho}{M(r)} = 3\frac{\rho(r)}{\bar{\rho}_r}$$

$$V \equiv -\frac{d \ln P}{d \ln r} = \frac{GM(r)\rho}{rP} = \frac{3}{2}\frac{GM(r)/r}{\frac{3}{2}P/\rho} \tag{4.13}$$

$$N + 1 \equiv \frac{d \ln P}{d \ln T} = \frac{16\pi ac}{3}\frac{GM(r)T^4}{P\kappa L(r)}$$

At the center of a star, $U \to 3$, $V \to 0$, and at the surface $U \to 0$, $V \to \infty$. The polytropic index N varies between 1.5 for a convective region and infinity for an isothermal region. Also

$$\frac{d \ln T}{d \ln r} = -\frac{V}{N+1}$$

$$\frac{d \ln \rho}{d \ln r} = -\frac{NV}{N+1} \tag{4.14}$$

Thus the r-dependence of the physical variables is given in terms of U, V, $N+1$ by

$$M(r) \sim r^{U}$$
$$P \sim r^{-V}$$
$$T \sim r^{-V/(N+1)} \tag{4.15}$$
$$\rho \sim r^{-NV/(N+1)}$$

From the continuity of the physical variables r, $M(r)$, P, T, and ρ/μ, the continuity conditions on U, V, $N+1$ are

$$\frac{U}{\mu} \qquad \frac{V}{\mu} \qquad \times L_r(N+1) \qquad \text{(continuous)} \tag{4.16}$$

The $U-V$ locus of a star is given by

$$\frac{d\ln U}{d\ln r} = 3 - U - \frac{NV}{N+1}$$
$$\frac{d\ln V}{d\ln r} = U - 1 + \frac{V}{N+1} \tag{4.17}$$

or

$$\frac{d\ln V}{d\ln U} = \frac{U + V/(N+1) - 1}{3 - U - NV/(N+1)}$$

The points on the $V-U$ curve with horizontal or vertical tangent are given by

$$U + V/(N+1) - 1 = 0 \qquad \text{(horizontal)}$$
$$U + NV/(N+1) - 3 = 0 \qquad \text{(vertical)} \tag{4.18}$$

These two lines intersect at the point

$$U = \frac{N-3}{N-1} \qquad V = 2\frac{N+1}{N-1} \tag{4.19}$$

Thus, for $N > 3$, the intersection point is in the physical region and there is a loop point.

Typical $U-V$ curves for homogeneous and inhomogeneous stars are shown in Figure 12.

A star's central condensation is increased by the tendency toward isothermality in the core, when nuclear energy generation there ceases. For an isothermal core, $N = \infty$, so the $U-V$ curve has a loop point at $U = 1$, $V = 2$. The maximum V thus occurs for $U = 1$ and is somewhat larger than 2:

$$U_{1+} \approx \frac{\mu_e}{\mu_c} U \approx 0.5 \tag{4.20}$$

However, an isothermal core, if too large, cannot support the weight of the envelope. The critical size of an isothermal core can be found from the virial theorem (McCrea, 1957),

$$3(\gamma - 1)U + \Omega - 3PV = 0$$

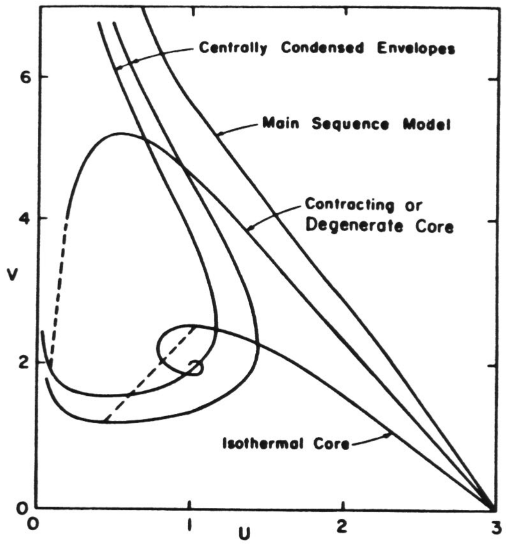

Figure 12.

so

$$P = (\gamma - 1)\frac{U}{V} + \frac{1}{3}\frac{\Omega}{V}$$

where U and V are now the internal energy and volume of the isothermal core, and P is the pressure on its surface. For an isothermal sphere the internal energy is, from equation (2.3)

$$U = \frac{1}{\gamma - 1}\frac{k}{\mu H}TM$$

and for a sphere of uniform density the gravitational energy is, from equation (1.19)

$$\Omega = -\frac{3}{5}\frac{GM^2}{R}$$

The pressure at the boundary of the isothermal core is therefore

$$P = \frac{3}{4\pi}\frac{k}{\mu H}\frac{TM}{R^3} - \frac{3}{5}\frac{1}{4\pi}\frac{GM^2}{R^4}$$

There is a maximum pressure consistent with the equilibrium virial theorem, which is given by

$$\frac{dP}{dR} = -\frac{9}{4\pi}\frac{kTM}{\mu H R^4} + \frac{12}{20\pi}\frac{GM^2}{R^5} = 0$$

Thus, there is a critical core radius

$$R_{crit} = \tfrac{4}{13}\frac{G\mu_c H}{k}\frac{M_1}{T_1} \tag{4.21}$$

with stability possible only for $R_{core} \geq R_{crit}$. The maximum possible pressure is

$$P_{max} = \frac{3}{16\pi}\left(\frac{15}{4}\right)^3\left(\frac{kT_1}{\mu_c H}\right)^4\frac{1}{G^3 M_1^2} \tag{4.22}$$

which decreases with increasing core mass. To determine the limiting mass of an isothermal core, this P_{max} must be compared with the pressure necessary to support a star. For the linear model, equation (3.4),

$$\bar{P} = \frac{103}{350}\left(\frac{5}{4\pi}\frac{GM^2}{R^4}\right)$$

Also for the linear model, equation (3.5),

$$T_1 \approx \bar{T} = \frac{104}{175}\left(\tfrac{5}{12}\frac{G\mu H}{k}\frac{M}{R}\right)$$

Thus, the condition for a stable star, that an isothermal nondegenerate core can support the surrounding envelope, is

$$P_{max} \gtrsim \bar{P}$$

or

$$\frac{M_1}{M} \lesssim 0.3\left(\frac{\mu_e}{\mu_c}\right)^2 \tag{4.23}$$

Accurate calculations (Schönberg and Chandrasekhar, 1942) give $q_1 < 0.37(\mu_e/\mu_c)^2$ with an accuracy of 8%.

If the mass of the core is below the isothermal core limiting mass (Schönberg–Chandrasekhar limit), the core becomes isothermal.

In massive stars, the core exceeds the Schönberg–Chandrasekhar limit and gravitational contraction begins when nuclear-energy generation ceases to support the star. In small-mass stars, the core is initially below the limiting size, but shell burning adds material to the core until in this case too the core exceeds the Schönberg–Chandrasekhar limit, and it contracts. If the core mass is less than the white-dwarf limiting mass [equation (5.9)], the electrons become degenerate, while larger cores continue to contract. The pressure gradient needed to support the envelope is thus supplied by the electron-degeneracy pressure in small-mass stars and by a combination of the density and temperature gradients in large-mass stars. The development of a degenerate core greatly increases the central condensation.

We now consider the structure of a stellar envelope with extreme central condensation, characterized by $U \to 0$ as $r \to r_{shell}$. Since r_{shell} is very small, the structure at the base of the envelope will not be too different from that in the limit $r \to 0$.

Since $U \to 0$, $M(r) \approx M_1$ near the shell [equation (4.15)], that is, as $r \to 0$, $\rho \to \infty$ and $M(r) \to$ finite value. Then

$$\frac{dP}{dr} = \frac{k}{\mu_e H}\frac{d}{dr}(\rho T) = -\frac{GM_1}{r^2}\rho$$

and assuming some polytropic law, $\rho \propto T^N$, gives

$$\frac{dT}{dr} = -\frac{G\mu_e H}{(N+1)k}\frac{M_1}{r^2}$$

Thus

$$T = \frac{G\mu_e H}{(N+1)k}\left(\frac{M_1}{r}\right) + \text{constant} \propto \frac{1}{r} \tag{4.24}$$

Thus, for an extremely centrally condensed envelope

$$V \to N + 1 \tag{4.25}$$

in the limit as $r \to 0$ [from equations (4.15) and (4.24)]. The loop condition [equation (4.19)] shows that the limit of extreme central condensation, $U \to 0$ as $r \to 0$, requires $N \leq 3$. The radial dependence of the physical variables is

$$P \propto r^{-(N+1)} \qquad T \propto r^{-1} \qquad \rho \propto r^{-N} \tag{4.26}$$

For $N > 3$, the limit as $r \to 0$ is a loop point given by equation (4.19). In this case, $U > 0$ and the envelope is not so centrally condensed. The radial dependence of the physical variables is

$$P \sim r^{-2(N+1)/(N-1)} \qquad T \sim r^{-2/(N-1)} \qquad \rho \sim r^{-2N/(N-1)} \tag{4.27}$$

We now determine the effective polytropic index at the base of a centrally condensed envelope. In terms of nondimensional variables,

$$P = p\frac{GM^2}{4\pi R^4}$$

$$T = t\frac{\mu_e H}{k}\frac{GM}{R}$$

$$M(r) = qM$$

$$r = xR$$

the hydrostatic equilibrium equations are

$$\frac{dp}{dx} = -\frac{pq}{tx^2}\beta l \qquad \frac{dq}{dx} = \frac{x^2 p}{t}\beta l \tag{4.28}$$

where $l = \mu/\mu_e$, and the flux equations are

$$\begin{aligned} \frac{dt}{dx} &= -C_K \frac{p^2}{x^2 t^{8.5}} \quad \text{(Kramer's opacity)} \\ \frac{dt}{dx} &= -C_E \frac{p}{x^2 t^4} \quad \text{(Electron scattering)} \end{aligned} \tag{4.29}$$

For Kramer's opacity, combining equations (4.28) and (4.29),

$$\frac{dp^2}{dt^{8.5}} = \frac{q}{4.25 C_K}$$

so, near the surface and near the shell at the base of the envelope, where U is very small and the mass fraction q is nearly constant, the polytropic index is

$$N = 3.25$$

Thus, at the shell

$$\rho \sim r^{-2.89} \quad \text{(Kramer's)} \tag{4.30}$$

Similarly, for electron scattering,

$$\frac{dp}{dt^4} = \frac{q}{4C_E}$$

so, near the surface and near the shell, the polytropic index is

$$N = 3$$

Thus, the density distribution at the shell is

$$\rho \sim r^{-3} \quad \text{(Electron scattering)} \tag{4.31}$$

The only envelope model which can be readily solved analytically is $\rho(r) \sim r^{-3}$. This, as was just shown, corresponds to the limiting case of extreme central condensation for both Kramer's and electron-scattering opacity. The internal structure will be well represented by such a model, but, because it is too centrally condensed, the stellar radii will be much too large. To calculate the radii, account must be taken of the fact that the mean polytropic index of the envelope is less than 3.

