VIRIAL THEOREM I.

Following Clayton §2 and Hansen §1.3

The overall behaviour of a star is largely an outcome of the virial theorem – it is worth going over in detail.

The theorem is a statistical statement about a system of masses/particles that are mutually interacting.

Recall forces can be written as $\vec{F_i} = \frac{d}{dt}\vec{p_i}$

The system is composed of mass points m_i at positions $\vec{r_i}$ susceptible to forces $\vec{F_i}$

Consider the quantity $\sum_i \vec{p_i}\cdot\vec{r_i}$ summed over all particles/masses

then $$\frac{d}{dt}\sum_i \vec{p_i}\cdot\vec{r_i} = \sum_i \frac{d\vec{p_i}}{dt}\vec{r_i} + \sum_i \vec{p_i}\frac{d\vec{r_i}}{dt}$$

$$\vec{p_i} = m\dot{\vec{r_i}}$$
$$\frac{d\vec{r_i}}{dt} = \dot{\vec{r_i}}$$

in non-relativistic case the RH term is just $2K = \sum_i m_i \dot{r}^2$

where <u>K = total kinetic energy</u> [K is sometimes split into kinetic energy of mass motion "T" and kinetic energy of random motions of atoms/molecules/nuclei "U"]

LH side can be rewritten

$$\frac{d}{dt}\sum_i \vec{p_i}\cdot\vec{r_i} = \frac{d}{dt}\sum_i m\dot{\vec{r_i}}\cdot\vec{r_i} = \frac{d}{dt}\sum_i \frac{1}{2}\frac{d}{dt}(m_i r_i^2) = \frac{1}{2}\ddot{I}$$

<u>$I = \sum_i m_i r_i^2$ is the spherical moment of inertia</u>

So now we have the identity $$\boxed{\frac{1}{2}\ddot{I} = 2K + \sum_i \vec{F_i}\cdot\vec{r_i}}$$

the sum "$\sum_i \vec{F_i}\cdot\vec{r_i}$" is called the "Virial of Clausius"

Only long-range forces and external forces are important – "collisions" between particles contribute two terms of $\vec{F_i}\cdot\vec{r_i}$ of equal and opposite sign – i.e. they contribute zero to the sum.

In <u>static</u> configurations, I = constant, $\dot{I} = \ddot{I} = 0$, so

$$\boxed{K = -\frac{1}{2}\sum_i \vec{F_i}\cdot\vec{r_i}}$$

(VIRIAL THEOREM) II.

considering forces between the particles

$$\sum_i \vec{F}_i \cdot \vec{r}_i = \sum_{pairs} \left(\vec{F}_{ij} \vec{r}_i + \vec{F}_{ji} \vec{r}_j \right)$$

For coulomb + gravity

$$\vec{F}_{ij} = -\vec{F}_{ji}$$

Coulomb forces are strong, but star is neutral overall and interactions at "point" — i.e. $\vec{r}_i = \vec{r}_j$, cancel each other, contributing negligibly to $\sum_i \vec{F}_i \cdot \vec{r}_i$. So we are left with gravity:

$$F_{ij} = -\frac{G m_i m_j}{(r_{ij})^3} (\vec{r}_i - \vec{r}_j)$$

so our "static" version of the virial theorem becomes

$$K = -\frac{1}{2} \sum_i \vec{F}_i \cdot \vec{r}_i = -\frac{1}{2} \sum_{pairs} F_{ij} (\vec{r}_i - \vec{r}_j) = \frac{1}{2} \sum_{pairs} \underbrace{\frac{G m_i m_j}{r_{ij}}}$$

i.e. negative of P.E. of pairs

so

$$\boxed{K = -\frac{\Omega}{2}}$$

$$\Omega = -\sum \frac{G m_i m_j}{r_{ij}} =$$

total potential energy of system (star in this case)

This is what is commonly called "the virial theorem" — appropriate for static configurations (I = constant) with interparticle forces with $\frac{1}{r^2}$ dependence.

Particle kinetic energy scales with temperature ...

