Relativistic Quantum Mechanics
Homework 8 (solution)

November 23, 2007

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$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 && (1)\\
\frac{\partial\mathcal{L}}{\partial\phi} &= -m^2\phi - \frac{\lambda}{3!}\phi^3\\
\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} &= \partial^\mu\phi\\
\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} &= 0 = -m^2\phi - \frac{\lambda}{3!}\phi^3 - \Box\phi && (2)
\end{aligned}$$

$$\begin{aligned}
\partial^\nu\mathcal{L} &= \partial^\nu(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4)\\
&= \frac{1}{2}(\partial^\nu\partial_\mu\phi)\partial^\mu\phi + \frac{1}{2}\partial_\mu\phi(\partial^\nu\partial^\mu\phi) - m^2\phi\partial^\nu\phi - \frac{\lambda}{3!}\phi^3\partial^\nu\phi\\
&= \frac{1}{2}(\partial^\nu\partial_\mu\phi)\partial^\mu\phi + \frac{1}{2}\partial^\mu\phi(\partial^\nu\partial_\mu\phi) - (m^2\phi + \frac{\lambda}{3!}\phi^3)\partial^\nu\phi\\
&= (\partial^\nu\partial_\mu\phi)\partial^\mu\phi - (m^2\phi + \frac{\lambda}{3!}\phi^3)\partial^\nu\phi && (3)
\end{aligned}$$

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu(\partial^\nu\phi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \eta^{\mu\nu}\mathcal{L})\\
&= \partial_\mu(\partial^\nu\phi\partial^\mu\phi - \eta^{\mu\nu}\mathcal{L})\\
&= (\partial_\mu\partial^\nu\phi)\partial^\mu\phi + \partial^\nu\phi(\Box\phi) - \partial^\nu\mathcal{L}\\
&\overset{Eq.3}{=} (\partial_\mu\partial^\nu\phi)\partial^\mu\phi + \partial^\nu\phi(\Box\phi) - \partial^\nu\mathcal{L} - (\partial^\nu\partial_\mu\phi)\partial^\mu\phi + (m^2\phi + \frac{\lambda}{3!}\phi^3)\partial^\nu\phi\\
&= \partial^\nu\phi(\Box\phi) + (m^2\phi + \frac{\lambda}{3!}\phi^3)\partial^\nu\phi\\
&= (\Box\phi + m^2\phi + \frac{\lambda}{3!}\phi^3)\partial^\nu\phi\\
&\overset{Eq.2}{=} 0 && (4)
\end{aligned}$$

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2.1

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}} (e^{-ikx}a(\mathbf{k}) + e^{ikx}a^\dagger(\mathbf{k})) \tag{5}$$

$$\overrightarrow{\nabla}\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}} i\mathbf{k}(e^{-ikx}a(\mathbf{k}) - e^{ikx}a^\dagger(\mathbf{k})) \tag{6}$$

$$\dot{\phi}(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}} ik^0(-e^{-ikx}a(\mathbf{k}) + e^{ikx}a^\dagger(\mathbf{k})) \tag{7}$$

and

$$\begin{aligned}
\mathbf{P} &= -\int d^3x \overrightarrow{\nabla}\phi(x)\dot{\phi}(x) \qquad (8)\\
&= \int \frac{d^3x d^3k d^3k'}{(2\pi)^3 2\sqrt{k^0 k'^0}} \mathbf{k}k'^0 (e^{-ikx}a(\mathbf{k}) - e^{ikx}a^\dagger(\mathbf{k}))(-e^{-ik'x}a(\mathbf{k}') + e^{ik'x}a^\dagger(\mathbf{k}'))\\
&= \int \frac{d^3x d^3k d^3k'}{(2\pi)^3 2\sqrt{k^0 k'^0}} \mathbf{k}k'^0 \cdot\\
&\quad (-e^{-i(k+k')x}a(\mathbf{k})a(\mathbf{k}') + e^{-i(k-k')x}a(\mathbf{k})a^\dagger(\mathbf{k}') + e^{i(k-k')x}a^\dagger(\mathbf{k})a(\mathbf{k}') - e^{i(k+k')x}a^\dagger(\mathbf{k})a^\dagger(\mathbf{k}'))\\
&= \int \frac{d^3k d^3k'}{2\sqrt{k^0 k'^0}} \mathbf{k}k'^0 \cdot\\
&\quad (-\delta^3(k+k')a(\mathbf{k})a(\mathbf{k}') + \delta^3(k-k')a(\mathbf{k})a^\dagger(\mathbf{k}') + \delta^3(k-k')a^\dagger(\mathbf{k})a(\mathbf{k}') - \delta^3(k+k')a^\dagger(\mathbf{k})a^\dagger(\mathbf{k}'))\\
&= \int \frac{d^3k}{2} \mathbf{k}(-ia(\mathbf{k})a(-\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) - ia^\dagger(\mathbf{k})a^\dagger(\mathbf{k}))\\
&= \int d^3k \frac{1}{2}\mathbf{k}(a(\mathbf{k})a^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k})) \qquad (9)
\end{aligned}$$

so

$$:\mathbf{P}:= \int d^3k \mathbf{k} a^\dagger(\mathbf{k})a(\mathbf{k}) \tag{10}$$

2.2

The momentum operator wouldn't change because the momentum is defined as the partial derivative of the Lagrangian with respect to the derivative of the field and not the field itself. The extra term $\lambda\phi^4$ wouldn't affect $\mathbf{P}$.

3

$$[\phi(x), \phi(y)]_{x^0=y^0} = 0 = [\Pi(x), \Pi(y)] \tag{11}$$

$$[\phi(x), \Pi(y)]_{x^0=y^0} = i\delta^3(x-y) \tag{12}$$

For $\mu = 0$:

$$P_0(y) = P^0(y) = \int d^3y T^{00} = \int d^3y [\frac{1}{2}\dot{\phi}^2(y) + \frac{1}{2}\overrightarrow{\nabla}\phi(y)\overrightarrow{\nabla}\phi(y) + \frac{m^2}{2}\phi^2(y)] \tag{13}$$

$$
\begin{aligned}
[P_0(y), \phi(x)] &= \int d^3y \frac{1}{2}(\dot{\phi}(y)[\dot{\phi}(y), \phi(x)] + [\dot{\phi}(y), \phi(x)]\dot{\phi}(y)) \\
&= \int d^3y \frac{1}{2}(-i\delta^3(y-x)\dot{\phi}(y) - i\delta^3(y-x)\dot{\phi}(y)) \\
&= -i\dot{\phi}(y) = -i\partial_0\phi(x) \qquad (14)
\end{aligned}
$$

For $\mu = i$:

$$
P_i(y) = -P^i(y) = \int d^3y T^{0i} = \int d^3y \overrightarrow{\nabla}\phi(y)\dot{\phi}(y) \qquad (15)
$$

$$
\begin{aligned}
[P_i(y), \phi(x)] &= \int d^3y \overrightarrow{\nabla}\phi(y)[\dot{\phi}(y), \phi(x)] \qquad (16) \\
&= \int d^3y \overrightarrow{\nabla}\phi(y)(-i\delta^3(y-x)) = -i\overrightarrow{\nabla}\phi(x) = -i\partial_i\phi(x) \qquad (17)
\end{aligned}
$$

Therefore,

$$
[P_\mu, \phi(x)] = -i\partial_\mu\phi(x) \qquad (18)
$$