Inhomogeneous Analytic Stellar Model

We now construct an analytic model of a star with one shell using a linear density distribution in the core and an r^{-3} density distribution in the envelope.

1. *Core*. In the core assume a linear density distribution:

$$\rho(r) = \rho_c - (\rho_c - \rho_1)\frac{r}{R_1} \tag{4.32}$$

where ρ_c is the central density and ρ_1 the density at the shell. Then the mass distribution in the core is, by equation (1.4),

$$\begin{aligned} M(r) &= \int_0^r 4\pi\rho(r)r^2\,dr \\ &= \frac{4\pi}{3} r^3 \left[\rho_c - \tfrac{3}{4}(\rho_c - \rho_1)\frac{r}{R_1}\right] \end{aligned} \tag{4.33}$$

and the mass of the core is

$$M_1 = \frac{\pi}{3} R_1^3(\rho_c + 3\rho_1) \tag{4.34}$$

This relation can be turned around to give the radius of the core:

$$\begin{aligned} \frac{R_1}{R_\odot} &= \left(\frac{3M_\odot}{\pi R_\odot{}^3}\right)^{1/3} \left(\frac{M_1}{M_\odot}\right)^{1/3} \left(\rho_c + 3\frac{\mu_c}{\mu_e}\rho_{1-}\right)^{-1/3} \\ &= 1.78\left(\frac{M_1}{M_\odot}\right)^{1/3} \left(\rho_c + 3\frac{\mu_c}{\mu_e}\rho_{1-}\right)^{-1/3} \end{aligned} \tag{4.35}$$

The pressure in the core is determined by hydrostatic equilibrium, equation (1.1),

$$\begin{aligned} P(r) &= P_c - G\int_0^r \frac{M(r)\rho(r)}{r^2}\,dr \\ &= P_c - \frac{2\pi}{3} G\rho_c^2 r^2 \left[1 - \tfrac{7}{6}\left(1 - \frac{\rho_{1-}}{\rho_c}\right)\frac{r}{R_1} + \tfrac{3}{8}\left(1 - 2\frac{\rho_{1-}}{\rho_c} + \frac{\rho_{1-}^2}{\rho_c^2}\right)\frac{r^2}{R_1^2}\right] \end{aligned}$$

Applying the boundary condition that $P = P_1$ at the core boundary $r = R_1$, we find that the pressure in the core is

$$\begin{aligned} P(r) = P_1 + \frac{\pi}{36} G\rho_c^2 R_1^2 \Bigg[& 5 + 10\frac{\rho_{1-}}{\rho_c} + 9\frac{\rho_{1-}^2}{\rho_c^2} \\ & - 24\frac{r^2}{R_1^2} + 28\left(1 - \frac{\rho_{1-}}{\rho_c}\right)\frac{r^3}{R_1^3} - 9\left(1 - 2\frac{\rho_{1-}}{\rho_c} + \frac{\rho_{1-}^2}{\rho_c^2}\right)\frac{r^4}{R_1^4}\Bigg] \end{aligned} \tag{4.36}$$

and the central pressure is

$$P_c = P_1 + \frac{5\pi}{36} G\rho_c^2 R_1^2 \left(1 + 2\frac{\rho_{1-}}{\rho_c} + 1.8\frac{\rho_{1-}^2}{\rho_c^2}\right) \tag{4.37}$$

For a perfect gas, with negligible radiation pressure, the temperature is, by equation (1.2),

$$T(r) = \frac{\mu H}{k}\frac{P(r)}{\rho(r)}$$

Thus, the temperature in the core is

$$T(r) = \left[1 - \left(1 - \frac{\rho_{1-}}{\rho_c}\right)\frac{r}{R_1}\right]^{-1}\left[\frac{\rho_{1-}}{\rho_c}T_1 + \frac{\pi}{36}\frac{G\mu_c H}{k}\rho_c R_1^2\left\{5 + 10\frac{\rho_{1-}}{\rho_c} + 9\frac{\rho_{1-}^2}{\rho_c^2} - 24\frac{r^2}{R_1^2} + 28\left(1 - \frac{\rho_{1-}}{\rho_c}\right)\frac{r^3}{R_1^3} - 9\left(1 - 2\frac{\rho_{1-}}{\rho_c} + \frac{\rho_{1-}^2}{\rho_c^2}\right)\frac{r^4}{R_1^4}\right\}\right] \tag{4.38}$$

and the central temperature is

$$T_c = \frac{\mu_c}{\mu_e}\frac{\rho_1}{\rho_c}T_1 + \frac{5\pi}{36}\frac{G\mu_c H}{k}R_1^2\rho_c\left(1 + 2\frac{\rho_{1-}}{\rho_c} + 1.8\frac{\rho_{1-}^2}{\rho_c^2}\right)$$

$$= \frac{\mu_c}{\mu_e}\frac{\rho_1}{\rho_c}T_1 + 0.17 \times 10^7\mu_c\left(\frac{R_1}{R_\odot}\right)^2\rho_c\left(1 + 2\frac{\rho_1}{\rho_c}\frac{\mu_c}{\mu_e} + 1.8\frac{\mu_c^2}{\mu_e^2}\frac{\rho_1^2}{\rho_c^2}\right) \tag{4.39}$$

However, when degenerate, the core is assumed to be isothermal, so $T_c = T_1$.

2. *Envelope*. In the envelope assume an r^{-3} density distribution:

$$\rho(r) = \rho_1\left(\frac{R_1}{r}\right)^3 \tag{4.40}$$

Notice that in this model the density does not go to zero at the surface of the star. The mass distribution is, from equation (1.4),

$$M(r) = M_1 + 4\pi\rho_1 R_1^3\int_{R_1}^{r}\frac{dr}{r}$$

$$= M_1 + 4\pi\rho_1 R_1^3 \ln\frac{r}{R_1}$$

where M_1 is the mass inside the shell. The mass of the envelope is

$$M - M_1 = 4\pi\rho_1 R_1^3 \ln\frac{R}{R_1} \tag{4.41}$$

The pressure is determined by hydrostatic equilibrium, equation (1.1),

$$P(r) = P_1 - G\int_{R_1}^{r}\frac{M(r)\rho(r)}{r^2}\,dr$$

$$= P_1 - G\rho_1 R_1^3\int_{R_1}^{r}\left(M_1 + 4\pi\rho_1 R_1^3 \ln\frac{r}{R_1}\right)\frac{1}{r^5}\,dr$$

so

$$P(r) = P_1 - \tfrac{1}{4}G\rho_1\frac{M_1}{R_1}\left[1 - \left(\frac{R_1}{r}\right)^4\right] - \frac{\pi}{4}G\rho_1^2R_1^2 + \pi G\rho_1^2R_1^6\frac{1}{r^4}\left(\tfrac{1}{4} + \ln\frac{r}{R_1}\right)$$

The boundary condition $P(R) = 0$ determines P_1, and we get

$$P(r) = \tfrac{1}{4}G\rho_1\frac{M_1}{R_1}\left[\left(\frac{R_1}{r}\right)^4 - \left(\frac{R_1}{R}\right)^4\right] + \frac{\pi}{4}G\rho_1^2R_1^2\left[\left(\frac{R_1}{r}\right)^4 - \left(\frac{R_1}{R}\right)^4 + 4\left(\frac{R_1}{r}\right)^4\ln\frac{r}{R_1} - 4\left(\frac{R_1}{R}\right)^4\ln\frac{R}{R_1}\right] \tag{4.42}$$

and the pressure at the shell is

$$P_1 = \tfrac{1}{4}G\rho_1\frac{M_1}{R_1}\left[1 - \left(\frac{R_1}{R}\right)^4\right] + \frac{\pi}{4}G\rho_1^2R_1^2\left[1 - \left(\frac{R_1}{R}\right)^4\right] - \pi G\rho_1^2R_1^2\left(\frac{R_1}{R}\right)^4\ln\frac{R}{R_1} \tag{4.43}$$

The temperature in the envelope, given by the equation of state with negligible radiation pressure (1.2), is

$$T(r) = \frac{G\mu_e H}{4k}\left[\frac{M_1}{R_1}\left\{\frac{R_1}{r} - \frac{R_1}{R}\left(\frac{r}{R}\right)^3\right\} + \pi\rho_1R_1^2\left\{\frac{R_1}{r} - \frac{R_1}{R}\left(\frac{r}{R}\right)^3 + 4\frac{R_1}{r}\ln\frac{r}{R_1} - 4\frac{R_1}{R}\left(\frac{r}{R}\right)^3\ln\frac{R}{R_1}\right\}\right] \tag{4.44}$$

Note that the temperature is proportional to r^{-1}, except near the surface. The temperature at the shell is

$$T_1 = \frac{G\mu_e H}{4k}\left[\frac{M_1}{R_1}\left\{1 - \left(\frac{R_1}{R}\right)^4\right\} + \pi R_1^2\rho_1\left\{1 - \left(\frac{R_1}{R}\right)^4\left(1 + 4\ln\frac{R}{R_1}\right)\right\}\right]$$

$$\approx 0.577 \times 10^7\mu_e\left(\frac{M_1}{M_\odot}\right)\left(\frac{R_\odot}{R_1}\right) + 0.3066 \times 10^7\mu_e\left(\frac{R_1}{R_\odot}\right)^2\rho_1 \tag{4.45}$$

The stellar radius is extremely sensitive to the degree of central condensation. For our envelope density distribution $\rho \sim r^{-3}$, the radius is obtained from the mass relation (4.41):

$$R = R_1\exp\left(\frac{M - M_1}{4\pi\rho_1}R_1^{-3}\right)$$

$$= R_1\exp\left[\frac{1 - q_1}{12q_1}\left(\frac{\rho_c}{\rho_1} + 3\frac{\mu_c}{\mu_e}\right)\right] \tag{4.46}$$

Thus, the radius depends exponentially on ρ_c/ρ_1. This leads to extremely large radii, much larger than are observed. This is to be expected, since this envelope corresponds to the maximum degree of central condensation.