Consider the "virial of Clausius" for a box of ideal gas at some pressure. Consider only external pressure force acting on the surface of the container — assume interparticle coulomb forces contribute negligibly to sum.

$$\sum \vec{F}_i \cdot \vec{r}_i = \int_S (-P) d\vec{S} \cdot \vec{r} = -P \int_S \vec{r} \cdot \vec{n} \, dS$$

$\vec{n}$ is unit vector normal to surface

Apply Gauss' theorem $= -P \int_V \nabla \cdot \vec{r} \, dV = -3PV$

so from before $K = -\frac{1}{2} \sum_i \vec{F}_i \cdot \vec{r}_i = \frac{3}{2} PV - \frac{1}{2} \underbrace{\sum_{pair} \vec{F}_{ij} (\vec{r}_i - \vec{r}_j)}$, ϕ for ideal gas

interparticle forces e.g. Coulomb

Ideal gas law $PV = NkT$
$P = nkT$

P : Pressure
V : Volume
N : # molecules
k : Boltzmann constant
1.38×10^{-23} J/K
$= R/N_A$
T : Temperature
n : # molecules/unit volume

so

$$K = \frac{3}{2}PV = \frac{3}{2}NkT$$

i.e. Kinetic energy per particle is $\frac{3}{2}kT$

Note that K only includes translational kinetic energy thus far. Additional degrees of freedom (rotation, vibration, excitation) and photon energy has not yet been included.

For a star in steady-state/equillibrium, we can write the total energy

$$E = K + \Omega = -\frac{\Omega}{2} + \Omega = \frac{\Omega}{2}$$

(K: kinetic, Ω: potential)

so for star composed of ideal gas

$$K = -\frac{\Omega}{2} = \frac{3}{2}NkT \qquad E = \frac{\Omega}{2}$$

since Ω negative, E is negative

if star formed from a very large configuration (through gravitational contraction) then Ω initially was ~0. Since observed stars have very negative Ω, and energy is conserved overall, then significant E must have been radiated away, out of the system, during collapse.

When star changes potential energy Ω, i.e. contracts, half of the Ω is converted to "heat" K. Since we have a thermal gradient ($T\downarrow$ as $R\uparrow$) – energy is transported via radiation or convection to the surface of the star and "lost" as luminosity.

VIRIAL THEOREM IV

Consequences of Virial theorem for adiabatic gas

From 1st law of thermo. for $dQ = 0$

$$dU = -PdV$$

internal energy — work

For adiabatic gas

$$PV^{\gamma} = \text{constant}$$
$$= TV^{\gamma-1}$$
$$= P^{1-\gamma}T^{\gamma}$$

$$\frac{dT}{T} + (\gamma-1)\frac{dV}{V} = \phi$$

recall $\gamma = \frac{C_P}{C_V}$ ratio of specific heats at constant P and V

$$dV = -\frac{dT}{T}\cdot\frac{V}{(\gamma-1)}$$

for completely ionized matter there are only 3 deg. of freedom

$$\gamma = 1 + \frac{2}{f} \qquad f = 3 \quad \gamma = \frac{5}{3}$$

$\gamma = 5/3$ for monatomic ideal gas

$$dU = -PdV = \frac{PV\,dT}{T(\gamma-1)}$$

$$dU = Nk\frac{dT}{\gamma-1}$$

recall
$P = nkT$
$PV = NkT$
$\frac{PV}{T} = Nk$

the total kinetic energy of our ideal gas is

$$K = \frac{3}{2}NkT$$

(N ← total particles)

so $dK = \frac{3}{2}Nk\,dT$

$$dT = \frac{dK}{\frac{3}{2}Nk} = \frac{2}{3Nk}dK$$

so $dU = Nk\frac{dT}{\gamma-1} = \frac{Nk}{\gamma-1}\cdot\frac{2\,dK}{3Nk} = \frac{2}{3}\frac{dK}{\gamma-1}$

For adiabatic ideal gas

integrating over the whole star $U = \frac{2}{3}\frac{K}{\gamma-1} \Rightarrow$ $\boxed{K = \frac{3}{2}(\gamma-1)U}$

Write a more general form of an equation for total energy E

$$E = U + \Omega = U - 2K = U - 3(\gamma-1)U = U(1-3\gamma+3)$$

$$\boxed{E = -(3\gamma-4)U = \frac{3\gamma-4}{3(\gamma-1)}\Omega}$$

total energy — generalized for additional degrees of freedom (for adiabatic index γ) — restatement of virial theorem

for ideal monatomic gas $\gamma = \frac{5}{3}$ $E = -U = \frac{\Omega}{2}$