# Emden Equation

# Purpose

- Describe the density of an isothermal hydrostatic sphere

- Such a sphere is called a “ bonner-ebert” sphere.

# Method

- Transform 2 coupled ODE's into 1 second-order non-linear ODE.

- Non-dimensionalize the equation to find general characteristics of the density function.

# The Equations

For this derivation one starts with the equations of :

- Hydrostatic Equilibrium--

$$\frac{dP}{dr} = -\frac{(GM_r\,\rho)}{r^2} \quad (1)$$

- Mass of a spherical shell---

$$\frac{dM_r}{dr} = 4\,\pi\,r^2\,\rho \quad (2)$$

# Combining the Equations

- By solving (1) for M(r),

$$M(r) = -\frac{r^2}{G\rho}\ \frac{dP}{dr} \quad (3)$$

we can couple these two equations by plugging M(r) into (2). This yields:

$$\frac{1}{r^2}\ \frac{d}{dr}\left[\frac{r^2\ dP}{\rho\ dr}\right] = -4\pi G\rho \quad (4)$$

- (4) is the 2nd-order non-linear ODE we were looking for. It is known as the "Emden - Equation", and its solution gives us $\rho$ as a function of r.
- Note that with the equation of state, we can relate P to T. By making the substitution,

$$P = nkT = V_s^2\,\rho\,,$$
$$\text{where } V_s^2 = \sqrt{\frac{kT}{m}} \quad (5)$$

we may specify the Temperature of the isothermal sphere and our D.E. will then only contain $\rho$, r as variables.

# Solving the Equation

By non-dimensionalizing this equation, we can find and plot a scaled numerical solution:

The fully specified problem becomes:

$$y'' + \frac{(y')^2}{y} + \frac{2\,y'}{x} + y^2 = 0 \quad (6)$$

$$y = \rho / \rho_c \quad (7)$$

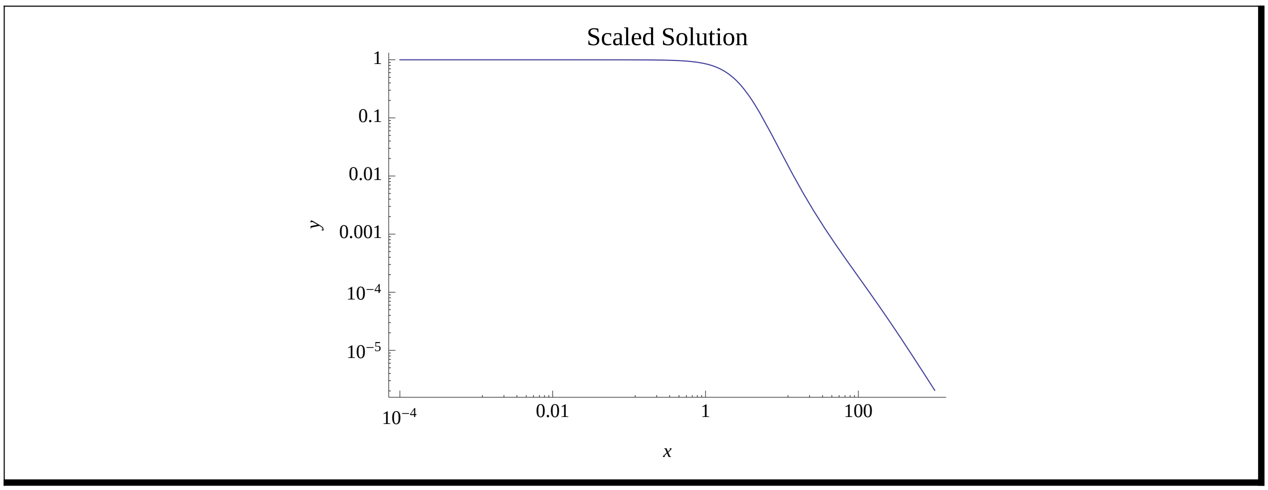
$$x = r \sqrt{\frac{4\,\pi \mathrm{Gm}\rho_c}{\mathrm{kT}}} \quad (8)$$

$$y[0] = 1, \; y'[0] = 0 \quad (9)$$

The solution of which is a Bonner-Ebert Sphere.

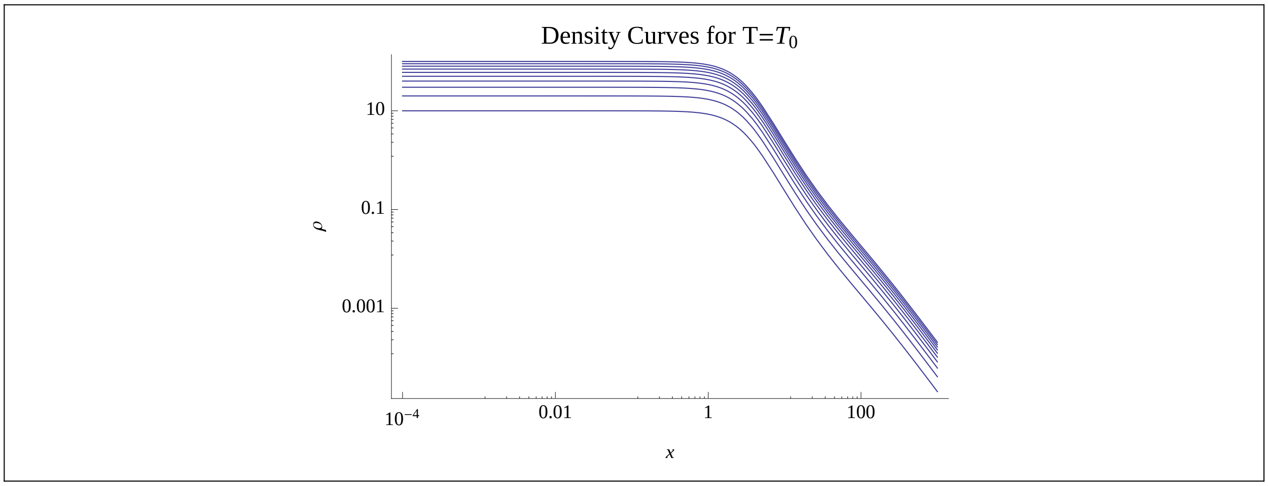
# Numerical Solution

The "general" shape of the curve looks like this--




The y-axis can easily be scaled to different physical densities by multiplying the y-values by $\rho_c$. I am now in the stage of trying to appropriately plot these curves before researching a bit more into sensible values for $\rho_c$ and T.

Once I do this, I can make a series of plots for different values of T, that will look like:




Here the different curves would represent the different values of $\rho_c$