Consider now less centrally condensed envelopes with density distributions

$$\rho(r) = \rho_1\left(\frac{R_1}{r}\right)^n \qquad (1.5 < n < 3) \tag{4.47}$$

The mass distribution in the envelope is then

$$M(r) = M_1 + 4\pi\rho_1 R_1^n \int_{R_1}^{r} r^{2-n}\,dr$$

$$= M_1 + \frac{4\pi}{3-n}\rho_1 R_1^n (r^{3-n} - R_1^{3-n})$$

and

$$M = M_1 + \frac{4\pi}{3-n}\rho_1 R_1^n (R^{3-n} - R_1^{3-n})$$

Thus, the radius is

$$R \approx \frac{R_1}{3-n} + \left[\frac{(3-n)(M-M_1)}{4\pi\rho_1 R_1^n}\right]^{1/(3-n)} \tag{4.48}$$

The dependence of the radius on ρ_c/ρ_1 varies from exponential for $n = 3$ to almost constant for $n = 1.5$. Thus, the radius of the star depends sensitively on the precise degree of central condensation. It is reasonable to treat the internal envelope structure and the radius using different effective polytropic indices n, because the internal properties of the envelope are determined by the behavior near the shell where $n \approx 3$, while the radius is determined by the entire envelope in which the polytropic index is less than 3 over large regions. In the radiative region near the shell $n = 3$ and the mass

$$M(r) \propto \ln \frac{r}{R_1}$$

increases very slowly with radius, while in a convective region $n = 1.5$ and the mass

$$M(r) \propto r^{3/2}$$

increases much faster with radius. As a rough approximation for all stars, we choose $\bar{n} = 2.4$. Then

$$\frac{R}{R_\odot} = \frac{5}{3}\frac{R_1}{R_\odot} + \left(\frac{3M_\odot}{20\pi R_\odot^3}\right)^{5/3}\left(\frac{M}{M_\odot}\right)^{5/3}(1-q_1)^{5/3}\left(\frac{R_1}{R_\odot}\right)^{-4}\rho_1^{-5/3}$$

$$= \frac{5}{3}\frac{R_1}{R_\odot} + 0.122\left(\frac{M}{M_\odot}\right)^{5/3}(1-q_1)^{5/3}\left(\frac{R_1}{R_\odot}\right)^{-4}\rho_1^{-5/3} \tag{4.49}$$

The internal structure of a typical inhomogeneous model is shown in Figure 13. Figure 14 shows the tremendous degree of central condensation of the mass as compared with the homogeneous model.

We now turn from the hydrostatics to the energy balance in the envelope. The rate of thermonuclear-energy generation in the shell is, from equation (1.6),

$$L = 4\pi\mathscr{E}_0 \int_{R_1}^{R} \rho^2 \left(\frac{T}{T_0}\right)^n r^2\,dr$$

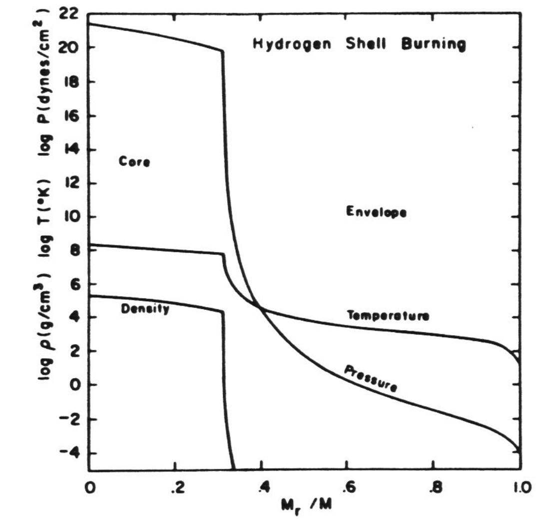

Figure 13.

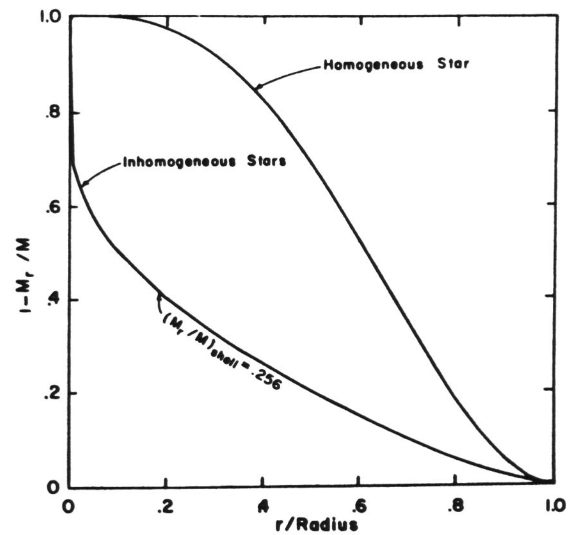

Figure 14.

where we have assumed that the composition in the shell source is the unevolved envelope composition and that the nuclear-energy-generation rate per gram has the form

$$\mathscr{E} = \mathscr{E}_0 \rho \left(\frac{T}{T_0}\right)^n$$

Then

$$L = 4\pi \mathscr{E}_0 \rho_1^2 \left(\frac{T_1}{T_0}\right)^n R_1^{6+n} \int_{R_1}^{R} \frac{dr}{r^{n+4}}$$

$$= \frac{4\pi R_1^3}{n+3} \mathscr{E}_0 \rho_1^2 \left(\frac{T_1}{T_0}\right)^n \left[1 - \left(\frac{R_1}{R}\right)^{n+3}\right]$$

Thus, the rate of thermonuclear-energy release from a shell source is

$$L = \frac{4\pi R_1^3}{n+3} \mathscr{E}_0 \rho_1^2 \left(\frac{T_1}{T_0}\right)^n$$

or

$$\frac{L}{L_\odot} = 1.12 \frac{\mathscr{E}_0}{n+3} \left(\frac{R_1}{R_\odot}\right)^3 \rho_1^2 \left(\frac{T_1}{T_0}\right)^n \tag{4.50}$$

The energy generation is confined to an extremely thin shell source as shown in Figure 15.

The luminosity for radiative energy transport is

$$L = -4\pi r^2 \frac{16\sigma}{3\varkappa_0} \frac{T^{3-b}}{\rho^{1+a}} \frac{dT}{dr}$$

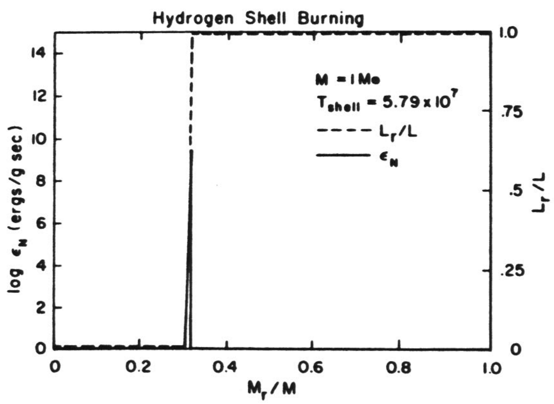

Figure 15.

assuming an opacity law of the form

$$\varkappa = \varkappa_0 \rho^a T^b$$

The temperature gradient given by the analytic model [equation (4.44)] is

$$\frac{dT}{dr} = -T_1 \frac{R_1}{r^2}$$

Thus, the radiative luminosity is

$$L = \frac{64}{3} \frac{\pi\sigma}{\varkappa_0} R_1 T_1 \frac{T^{3-b}}{\rho^{1+a}}$$

Evaluating it at the shell gives

$$L = \frac{64}{3} \frac{\pi\sigma}{\varkappa_0} R_1 \frac{T_1^{4-b}}{\rho_1^{1+a}}$$

$$= \frac{64}{3} \frac{\pi\sigma}{\varkappa_0} R_1 \frac{T_1^{7.5}}{\rho_1^2} \quad \text{(Kramer's)} \tag{4.51}$$

$$= \frac{64}{3} \frac{\pi\sigma}{\varkappa_0} R_1 \frac{T_1^4}{\rho_1} \quad \text{(Electron scattering)}$$

Thus, the radiative luminosity of the envelope is, for population I ($X = 0.6$, $Y = 0.38$, $Z = 0.02$),

$$\frac{L}{L_\odot} = \begin{cases} 4.25 \times 10^3 \left(\frac{R_1}{R_\odot}\right) \frac{T_{1(7)}^{7.5}}{\rho_1^2} & \text{(Kramer's)} \\ 2.18 \times 10^3 \left(\frac{R_1}{R_\odot}\right) \frac{T_{1(7)}^4}{\rho_1} & \text{(Electron scattering)} \end{cases} \tag{4.52}$$

and for population II ($X = 0.9$, $Y = 0.099$, $Z = 0.001$),

$$\frac{L}{L_\odot} = \begin{cases} 2.27 \times 10^4 \left(\frac{R_1}{R_\odot}\right) \frac{T_{1(7)}^{7.5}}{\rho_1^2} & \text{(Kramer's)} \\ 1.84 \times 10^3 \left(\frac{R_1}{R_\odot}\right) \frac{T_{1(7)}^4}{\rho_1} & \text{(Electron scattering)} \end{cases} \tag{4.53}$$

The effective temperature is given by equation (3.17).

For a fully convective envelope, the luminosity and effective temperature are determined by the surface condition, equations (3.24) and (3.28). The track is the same as for pre-main sequence fully convective contraction, but traversed in the opposite direction.

The time scale of evolution is determined by the rate of release of energy:

$$L = \frac{dE}{dt} \quad \text{so} \quad \Delta t = \frac{\Delta E}{L} \tag{4.54}$$

The time scale during stages of core contraction is determined by the gravitational energy release:

$$\Delta E = \tfrac{1}{2}|\Delta\Omega| = \tfrac{1}{2}\,\tfrac{6}{7}\,\frac{G\bar{M}_c^2}{\bar{R}_c}\left(-\frac{\Delta R_c}{\bar{R}_c} + 2\frac{\Delta M_c}{M_c}\right)$$

where the factor $\tfrac{6}{7}$ arises in evaluating Ω using the linear model. The luminosity of the contracting core is determined by the rate at which material is added to the core by the hydrogen-burning shell source:

$$L_c = \Phi_c \frac{dM_c}{dt}$$

where Φ_c is the gravitational potential at the core surface

$$\Phi_c = \frac{GM_c}{R_c}$$

and

$$\frac{dM_c}{dt} = \frac{L_{\text{shell}}}{E_H X}$$

where E_H is the energy released per gram of hydrogen consumed. Thus the contracting core's luminosity is

$$L_c = \frac{GM_c}{R_c}\frac{L}{E_H X}$$

or

$$\frac{L_c}{L_\odot} = \frac{GM_\odot}{E_H R_\odot}\frac{1}{X}\left(\frac{M_c}{M_\odot}\right)\left(\frac{R_\odot}{R_c}\right)\left(\frac{L}{L_\odot}\right)$$

$$= 3.18 \times 10^{-4}\frac{1}{X}\left(\frac{M_c}{M_\odot}\right)\left(\frac{R_\odot}{R_c}\right)\left(\frac{L}{L_\odot}\right)$$

