# Virial Theorem and Cloud Stability

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## Virial Equilibrium

- **From combining the equations of mass and momentum conservation, one can arrive at the virial theorem for a stationary cloud :**

$$\frac{1}{2}\ \frac{D^2 I}{Dt^2} = \int_V d\mathrm{V} 3\mathrm{P} + \int_V \frac{B^2}{8\pi}\, dV - \int_S \left(P + \frac{B^2}{8\pi}\right) \vec{r}\, d\vec{S} + \frac{1}{4\pi} \int_S \left(\vec{r}\, \vec{B}\right)\left(\vec{B}\, d\vec{S}\right) - \int_V d\mathrm{V} \rho\, \vec{r}\, \nabla \phi$$

- **The condition for virial equilibrium in this equation is:**

$$\frac{D^2 I}{Dt^2} = 0$$

**When we consider this equilibrium condition for ISOTHERMAL, SPHERICAL, NON-MAGNETIZED clouds, we can study stability. The virial equilibrium condition becomes:**

$$\int_S \left(P + \frac{B^2}{8\pi}\right) \vec{r}\, d\vec{S} = \int_V d\mathrm{V} 3\mathrm{P} - \int_V d\mathrm{V} \rho\, \vec{r}\, \nabla \phi$$

**Which integrates to --**

$$4\pi R_{cl}^3 P = 3 C_s^2 M_{cl} - \frac{3}{5}\ \frac{GM_{cl}^{\ 2}}{R_{cl}}$$

**This is the equation of <u>virial equilibrium</u> of a spherical, isothermal, non-magnetized cloud. Since the LHS constitutes a surface pressure-"like" term, and the RHS is the sum of an internal pressure-like term and gravitational potential energy, it seems intuitively correct that this is an equilibrium equation if we consider the LHS external parameters and the RHS internal. Thus, let's call P on the LHS $P_o$ for outside pressure:**

$$4\pi R_{cl}^3 P_o = 3 C_s^2 M_{cl} - \frac{3}{5}\ \frac{GM_{cl}^{\ 2}}{R_{cl}} (*)$$

- **In the limit R → ∞, gravity becomes neglible. Thus at large radii, equilibrium is established by a balance of internal and external pressures:**

$$4\pi R_{cl}^3 P_o = 3 C_s^2 M_{cl}$$

**In the small radii limit though, gravity is important and so that second term does not vanish. We can solve for a minimum radius of a cloud in virial equilibrium by solving for an R such that the external pressure of (*) equals 0:**

$$R_{min} = \frac{1}{5} \frac{GM_{cl}}{C_s^2}$$

**Any cloud with a smaller radius than this will not be in virial equilibrium. It is important to note that the $R_{cl} > R_{min}$ alone is <u>not enough to insure stability</u>.**

## Cloud Stability

- **$R_{cl} > R_{min}$ is not enough to insure stability. That is, while $R_{cl} > R_{min}$will satisfy virial equilibrium (it was after all, derived from virial theorem considerations), NOT ALL EQUILIBRIA ARE STABLE. To find stable equilibria, we seek to satisfy the condition:**

$$\frac{dP_o}{dR_{cl}} < 0$$

**Since,**

$$\frac{dP_o}{dR_{cl}} = \frac{1}{4\pi}\left(-\frac{9\,C_s^{\,2}\,M_{cl}}{R_{cl}^{\,4}} + \frac{12}{5}\,\frac{GM_{cl}^{\,2}}{R_{cl}^{\,5}}\right)$$

**we see that at large enough radii, the derivative becomes negative (the second term < first term). Thus the derivative changes sign, defining a max pressure. Solving for $R_{crit}$ and $P_{crit}$ gives:**

$$R_{crit} = \frac{4}{15}\,\frac{GM_{cl}}{C_s^{\,2}},\ P_{crit} = 3.15\,\frac{C_s^{\,8}}{G^3\,M_{cl}^{\,2}}$$

**Higher pressures, produce no equilibria.**