Relativistic Quantum Mechanics
Homework 9 (solution)

December 4, 2007

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1.1

$$|j_1 m_1; j_2 m_2\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle \tag{1}$$

$$\begin{pmatrix} |+\rangle \otimes \langle +| & |+\rangle \otimes \langle -| & |+\rangle \otimes \langle +| & |+\rangle \otimes \langle -| \\ |-\rangle \otimes \langle +| & |-\rangle \otimes \langle -| & |-\rangle \otimes \langle +| & |-\rangle \otimes \langle -| \\ |+\rangle \otimes \langle +| & |+\rangle \otimes \langle -| & |+\rangle \otimes \langle +| & |+\rangle \otimes \langle -| \\ |-\rangle \otimes \langle +| & |-\rangle \otimes \langle -| & |-\rangle \otimes \langle +| & |-\rangle \otimes \langle -| \end{pmatrix} \tag{2}$$

$$\overrightarrow{J} = \overrightarrow{J_1} + \overrightarrow{J_2} \tag{3}$$

$$J_i = J_{1i} + J_{2i} = \frac{J_{2+} + J_{2-}}{2} + \frac{J_{2+} + J_{2-}}{2} \tag{4}$$

$$J_x|++\rangle = \frac{1}{2}|-+\rangle + \frac{1}{2}|+-\rangle$$
$$J_x|+-\rangle = \frac{1}{2}|--\rangle + \frac{1}{2}|++\rangle$$
$$J_x|-+\rangle = \frac{1}{2}|++\rangle + \frac{1}{2}|--\rangle$$
$$J_x|--\rangle = \frac{1}{2}|+-\rangle + \frac{1}{2}|-+\rangle$$

$$J_x = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{5}$$

Similarly,

$$J_y = \frac{1}{2}\begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix} \tag{6}$$

$$J_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{7}$$

We also have

$$\begin{aligned} J_{1x} \otimes \mathbf{1} + \mathbf{1} \otimes J_{1x} &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{aligned} \tag{8}$$

$$\begin{aligned} J_{1y} \otimes \mathbf{1} + \mathbf{1} \otimes J_{1y} &= \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix} \end{aligned} \tag{9}$$

$$\begin{aligned} J_{1z} \otimes \mathbf{1} + \mathbf{1} \otimes J_{1z} &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned} \tag{10}$$

1.2

$$\begin{pmatrix} |j=1,m=1\rangle \\ |j=1,m=0\rangle \\ |j=1,m=-1\rangle \\ |j=0,m=0\rangle \end{pmatrix} = U \begin{pmatrix} |m_1=\frac{1}{2};m_2=\frac{1}{2}\rangle \\ |m_1=\frac{1}{2};m_2=-\frac{1}{2}\rangle \\ |m_1=-\frac{1}{2};m_2=\frac{1}{2}\rangle \\ |m_1=-\frac{1}{2};m_2=-\frac{1}{2}\rangle \end{pmatrix} \tag{11}$$

$$|j;m\rangle = \sum_{m_1,m_2} |m_1;m_2\rangle\langle m_1;m_2|j,m\rangle \tag{12}$$

$$U = \begin{pmatrix} \langle m_1=\frac{1}{2};m_2=\frac{1}{2}|j=1,m=1\rangle & \langle m_1=\frac{1}{2};m_2=-\frac{1}{2}|j=1,m=1\rangle & ... & ... \\ \langle m_1=\frac{1}{2};m_2=\frac{1}{2}|j=1,m=0\rangle & ... & ... & \\ ... & ... & ... & ... \\ ... & ... & ... & ... \end{pmatrix} \tag{13}$$

The only non-vanishing components of U are

$$\begin{aligned} \langle\frac{1}{2};\frac{1}{2}|1,1\rangle &= \langle-\frac{1}{2};-\frac{1}{2}|1,-1\rangle = 1 \\ \langle\frac{1}{2};-\frac{1}{2}|1,0\rangle &= \langle-\frac{1}{2};\frac{1}{2}|1,0\rangle = \frac{1}{\sqrt{2}} \\ \langle\frac{1}{2};-\frac{1}{2}|0,0\rangle &= -\langle-\frac{1}{2};\frac{1}{2}|0,0\rangle = \frac{1}{\sqrt{2}} \end{aligned} \tag{14}$$

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad \text{and} \quad U^{-1} = U^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$J'_x = UJ_xU^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{15}$$

$$J'_y = UJ_yU^{-1} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{16}$$

$$J'_z = UJ_zU^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

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$$\phi(x)\phi(y) = \,:\phi(x)\phi(y): - iG^+(x-y) \tag{18}$$
$$T(\phi(x)\phi(y)) = \,:\phi(x)\phi(y): + iG_F(x-y) \tag{19}$$
$$G_F(x) = -\theta(x^0)G^+(x) + \theta(-x^0)G^-(x) \tag{20}$$
$$G^A(x) = \theta(-x^0)G(x) \tag{21}$$