The time scale for contraction of the core is

$$\Delta t = \tfrac{6}{7}\frac{E_H X M_c}{2L}\left(2\frac{\Delta M_c}{M_c} - \frac{\Delta R_c}{R_c}\right)$$

$$= 4.31 \times 10^{10} X_e\left(\frac{M_c}{M_\odot}\right)\left(\frac{L}{L_\odot}\right)^{-1}\left(-\frac{\Delta R_c}{R_c} + 2\frac{\Delta M_c}{M_c}\right) \text{ years} \qquad (4.55)$$

The amount of material added to the core by the hydrogen burning shell during this time is

$$\Delta M_c = \frac{\Delta t L}{E_H X_e}$$

so

$$\Delta q_1 = \frac{\Delta M_1}{M} = \left(\frac{L_\odot}{M_\odot E_H X_e}\right)\frac{M_\odot}{M}\frac{L}{L_\odot}\Delta t \qquad (4.56)$$

Evolution During the Hydrogen Shell-Burning Phase

Evolution during the hydrogen shell-burning phase is toward greater central density and temperature and larger central condensation. The evolution of the shell structure can be expressed in terms of the central condensation U_1, the core mass M_1 and the ratio $\beta_1 = P_{g1}/P_1$ (Hayashi, Hoshi, and Sugimoto, 1962). From the definitions (4.13) of U, V, and $(N + 1)$

$$R_1 = \frac{G\mu\beta_1 H}{V_1 k}\frac{M_1}{T_1}$$

$$\rho_1 = \frac{1}{4\pi}\frac{M_1 U_1}{R_1^3}$$

$$= \frac{1}{4\pi}\left(\frac{k}{G\mu\beta_1 H}\right)^3\frac{T_1^3 V_1^3 U_1}{M_1^2}$$

$$L = \frac{16\pi c G(1 - \beta_1)M_1}{(N + 1)_1 \varkappa_1}$$

The rate of energy generation in the shell, equation (4.50), is

$$L_{\text{shell}} = \frac{4\pi R_1^3}{n + 3}\mathscr{E}_0 \rho_1^2 T_1^n$$

Thus, if the energy release in the core is negligible, the shell temperature is

$$T_1^{n+3} = 64\pi^2 c G\left(\frac{G\mu\beta_1 H}{k}\right)^3\left(\frac{n + 3}{\mathscr{E}_0 \varkappa_1}\right)\frac{(1 - \beta_1)M_1^2}{V_1^3 U_1^2 (N + 1)_1}$$

We have seen that for a centrally condensed envelope with electron-scattering opacity dominant $V_1 = (N + 1)_1 = 4$. Thus, the shell structure is

$$R_1 = \frac{G\mu_e H\beta_1 M_1}{4kT_1}$$

$$\rho_1 = \frac{16}{\pi}\left(\frac{k}{G\mu_e H}\right)^3\frac{T_1^3 U_1}{\beta_1^3 M_1^2}$$

$$T_1^{n+3} = \frac{\pi^2}{4}cG\left(\frac{G\mu_e H}{k}\right)^3\frac{n + 3}{\mathscr{E}_0 \varkappa_1}\frac{\beta_1^3(1 - \beta_1)M_1^2}{U_1^2} \qquad (4.57)$$

$$L = 4\pi c G\frac{(1 - \beta_1)M_1}{\varkappa_1}$$

As hydrogen shell-burning proceeds, the core contracts, U_1 decreases, and M_1 grows. The shell temperature T_1 increases, but only slightly, due to the high temperature sensitivity of nuclear reactions. The shell radius R_1 is approximately constant; it may increase slightly in the late red-giant phase, where the core mass increases rapidly, and it may decrease slightly in massive stars. The shell density ρ_1 decreases. The luminosity L is approximately constant, except in low-mass stars where $\beta_1 \approx 1$, so

$$(1 - \beta_1) = \frac{\pi}{48}\frac{a}{G}\left(\frac{G\mu H}{k}\right)^4\frac{\beta_1^4 M_1^2}{U_1}$$

increases substantially, and

$$L = \frac{\pi^2}{3}\sigma\left(\frac{G\mu H}{k}\right)^4 \frac{\beta_1^4 M_1^3}{\varkappa_1 U_1}$$

The stellar structure can be expressed as an explicit function of ρ_c, q_1, and M, using our analytic core and envelope models when $\rho_c \gg \rho_1$ and $R_1 \ll R$. The central density ρ_c is chosen as the parameter labeling the course of evolution, since ρ_c increases monotonically during the contraction of the helium core. A sequence of models with increasing ρ_c describes the course of evolution. It is necessary to choose an initial core size to start the sequence, since the details of the setting up of a shell source cannot be followed analytically.

The core radius is, from equation (4.35),

$$\frac{R_1}{R_\odot} = 0.178\left(\frac{M_1}{M_\odot}\right)^{\frac{1}{3}}\left(\frac{\rho_c}{10^3}\right)^{-\frac{1}{3}} \tag{4.58}$$

so R_1 shrinks with increasing ρ_c.

The central temperature for a nondegenerate core is found by substituting this expression for R_1 in the equation for T_c (4.39) and omitting the first term, which is negligible,

$$T_c = 5.4 \times 10^7 \mu_c\left(\frac{M_1}{M_\odot}\right)^{\frac{2}{3}}\left(\frac{\rho_c}{10^3}\right)^{\frac{1}{3}} \tag{4.59}$$

The shell temperature is found by substituting equation (4.58) for R_1 in the equation for T_1 (4.45) and neglecting the second term, which is small,

$$T_1 = 3.24 \times 10^7 \mu_e\left(\frac{M_1}{M_\odot}\right)^{\frac{2}{3}}\left(\frac{\rho_c}{10^3}\right)^{\frac{1}{3}} \tag{4.60}$$

For a nondegenerate core

$$\frac{T_c}{T_1} = 1.67\frac{\mu_c}{\mu_e}$$

In small-mass stars, $M \lesssim 3M_\odot$, the core is degenerate and isothermal, so instead of equation (4.59),

$$T_c = T_1$$

Thus, the core and shell temperatures increase during hydrogen shell-burning as $\rho_c^{\frac{1}{3}}$.

The shell density is determined by the energy balance: luminosity = energy-generation rate. The energy-generation rate per gram is assumed to be of the form

$$\mathscr{E} = \mathscr{E}_0\rho\left(\frac{T}{T_0}\right)^n$$

and all constants are evaluated for the CNO cycle at $T_0 = 3 \times 10^7$ °K, so $\mathscr{E}_0 = 3.87 \times 10^5 X_H X_{CNO}$ and $n = 16$. The total energy generation rate is given by equation (4.50). The luminosity depends on the opacity and the energy-transport mechanism, and two cases are considered: Radiative transfer with electron-scattering opacity, equation (4.51), and convective transport with low surface density,

equation (3.23) with equation (3.22). For electron scattering

$$\rho_1 = C_1\left(\frac{M_1}{M_\odot}\right)^{-2(n-3)/9}\left(\frac{\rho_c}{10^3}\right)^{-(n-6)/9} \tag{4.61a}$$

where

$$C_1 = \begin{cases} 217 & \text{(population I)} \\ 583 & \text{(population II)} \end{cases}$$

For a convective envelope

$$\rho_1 = C_2\left(\frac{M}{M_\odot}\right)^{-\frac{1}{9}\{[(2n+1)A-4]/(2A-5)\}}(1-q_1)^{\frac{3}{2}[(A-4)/(2A-5)]}q_1^{-\frac{1}{9}\{[(2n+11)A-32]/(2A-5)\}} \times \left(\frac{\rho_c}{10^3}\right)^{-\frac{1}{9}\{[(n-11)A+32]/(2A-5)\}} \tag{4.61b}$$

where $A = b + 3.5(a+1)$ for an H^- opacity law of the form $\varkappa = \varkappa_0 P^a T^b$,

$$A = \begin{cases} 11 & \text{(population I)} \\ 15 & \text{(population II)} \end{cases}$$

and

$$C_2 = \begin{cases} 31.6 & \text{(population I)} \\ 63.5 & \text{(population II)} \end{cases}$$

The radius of the star depends on the mean polytropic index of the envelope. It is given by substituting equation (4.58) for R_1 and equation (4.61) for ρ_1 in the equation for the radius (4.49) and omitting the small first term. For electron scattering

$$\frac{R}{R_\odot} = C_3\left(\frac{M}{M_\odot}\right)^{(10n-21)/27}(1-q_1)^{\frac{1}{3}}q_1^{(10n-66)/27}\left(\frac{\rho_c}{10^3}\right)^{(5n+6)/27} \tag{4.62a}$$

where

$$C_3 = \begin{cases} 1.54 \times 10^{-2} & \text{(population I)} \\ 2.97 \times 10^{-3} & \text{(population II)} \end{cases}$$

For a convective envelope

$$\frac{R}{R_\odot} = C_4\left(\frac{M}{M_\odot}\right)^{\{[(10n+21)A-60]/24(2A-5)\}}(1-q_1)^{\frac{3}{4}[A/(2A-5)]} \times q_1^{\{[(10n-9)A]/24(2A-5)\}}\left(\frac{\rho_c}{10^3}\right)^{\{[(5n+9)A]/24(2A-5)\}} \tag{4.62b}$$

where

$$C_4 = \begin{cases} 0.383 & \text{(population I)} \\ 0.120 & \text{(population II)} \end{cases}$$

Thus, the stellar radius increases with increasing central density.

The luminosity of the star is given by substituting equation (4.58) for R_1, equation (4.61) for ρ_1, and equation (4.60) for T_1 in the equation for the energy generation (4.50). For electron scattering

$$\frac{L}{L_\odot} = C_5\left(\frac{M_1}{M_\odot}\right)^{(2n+21)/9}\left(\frac{\rho_c}{10^3}\right)^{(n+3)/9} \tag{4.63a}$$

where

$$C_5 = \begin{cases} 39.5 & \text{(population I)} \\ 5.01 & \text{(population II)} \end{cases}$$

For a convective envelope

$$\frac{L}{L_\odot} = C_6\left(\frac{M}{M_\odot}\right)^{\{[(10n+21)A-40n-48]/12(2A-5)\}}(1-q_1)^{\frac{3}{4}[(A-4)/(2A-5)]}$$
$$\times\, q_1^{\{[(10n-9)(A-4)]/12(2A-5)\}}\left(\frac{\rho_c}{10^3}\right)^{\{[(5n+9)(A-4)]/12(2A-5)\}} \tag{4.63b}$$

where

$$C_6 = \begin{cases} 0.835 & \text{(population I)} \\ 0.0594 & \text{(population II)} \end{cases}$$

The effective temperature of the star is found from equation (3.17) with equations (4.63) and (4.62). For electron scattering

$$T_e = C_7\left(\frac{M}{M_\odot}\right)^{-7(2n-15)/108}(1-q_1)^{-\frac{1}{4}}q_1^{-[(14n-195)/108]}\left(\frac{\rho_c}{10^3}\right)^{-(7n+3)/108} \tag{4.64a}$$

where

$$C_7 = \begin{cases} 1.16 \times 10^5 & \text{(population I)} \\ 1.58 \times 10^5 & \text{(population II)} \end{cases}$$

For a convective envelope

$$T_e = C_8\left(\frac{M}{M_\odot}\right)^{-[(10n-3)/12(2A-5)]}(1-q_1)^{-\frac{3}{8}[1/(2A-5)]}$$
$$\times\, q_1^{-[(10n-9)/12(2A-5)]}\left(\frac{\rho_c}{10^3}\right)^{-[(5n+9)/12(2A-5)]} \tag{4.64b}$$

where

$$C_8 = \begin{cases} 8.89 \times 10^3 & \text{(population I)} \\ 8.22 \times 10^3 & \text{(population II)} \end{cases}$$

For radiative envelopes, the luminosity increases ($\sim\rho_c^2$), and the effective temperature decreases ($\sim\rho_c^{-1}$), with increasing central density. For convective envelopes, the luminosity increases rapidly ($\sim\rho_c^3$), and the effective temperature decreases slowly ($\sim\rho_c^{-0.4}$), with increasing central density. The mode of energy transport in the envelope switches from radiative to convective when the convective flux becomes larger than the radiative flux.

All of these relations simplify when the core is degenerate, for then the central density is a function of the core mass.

The tip of the red-giant sequence occurring in small-mass stars is determined by the onset of helium-burning in the center of a star when $T_c \approx 10^8$ °K. In small-mass stars with isothermal degenerate cores,

$$T_c = T_1 = 3.24 \times 10^7\mu_e\left(\frac{M_1}{M_\odot}\right)^{\frac{2}{3}}\left(\frac{\rho_c}{10^3}\right)^{\frac{1}{3}} \tag{4.60}$$

so the central density at the start of helium-burning is

$$\frac{\rho_c}{10^3} = 29.4\mu_e^{-3}\left(\frac{M_1}{M_\odot}\right)^{-2}$$

Thus, the maximum luminosity at the tip of the red-giant branch, where the envelope is convective, is, from equation (4.63b),

$$\left(\frac{L}{L_\odot}\right)_{\max} = C_9\left(\frac{M}{M_\odot}\right)^{[(A+8)/4(2A-5)]}\left(\frac{1-q_1}{q_1^{0.9}}\right)^{\frac{3}{4}[(A-4)/(2A-5)]} \tag{4.65a}$$

At $T_0 = 9.5 \times 10^7$

$$\left(\frac{L}{L_\odot}\right)_{\max} = \begin{cases} 1.485 \times 10^5\left(\dfrac{M}{M_\odot}\right)^{0.276}\left[\dfrac{(1-q_1)^{1.033}}{q_1^{0.931}}\right] & \text{(population I)} \\ 1.95 \times 10^5\left(\dfrac{M}{M_\odot}\right)^{0.23}\left[\dfrac{(1-q_1)^{1.1}}{q_1^{0.99}}\right] & \text{(population II)} \end{cases} \tag{4.65b}$$

The maximum luminosity is insensitive to the mass of the star and is much higher than obtained from accurate calculations (see Figure 20). The mean envelope index $\bar{n} = 2.4$ is much greater than the value $n = 1.5$ appropriate for a convective envelope. Therefore, in this model, the radius and luminosity are much too large.

The evolutionary changes in the central conditions are shown in Figure 16. For low-mass stars ($M \lesssim 3\,M_\odot$), the core is isothermal during hydrogen shell-burning; the central density increases with nearly constant central temperature and the electrons in the core become degenerate. The increasing luminosity along the red-giant branch eventually causes the shell temperature, and therefore the central temperature, to rise much more rapidly than $\rho_c^{\frac{1}{3}}$ until it approaches 10^8 °K and helium is ignited. In massive stars, an isothermal condition does not develop. The core contraction provides an appreciable part of the star's luminosity from the beginning of hydrogen shell-burning. The central temperature and density increase, with T_c increasing only slightly less rapidly than $\rho_c^{\frac{1}{3}}$.

The evolutionary tracks of stars in the H–R diagram during hydrogen shell-burning are shown in Figure 17 to 21. At the beginning of hydrogen shell-burning, the luminosity, except for very small mass population I stars, is much too low because we have taken the most centrally condensed model throughout and have not allowed the degree of central condensation of the envelope solutions to increase gradually. The temperature falls off extremely rapidly outside the shell and the hydrogen shell-burning region is therefore very thin, covering about 1% instead of an initial 10% of the mass, as found in accurate calculations. The total amount of energy generated is therefore too small. Since the thickness of the shell is constant, the rising shell-temperature during hydrogen shell-burning raises the luminosity in our models. Accurate calculations show, however, that the shells are originally much thicker than ours and the narrowing of the shells, due to the steepening temperature gradient on their outside and the exhaustion of fuel on their inside, counteracts the rising shell-temperature and the luminosity stays fairly constant unless a fully convective envelope is developed.

As stars evolve, their radii increase and they move to the right in the H–R diagram. The tracks depend in their grossest features on whether or not the star is small enough to develop a degenerate core. Those stars that develop isothermal degenerate cores must evolve to much higher central densities and much greater central condensation than those that do not. Thus, very small mass stars develop quite extensive envelopes, which are fully convective and very luminous. These form the red-giant branch (Figure 20). Intermediate mass stars develop fully convective but not so extensive envelopes, and their luminosity does not greatly increase (Figure 21). The very massive stars do not develop a very great central condensation before their central temperature has reached 10^8 °K, so they do not develop

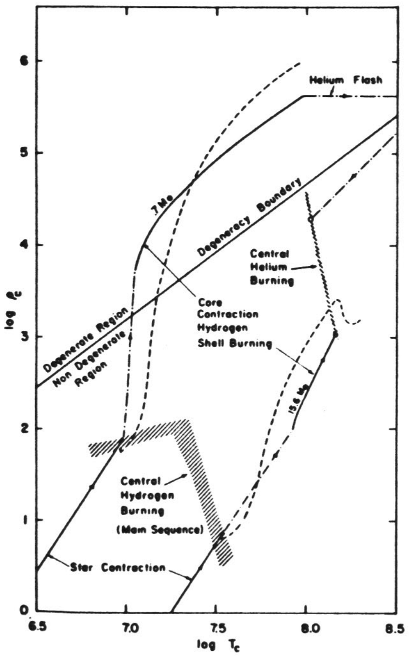

Figure 16. Evolution of central conditions during pre-main sequence contraction, central hydrogen-burning, helium core contraction, and central helium-burning. The solid lines and shaded regions are from the analytic models, the dashed-dot lines are interpolations. The dashed lines are from Hayashi, Hoshi, and Sugimoto (1962).

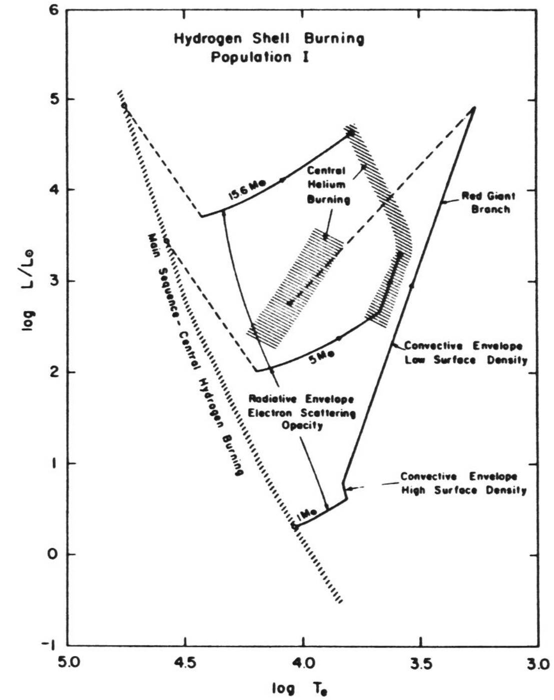

Figure 17. Evolutionary tracks of stars in H–R diagram during hydrogen-shell burning with helium core contraction. The nature of the energy-transport mechanism in the envelope, which determines the slopes of the tracks, is shown. The shaded area is the region where stars have just started to burn helium in their cores. The large extent of the central helium-burning region for low-mass stars that have passed through the red-giant phase is due to uncertainty in the mass of the helium core. The uncertainty in position of a given star is roughly parallel to the shading.

convective envelopes before helium-burning, and their luminosity increases only slightly.

The time scales for evolution during the hydrogen shell-burning–contracting helium-core stage are given in Table I.

To summarize: The cause of the extended envelopes of hydrogen-shell burning stars is their central condensation; the cause of their central condensation is as follows: When the core hydrogen is exhausted and the thermonuclear energy generation occurs in the shell, the core tends toward an isothermal state, that is,

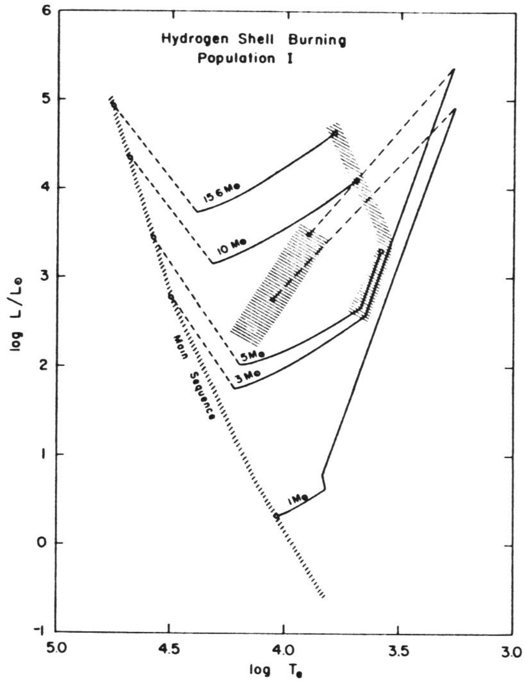

Figure 18.

as the core contracts the density rises by more than T^3. Thus, the density gradient in the core increases, which increases the star's central condensation. As a star's central density rises, its shell temperature tends to rise, since $T_1 \propto \rho_c^{1/3}$ for homologous contraction (equation 4.60). If the shell were not nuclear burning, its density would also rise. If, however, the shell is nuclear burning, the shell density must be low in order to keep down the rate of thermonuclear energy generation and increase the envelope's transparency, i.e., to bring the rate of energy generation and luminosity into balance. That is, the temperature sensitivity of the nuclear burning process keeps the shell temperature and radius nearly constant, so the shell density falls as the central density rises, in order to conserve mass. Thus, the star's central condensation increases. The greater the central density, the smaller the core radius; the higher the shell temperature, the lower the shell density [for nuclear burning processes with temperature sensitivity $n = d(\ln \mathscr{E})/d(\ln T) > 9$], and the larger the envelope.