also

$$(\Box_x + m^2)\phi(x) = 0 \tag{22}$$

$$(\Box_x + m^2)G^A(x) = -\delta^4(x) \tag{23}$$

$$\begin{aligned}
T(\phi(x)\phi(y)) &= :\phi(x)\phi(y): + iG_F(x-y) \\
&= \phi(x)\phi(y) + iG^+(x-y) + iG_F(x-y) \\
&= \phi(x)\phi(y) + i[G^+(x-y) + G_F(x-y)] \\
&= \phi(x)\phi(y) + i[G^+(x-y) - \theta(x^0-y^0)G^+(x-y) + \theta(y^0-x^0)G^-(x-y)] \\
&= \phi(x)\phi(y) + i[(1-\theta(x^0-y^0))G^+(x-y) + \theta(y^0-x^0)G^-(x-y)] \\
&= \phi(x)\phi(y) + i\theta(y^0-x^0)[G^+(x-y) + G^-(x-y)] \\
&= \phi(x)\phi(y) + i\theta(y^0-x^0)G(x-y) \\
&= \phi(x)\phi(y) + iG^A(x-y)
\end{aligned}$$

$$\begin{aligned}
(\Box_x + m^2)T(\phi(x)\phi(y)) &= (\Box_x + m^2)\phi(x)\phi(y) + i(\Box_x + m^2)G^A(x) \\
&= -i\delta^4(x-y) \qquad (24)
\end{aligned}$$

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3.1

$$\delta x^\mu = \epsilon^\mu{}_\nu x^\nu \tag{25}$$

$$\delta\phi(x) = \phi'(x) - \phi(x) = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \phi(x) \tag{26}$$

$$\begin{aligned}
\delta\mathcal{L} &= \delta\phi(x)\frac{\partial\mathcal{L}}{\partial\phi(x)} + \partial_\alpha\delta\phi(x)\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi(x))} \\
&= -\epsilon^\mu{}_\nu(x^\nu\partial_\mu\phi(x)\frac{\partial\mathcal{L}}{\partial\phi(x)} + (\partial_\alpha x^\nu)(\partial_\mu\phi(x))\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi(x))} + x^\nu(\partial_\alpha\partial_\mu\phi)\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi(x))}) \\
&= -\epsilon^\mu{}_\nu(x^\nu[\partial_\mu\phi(x)\frac{\partial\mathcal{L}}{\partial\phi(x)} + (\partial_\alpha\partial_\mu\phi)\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi(x))}] + \delta^\nu_\alpha\partial_\mu\phi\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi(x))}) \\
&= -\epsilon^\mu{}_\nu(x^\nu\partial_\mu\mathcal{L} + (\partial_\mu\phi)\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi(x))}) \\
&= -\epsilon^\mu{}_\nu\partial_\mu(x^\nu\mathcal{L} + \phi\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi(x))}) + \epsilon^\mu{}_\nu(\delta^\nu_\mu\mathcal{L} + \phi\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi(x))}) \\
&= -\epsilon^\mu{}_\nu\partial_\mu(x^\nu\mathcal{L} + \phi\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi(x))}) + \epsilon^\mu{}_\mu\mathcal{L} + \epsilon^\mu{}_\nu\phi\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi(x))} \\
&= -\epsilon^\mu{}_\nu\partial_\mu(x^\nu\mathcal{L} + \phi\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi(x))}) \qquad (27)
\end{aligned}$$

$$\delta\mathcal{L} = \partial_\mu K^\mu \tag{28}$$

$$K^\mu = -\epsilon^\mu{}_\nu(x^\nu\mathcal{L} + \phi\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi(x))}) \tag{29}$$

$$\begin{aligned} J^\mu &= \delta\phi(x)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))} - K^\mu \\ &= -\epsilon^\nu{}_\sigma x^\sigma\partial_\nu\phi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))} + \epsilon^\mu{}_\sigma(x^\sigma\mathcal{L} + \phi\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi(x))}) \\ &= -\epsilon_{\nu\sigma}[x^\sigma\partial^\nu\phi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))} - \eta^{\mu\nu}(x^\sigma\mathcal{L} + \phi\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi(x))})] \\ &= -\epsilon_{\nu\sigma}T^{\mu\nu\sigma} \end{aligned} \tag{30}$$

$$T^{\mu\nu\sigma} = x^\sigma\partial^\nu\phi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \eta^{\mu\nu}(x^\sigma\mathcal{L} + \phi\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi)}) \tag{31}$$

3.2

It's clear that $T^{\mu\nu\sigma}$ is symmetric in μ, ν

$$T^{\mu\nu\sigma} = T^{\nu\mu\sigma} \tag{32}$$

Since $\epsilon_{\nu\sigma}$ is antisymmetric $T^{\mu\nu\sigma}$ is also antisymmetric in μ, σ (or ν, σ).

3.3

$$\begin{aligned} Q &= \int d^3x J^0(\overrightarrow{x}, t) \\ &= -\epsilon_{\nu\sigma}\int d^3x T^{0\nu\sigma} \\ &= \int d^3x(-\epsilon_{\nu\sigma}[x^\sigma\partial^\nu\phi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi(x))} - \eta^{0\nu}(x^\sigma\mathcal{L} + \phi\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi(x))})]) \end{aligned} \tag{33}$$

For $\nu = j, \sigma = k$

$$Q = -\int d^3x\epsilon_{jk}x^k\partial^j\phi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi(x))} \tag{34}$$