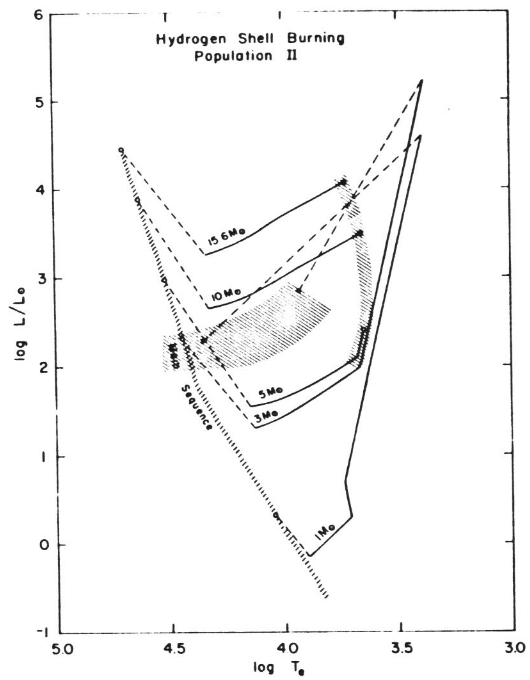

Figure 19.

CENTRAL HELIUM-BURNING

When the central temperature of the helium core is raised to about 10^8 °K, helium will begin to burn at the center of the star. In small-mass stars with degenerate cores, a helium flash occurs. The pressure of degenerate matter depends only on the density, not the temperature, so that the energy released by the onset of helium-burning increases the temperature without a corresponding increase in pressure. The increased temperature speeds up the helium reactions, which further increases the temperature, until the temperature is high enough (kT in the central degenerate region rises above the Fermi level) for the matter to become nondegenerate and the perfect gas law again holds. In nondegenerate material, increasing the temperature increases the pressure, which causes the core to expand, thereby reducing the density and temperature and damping the reaction. The core settles down to burning helium at a much lower density and slightly higher temperature than at the onset of the flash. In large-mass stars with nondegenerate cores, the

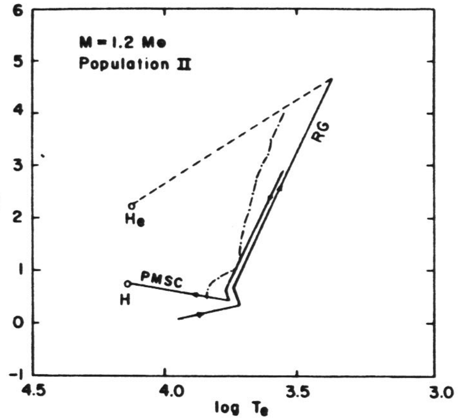

Figure 20. Evolutionary track in H–R diagram of 1.2 $M_\odot$ star. Solid lines are from analytic models for pre-main-sequence contraction (PMSC), central hydrogen-burning (H), hydrogen shell-burning red giants (RG), and central helium-burning (He). Dashed line is from Hoyle and Schwarzschild (1955), Schwarzschild and Selberg (1962), and Härm and Schwarzschild (1964).

pressure adjusts the structure to the increase in temperature and there is a slight decrease in central density.

A star burning helium at its center will be much more centrally condensed than a main sequence hydrogen-burning star. The density at the shell where the

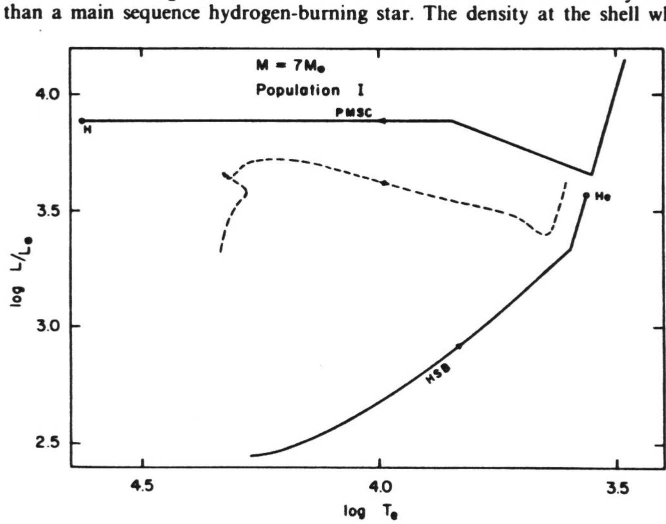

Figure 21. Evolutionary track in H–R diagram of $7M_\odot$ star. Solid lines are from analytic model for pre-main-sequence contraction (PMSC), central hydrogen-burning (H), hydrogen shell burning (HSB), and central helium-burning (He). The dashed curve is from Hofmeister, Kippenhahn, and Weigert (1965).

composition discontinuity, and possibly hydrogen-burning, occurs is much less than the central density. Thus the core by itself may be treated as a star with the density, but not the temperature, going to zero at its surface. The luminosity of the core is determined by the balance between the radiative energy transport and the helium energy-generation rate. This energy balance determines the central temperature, which together with hydrostatic equilibrium determines the central density. The radius of the core is determined by the density distribution, which we assume is linear. Thus, for the model of a star burning helium at its center, assume a linear density distribution in its core, a $\rho \sim r^{-3}$ density distribution in its envelope, and treat the core by itself as a star.

The helium energy-generation rate is

$$\mathscr{E}_{3\alpha \to C^{12}} = \mathscr{E}_0 \rho^2 \left(\frac{T}{T_0}\right)^n \tag{4.66}$$

for

$$T_8 \sim 1 \qquad (n = 41,\ \mathscr{E}_0 = 4.4 \times 10^{-8} X_{He}^2)$$

$$T_8 \sim 2 \qquad (n = 19,\ \mathscr{E}_0 = 15 X_{He}^2)$$

The total helium-burning energy-generation rate is

$$L = 4\pi R_1^3 \mathscr{E}_0 \rho_c^3 T_c^n J_n \tag{4.67}$$

where

$$J_n = \int_0^1 x^2 (1 - x)^{n+3} (1 + 2x - 1.8x^2)^n \, dx$$

is the integral of $[\rho(r)/\rho_c]^3 [T(r)/T_c]^n$ over the star. Then

$$\frac{L}{L_\odot} = 201 \mathscr{E}_0 J_n \mu_c^n \left(\frac{M_1}{M_\odot}\right)^{n+3} \left(\frac{R_\odot}{R_1}\right)^{n+6} \left(\frac{9.6 \times 10^6}{T_0}\right)^n \tag{4.68}$$

The luminosity with electron-scattering opacity is given by equation (3.16). The central temperature and density are, from equations (3.6) and (3.3),

$$T_c = 9.62 \times 10^6 \mu_c \left(\frac{M_1}{M_\odot}\right) \left(\frac{R_\odot}{R_1}\right)$$

$$\rho_c = 5.65 \left(\frac{M_1}{M_\odot}\right) \left(\frac{R_\odot}{R_1}\right)^3$$

The core structure, for $T_c \sim 1 \times 10^8$ °K, is

$$\begin{aligned} T_c &= 1.16 \times 10^8 \mu_c^{0.213} \left(\frac{M_1}{M_\odot}\right)^{0.128} \\ \rho_c &= 9.9 \times 10^3 \mu_c^{-2.36} \left(\frac{M_1}{M_\odot}\right)^{-1.62} \\ \frac{R_1}{R_\odot} &= 8.27 \times 10^{-2} \mu_c^{0.787} \left(\frac{M_1}{M_\odot}\right)^{0.872} \\ \frac{L_c}{L_\odot} &= 179 \mu_c^4 \left(\frac{M_1}{M_\odot}\right)^3 \end{aligned} \tag{4.69}$$

The density and temperature at the shell are determined by the conditions of hydrostatic equilibrium and energy conservation. The shell temperature is given by equation (4.45) with the small second term neglected and equation (4.69) for R_1,

$$T_1 = 6.97 \times 10^7 \left(\frac{\mu_e}{\mu_c^{0.787}}\right) \left(\frac{M_1}{M_\odot}\right)^{0.128} \tag{4.70}$$

The shell density is the solution of

$$\text{Luminosity} = L_{\text{core}} + L_{\text{shell}} \tag{4.71}$$

where L_{core} is the core luminosity, equation (4.69); L_{shell} is the shell energy-generation rate, equation (4.50); and the luminosity is given by equations (4.51), (3.24), or (3.28). Once ρ_1 is known, the luminosity is found from equation (4.71). The effective temperature is given by equation (3.17), and the radius is given by equation (4.49) as in the case of hydrogen-shell burning.

The locus of points in the H–R diagram where initial central helium-burning occurs is shown as the shaded regions in Figures 17 to 19. The relative contributions of hydrogen- and helium-burning to the luminosity are found to be initially: $L_H > L_{He}$ for population I stars, while $L_H \ll L_{He}$ for population II stars. The contribution of the hydrogen shell source decreases in massive stars. We see that small-mass population II stars lie at the onset of central helium-burning in a strip of nearly constant luminosity, but with varying effective temperature depending on the mass; the smaller the mass, the higher the effective temperature. The time scales for evolution during the central helium-burning stage are given in Table I.

In the more advanced stages of evolution—helium-burning, carbon-burning, neon- and oxygen-burning—the core of the star continues to become denser and hotter, a complicated shell structure develops, with some shells active and others inactive, and the radius continues to grow. A schematic picture of the stages of central nuclear burning and shell formation is given in Figure 5 of Hayashi (1965). How far a star progresses through these stages of nuclear burning depends, as we have shown, on its mass.

FINAL STAGES OF EVOLUTION

After a star has exhausted all the nuclear fuels it is capable of burning, its only remaining sources of energy are its gravitational potential energy, which it can release by contracting, and its thermal energy, which it can release by cooling. Such a star will contract, increasing its central density and temperature. The core will, however, tend to be cooled off by energy losses from neutrino emission. The rate of emission of neutrinos increases with temperature, and since their mean free path is larger than the radius of the star they remove energy from the star. If neutrino emission is intense, all stars in the stage of gravitational contraction after the exhaustion of nuclear fuel will develop degenerate cores.

If the central density resulting from the gravitational contraction is low, only electrons, not nucleons, are degenerate and supply the pressure to support the star. There is a maximum density possible for a stable star supported by degenerate electron pressure. At higher densities, the electrons are forced onto the protons,

creating neutrons. This process is a phase change and absorbs a great deal of energy, causing instability. The gravitational collapse of massive stars produces cores with densities above the critical density. The core of such a star will be composed of free degenerate neutrons and other baryons. If the mass of the remnant from the collapse is small enough, it can be supported by the pressure of the degenerate neutrons and a stable neutron star will be formed. If the mass is too large, the gravitational force, augmented by the relativistic effect that the pressure contributes to the effective mass, overwhelms the nuclear forces and the star collapses indefinitely. What happens to such core remnants remains to be discovered.

Structure of White Dwarfs

White dwarfs are stars whose support is provided by the pressure of degenerate electrons (only electrons, not nucleons, are degenerate) throughout most of the mass of the star. We assume that the electrons are completely degenerate throughout the star. This is, of course, not really possible, since the density in the surface layers is very low and the electrons are nondegenerate. However, the surface layers are extremely thin.

The equation of state of a degenerate gas is a complicated function

$$P = P(\rho)$$

approaching the limiting forms

$$P = K_1 \rho^{5/3} = 9.91 \times 10^{12} \left(\frac{\rho}{\mu_e}\right)^{5/3} \tag{5.1}$$

at low density, where the electrons are nonrelativistic ($p \ll m_e c$, where p is the electron Fermi momentum), and

$$P = K_2 \rho^{4/3} = 1.23 \times 10^{15} \left(\frac{\rho}{\mu_e}\right)^{4/3} \tag{5.2}$$

at high density where the electrons are relativistic, ($p \gg m_e c$). Thus, the temperature disappears from the equation of state, and the internal structure of a white dwarf is independent of the temperature.

Applying dimensional analysis to the equation of hydrostatic equilibrium gives, for a nonrelativistic degenerate gas (Osterbrock, 1963),

$$\rho_c \propto \frac{M}{R^3}$$

$$P_c \propto g\rho R \propto \frac{M^2}{R^4}$$

$$\propto K_1 \rho_c^{5/3} \propto \frac{M^{5/3}}{R^5}$$

Thus, for a nonrelativistic degenerate gas

$$R \propto M^{-1/3}$$
$$\rho_c \propto M^2 \tag{5.3}$$

There is thus a definite relation between the mass and radius of a white dwarf—the larger the mass, the smaller the radius. Since the central density increases with mass, for large-mass white dwarfs the electrons become relativistic. For an extreme relativistic degenerate gas, the pressure force is

$$\frac{dP}{dr} \propto \frac{\rho^{4/3}}{R} \propto \frac{M^{4/3}}{R^5}$$

and the gravitational force is

$$g\rho \propto \frac{GM(r)}{r^2}\rho \propto \frac{M^2}{R^5}$$

Thus, the pressure force and the gravitational force depend on the radius in the same way, but depend differently on the mass, so the two forces will be in balance for only one mass, the limiting mass of a white-dwarf star. For larger masses, the gravitational force always exceeds the pressure force.

The mass–radius relation for a white dwarf can be obtained from the virial theorem (Salpeter, 1964):

$$3(\gamma - 1)U + \Omega = 0 \qquad (2.1')$$

where Ω is the gravitational potential energy, given by equation (1.19),

$$\Omega \approx -\frac{GM^2}{R}$$

and the internal energy U is the electron kinetic energy

$$U = NK_e$$

Here N is the number of electrons and K_e is the kinetic energy per electron. The mass of the star is

$$M = N\mu_e m_p$$

where m_p is the proton mass and μ_e is the molecular weight per electron

$$\mu_e = \left[Xx_1 + \frac{Y}{4}(y_1 + 2y_2) + \frac{Z}{2}\right]^{-1}$$

where x_1, y_1 are the fractions of singly and y_2 of doubly ionized hydrogen and helium, respectively. $\mu_e = 2$ for a fully ionized gas if $X = 0$. Thus from the virial theorem

$$K_e = \frac{Gm_p^2\mu_e^2 N}{3(\gamma - 1)R} \qquad (5.4)$$

The electron kinetic energy is related to its momentum by

$$K_e = \frac{p_e^2}{2m_e} \qquad p_e \ll m_e c$$

$$K_e = p_e c \qquad p_e \gg m_e c$$

The average electron momentum p_e is related to the average interelectron spacing r_e by the uncertainty principle:

$$r_e p_e \geq \hbar$$

Using the equality sign gives for the kinetic energy

$$K_e = \frac{p_e^2}{2m_e} = \tfrac{1}{2}m_e c^2\left(\frac{r_0}{r_e}\right)^2 \qquad [p_e \ll m_e c]$$

$$K_e = p_e c = m_e c^2\left(\frac{r_0}{r_e}\right) \qquad [p_e \gg m_e c] \qquad (5.5)$$

where $r_0 = \hbar/m_e c$ is the electron Compton wavelength.

These two limiting equations can be combined in the interpolation formula (Wheeler, 1964)

$$K_e = m_e c^2\left(\frac{1}{s + 2s^2}\right) \qquad (5.6)$$

where $s = r_e/r_0$. This formula is accurate to within 8%. The radius of the star is expressed in terms of r_e by

$$R = N^{1/3} r_e \qquad (5.7)$$

Equating the expressions (5.4) and (5.6) for the electron kinetic energy gives the relation

$$1 + 2s = \left[\frac{3(\gamma - 1)r_0 m_e c^2}{Gm_p^2\mu_e^2}\right]N^{-2/3}$$

$$\equiv \frac{3(\gamma - 1)}{\mu_e^2}\left(\frac{N_0}{N}\right)^{2/3} \equiv \frac{3(\gamma - 1)}{\mu_e^{4/3}}\left(\frac{M_0}{M}\right)^{2/3} \qquad (5.8)$$

where

$$N_0 \equiv \left[\frac{Gm_p^2}{\hbar c}\right]^{-3/2} = 2.2 \times 10^{57}$$

$$M_0 \equiv \left(\frac{G}{\hbar c}\right)^{-3/2} m_p^{-2} = N_0 m_p = 3.7 \times 10^{33}\ \text{g}$$

$$= 1.85\, M_\odot$$

For a nonrelativistic electron gas $\gamma = \frac{5}{3}$, and for an extreme relativistic electron gas $\gamma = \frac{4}{3}$, so $3(\gamma - 1)$ varies between 1 and 2. We use the interpolation formula

$$3(\gamma - 1) = \frac{1 + 2s}{1 + s}$$

which has a maximum error of 30% (Schatzman, 1958). Thus

$$1 + s = \frac{1}{\mu_e^{4/3}}\left(\frac{M_0}{M}\right)^{2/3} \qquad (5.8')$$

First note that the minimum value of the left-hand side of the above relation is 1, so there is a maximum mass for a white dwarf

$$M_{max} = \mu_e^{-2} M_0 = \frac{1.85 M_\odot}{\mu_e^2} \quad (5.9)$$

However, long before the density becomes infinite, inverse β-reactions will occur, and the above analysis will cease to apply. The increasing density causes instability of the white dwarf before the singularity is reached.

Second, the relation (5.8′) can be written as a mass–radius relation

$$R = R_0 \mu_e^{-\frac{4}{3}} \left[\mu_e^{-\frac{4}{3}} \left(\frac{M}{M_0}\right)^{-\frac{2}{3}} - \left(\frac{M}{M_0}\right)^{\frac{2}{3}} \right] \quad (5.10)$$

where

$$R_0 = N_0^{\frac{2}{3}} r_0 = 5 \times 10^8 \text{ cm}$$

Thus, the radius of a white dwarf is very small, and it decreases as the mass increases.

The mean density of a white dwarf is

$$\bar{\rho} = \frac{M}{(4\pi/3)R^3}$$

$$= \frac{3M_0\mu_e}{4\pi R_0^3} \left[\mu_e^{-\frac{4}{3}} \left(\frac{M}{M_0}\right)^{-\frac{4}{3}} - 1 \right]^{-3}$$

$$= 7.06 \times 10^6 \mu_e \left[\mu_e^{-\frac{4}{3}} \left(\frac{M}{M_0}\right)^{-\frac{4}{3}} - 1 \right]^{-3}$$

Since a white dwarf has a very thin nondegenerate surface layer, we may approximate such a star by a homogeneous model with a linear density distribution. Then the central density is

$$\rho_c = 4\bar{\rho} = 2.83 \times 10^7 \mu_e \left[\mu_e^{-\frac{4}{3}} \left(\frac{M}{M_0}\right)^{-\frac{4}{3}} - 1 \right]^{-3} \quad (5.11)$$

There is a maximum density possible for a stable white dwarf. As the density increases, the electron Fermi energy increases. An electron with energy greater than the β-decay energy for electron emission from a nucleus $(Z-1, A)$ will produce inverse β-reactions

$$e^- + (Z, A) \rightarrow (Z-1, A) + \nu$$

This process increases the value of μ_e in the interior, and thus the maximum stable mass is reduced. The predominant nuclei under white-dwarf conditions are elements in the range neon to iron, for which inverse β-decay will occur at densities of about 10^9 g/cm^3. Thus, the critical density for a white dwarf is about 10^9 g/cm^3. The relation between central density and mass for a white dwarf is shown in Figure 22, from Wheeler (1964). The stable configurations shown at higher densities are the neutron stars.

The degenerate interior of a white dwarf is practically isothermal because heat conduction by degenerate electrons is very efficient. This isothermal interior is blanketed by a nondegenerate surface layer, which is very thin and contains only

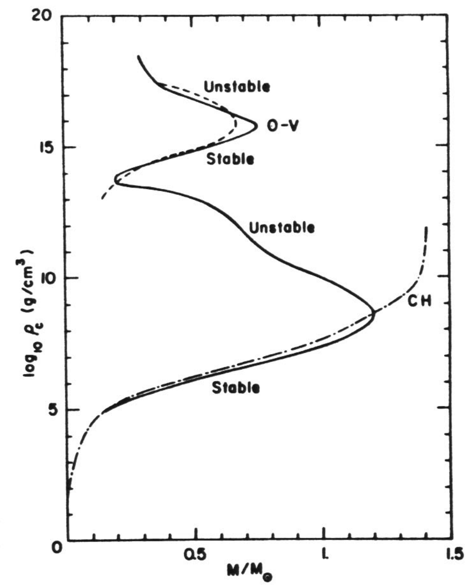

Figure 22. Schematic mass–density relation for white dwarfs and denser configurations from Wheeler (1964), calculated for cold catalyzed matter.

a minute fraction of the mass of the star. The small extent of the surface layer is easily seen by considering the scale height:

$$l = \frac{P}{\rho g} = \frac{RT}{\mu g}$$

The temperature at the transition layer is of the order of a million degrees, but $g = GM/R^2$ is extremely large because R is very small. Assuming $M \approx M_\odot$, $R \approx R_0$, and $T \approx 10^6$, then $g \approx 5 \times 10^8$ and $l \approx 10^6$ cm $= 10$ km. The density in the surface layer is less than about 10^3 g/cm^3, since it is nondegenerate. Again assuming $T \approx 10^6$, the mass of the surface layer will be

$$M_s = 4\pi R^2 \rho \Delta R \approx \pi 10^{27} \approx 10^{-6} M_\odot$$

Therefore, the equations for the surface layers may be integrated explicitly, since g, M, and L are practically constant.

The surface layer is in hydrostatic equilibrium, and energy transport is by radiation. We will assume Kramer's opacity (equation 3.8), with the quantum mechanical correction factor $(t/\bar{g}) \approx 10$. Then the equations for the structure of the envelope are (Schwarzschild, 1958, and Chandrasekhar, 1939)

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho$$

$$\frac{dT}{dr} = -\frac{3}{16\sigma}\frac{\varkappa\rho}{T^3}\frac{L}{4\pi r^2}$$

so

$$\frac{dP}{dT} = \frac{64\pi\sigma GM}{3\varkappa L} T^3$$

$$= \frac{64\pi\sigma GM}{3\varkappa_0 L} \frac{k}{\mu H} P^{-1} T^{7.5}$$

Thus, the pressure and density are related to the temperature by

$$P = \left(\frac{2}{8.5} \frac{64\pi\sigma GMk}{3\varkappa_0 L\mu H}\right)^{\frac{1}{2}} T^{4.25}$$
$$\rho = \left(\frac{2}{8.5} \frac{64\pi\sigma GM\mu H}{3\varkappa_0 Lk}\right)^{\frac{1}{2}} T^{3.25} \qquad (5.12)$$

The radial dependence of T can be found from the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{P}{T}\frac{\mu H}{k} GM \frac{1}{r^2}$$

and from equation (5.12)

$$\frac{dP}{P} = 4.25 \frac{dT}{T}$$

Thus

$$T = \frac{1}{4.25} \frac{\mu H}{k} GM \left(\frac{1}{r} - \frac{1}{R}\right) \qquad (5.13)$$

These equations for T, P, and ρ can be used throughout the nondegenerate surface layer.

The properties of the transition layer between the degenerate interior and the nondegenerate surface layer can be found as a function of the luminosity of the white dwarf (Schwarzschild, 1958). The isothermal nature of the interior gives a relation between interior temperature and the luminosity, which is constant through the surface layers, as follows:

$$L = \frac{2}{8.5} \frac{64\pi\sigma GM}{3\varkappa_0} \frac{\mu H}{k} \frac{T^{6.5}}{\rho^2} \qquad (5.14)$$

Apply this to the transition layer. The boundary condition is the equality of the electron pressures in the two regions:

$$\frac{k}{\mu_e H} \rho T = K_1 \left(\frac{\rho}{\mu_e}\right)^{\frac{5}{3}}$$

$$K_1 = \frac{h^2}{2m}\left(\frac{3}{\pi}\right)^{\frac{2}{3}} H^{-\frac{5}{3}} = 9.91 \times 10^{12}$$

so the boundary condition is

$$\rho_{tr} = \mu_e \left(\frac{kT_{tr}}{HK_1}\right)^{\frac{3}{2}} = 2.4 \times 10^{-8} \mu_e T_{tr}^{\frac{3}{2}} \qquad (5.15)$$

Then the luminosity and internal temperature, $T_c = T_{tr}$, are related by

$$L = \frac{2}{8.5} \frac{64\pi\sigma GM}{3\varkappa_0 K_1^3} \left(\frac{H}{k}\right)^4 \frac{\mu}{\mu_e^2} T_c^{3.5}$$
$$= 5.7 \times 10^{25} \left(\frac{t/\bar{g}}{Z}\right) \frac{\mu}{\mu_e^2} \frac{M}{M_\odot} T_{c(6)}^{3.5} \qquad (5.16)$$

The internal temperature, transition density, and extent of surface layer as a function of luminosity are shown in Table II for a 1 solar mass star with composition $X = 0$, $Y = 0.9$, $Z = 0.1$.

The source of energy for white dwarfs is the thermal energy of the nondegenerate nuclei. The energy source cannot be nuclear reactions. At the high densities found in white-dwarf interiors, the Coulomb barriers of nuclei are reduced. At densities greater than about 5×10^4 g/cm^3, hydrogen reactions occur, and at densities greater than about 5×10^8 g/cm^3, helium reactions occur. However, during a star's evolution before becoming a white dwarf, all the hydrogen in its core will have been exhausted, while white dwarfs with central densities great enough for helium reactions are massive enough to have exhausted the helium in their cores. In the surface layer, where hydrogen might be abundant, nuclear reactions would cause instability because of their temperature sensitivity. During a contraction, the rate of energy generation would increase above its equilibrium value, and during an expansion, it would decrease below its equilibrium value, thus feeding energy into the pulsations. The energy source cannot be gravitational, because a star's radius is fixed by the mass–radius relation after it has become almost completely degenerate, and no further contraction is possible. The energy source cannot be the thermal energy of the electrons, because they are degenerate and most are already in their lowest possible energy state.

The evolution of a white dwarf is a continual slow cooling at constant radius; its luminosity and effective temperature decrease in time. Evolutionary paths in the H–R diagram are shown for several masses in Figure 23.

The luminosity of a white dwarf is the rate of change of the thermal energy of the nondegenerate nuclei:

$$L = -\frac{d}{dt}\left(\tfrac{3}{2} kT \frac{M}{\mu_A H}\right) \qquad (5.17)$$

where μ_A is the molecular weight of the nuclei, $\mu_A^{-1} = X + \frac{1}{4}Y$. This equation can be integrated to obtain the cooling time of a white dwarf (Schwarzschild, 1958,

Table II

$L/L_\odot$	$T_c(10^6\ ^\circ\text{K})$	$\log \rho_{tr}$	$\frac{R - r_{tr}}{R}$
10^{-2}	17	3.5	0.011
10^{-3}	9	3.1	0.006
10^{-4}	4	2.6	0.003

This table is taken from Schwarzschild (1958), *Structure and Evolution of Stars*, p. 238.

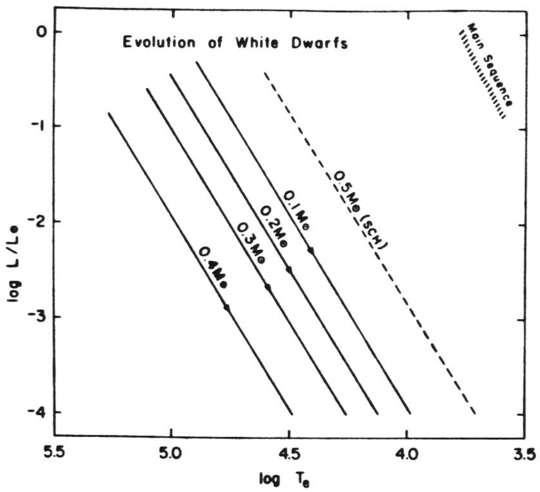

Figure 23. Evolutionary tracks of white dwarfs in the H–R diagram. Solid curves are from analytic expression (5.10); the dashed curve is from Schwarzschild (1958).

and Savedoff, 1965). Using the expression for the luminosity, equation (5.16),

$$L = K(\mu, M)T^{3.5}$$

gives

$$\frac{dT}{dt} = CT^n$$

where $n = 3.5$ and $C = -\frac{2}{3}(K\mu_A H/kM)$. Integration gives the cooling time from "infinite" temperature, setting the integration constant equal to zero, which is the time scale of evolution of white dwarfs

$$t = \frac{\left(\frac{3}{2}\frac{kT}{\mu_A H}M\right)}{(2.5L)}$$

$$= 1.73 \times 10^{11}\left(\frac{Z}{t/\bar{g}}\right)\left(\frac{\mu_e^2}{\mu\mu_A}\right)T_{c(7)}^{-2.5} \text{ years} \qquad (5.18)$$

$$= 8.92 \times 10^{7}\left(\frac{Z}{t/\bar{g}}\right)^{2/7}\left(\frac{\mu_e^2}{\mu}\right)^{2/7}\frac{1}{\mu_A}\left(\frac{M}{M_\odot}\right)^{5/7}\left(\frac{L}{L_\odot}\right)^{-5/7} \text{ years}$$

ACKNOWLEDGMENTS

I would like to thank A. G. W. Cameron, R. Stothers, E. Spiegel, and D. E. Osterbrock for stimulating discussions during the course of this work.

REFERENCES

A. G. W. Cameron (1963), *Space Physics*, lecture notes, Fairleigh Dickinson University, Rutherford, N.J.
A. G. W. Cameron (1965), *Galactic and Stellar Physics*, lecture notes, Yale University, New Haven, Conn.
S. Chandrasekhar (1939), *An Introduction to the Study of Stellar Structure*, Chicago University Press, Chicago, Ill.
A. N. Cox (1965), "Radiative Absorption and Opacity Calculations," this volume, p. 123.
D. Ezer and A. G. W. Cameron (1965), "The Contraction Phase of Solar Evolution," this volume, p. 203.
R. Härm and M. Schwarzschild (1964), *Astrophys. J.* **139**: 594.
C. Hayashi (1965), "Advanced Stages of Stellar Evolution," this volume p. 253.
C. Hayashi, R. Hoshi, and D. Sugimoto (1962), *Progr. Theoret. Phys. Suppl.* 22.
E. Hofmeister, R. Kippenhahn, and A. Weigert (1965), "Evolution of a Star of Seven Solar Masses," this volume, p. 263.
F. Hoyle and M. Schwarzschild (1955), *Astrophys. J. Suppl.* **2**: 1.
W. H. McCrea (1957), *Monthly Notices Roy. Astron. Soc.* **117**: 562.
P. Naur and D. E. Osterbrock (1953), *Astrophys. J.* **117**: 306.
D. E. Osterbrock (1963), Lecture notes, Summer Study Program in Geophysical Fluid Dynamics, Woods Hole Oceanographic Institution, Vol. II.
H. Reeves (1963), *Stellar Energy Sources*, Goddard Institute for Space Studies, NASA.
H. Reeves (1965), "Nuclear Energy Generation in Stars, and Some Aspects of Nucleosynthesis," this volume, p. 83.
E. E. Salpeter (1964) (see H. Y. Chiu, *Ann. Phys.* **26**: 364).
M. P. Savedoff (1965), "Cooling of White Dwarfs," this volume, p. 303.
E. Schatzman (1958), *White Dwarfs*, North Holland Publishing Co., Amsterdam.
M. Schönberg and S. Chandrasekhar (1942), *Astrophys. J.* **96**: 161.
M. Schwarzschild (1958), *Structure and Evolution of Stars*, Princeton University Press, Princeton, N.J.
M. Schwarzschild and H. Selberg (1962), *Astrophys. J.* **136**: 150.
E. A. Spiegel (1965), "Energy Transport by Turbulent Convection," this volume, p. 143.
J. W. Truran (1964), Private communication.
J. A. Wheeler (1964), in: H. Y. Chiu and W. F. Hoffman (eds.), *Gravitation and Relativity*, Benjamin, New York.
M. Wrubel (1958), *Handbuch der Physik, Vol. 51*, Springer-Verlag, Berlin.