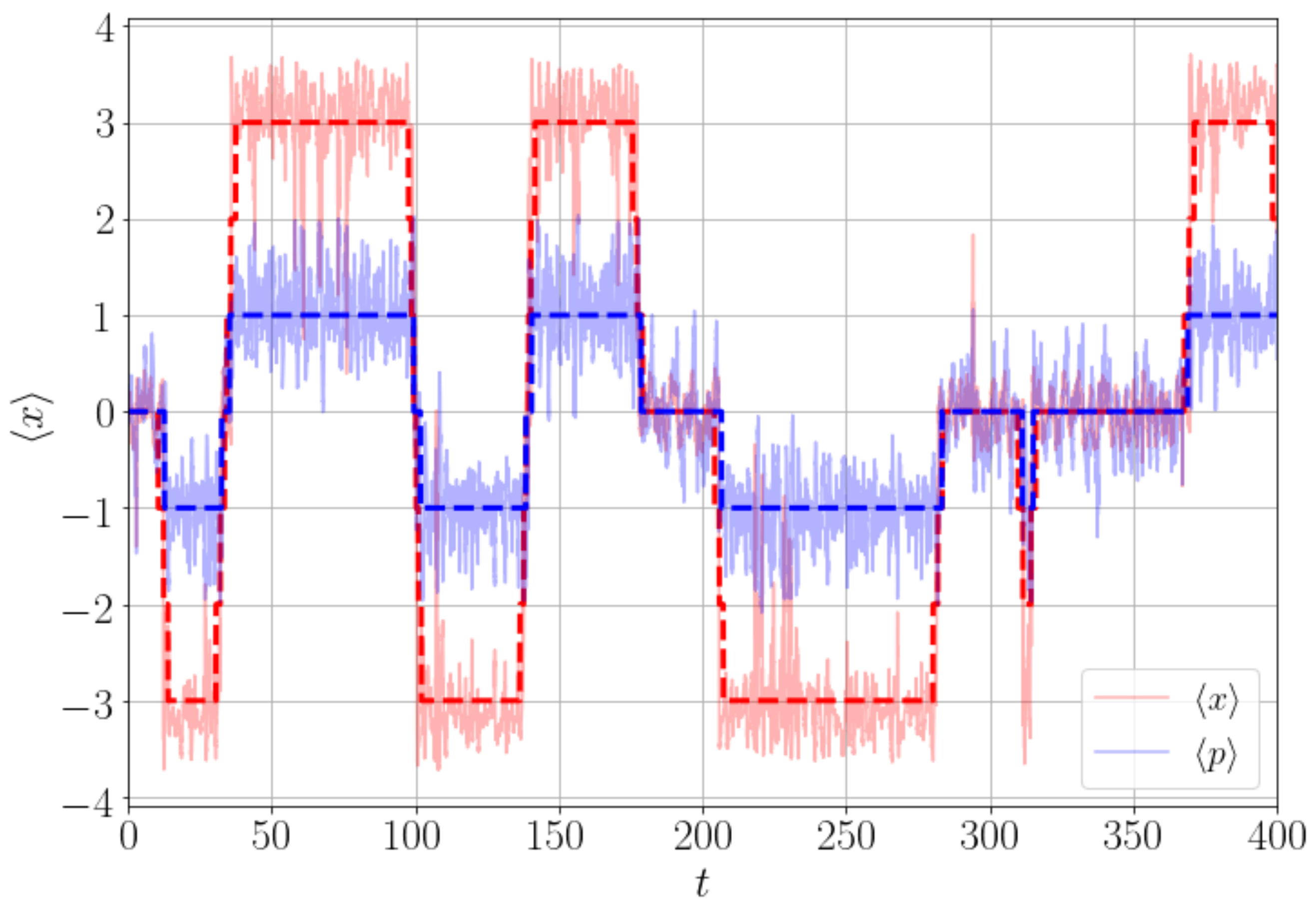
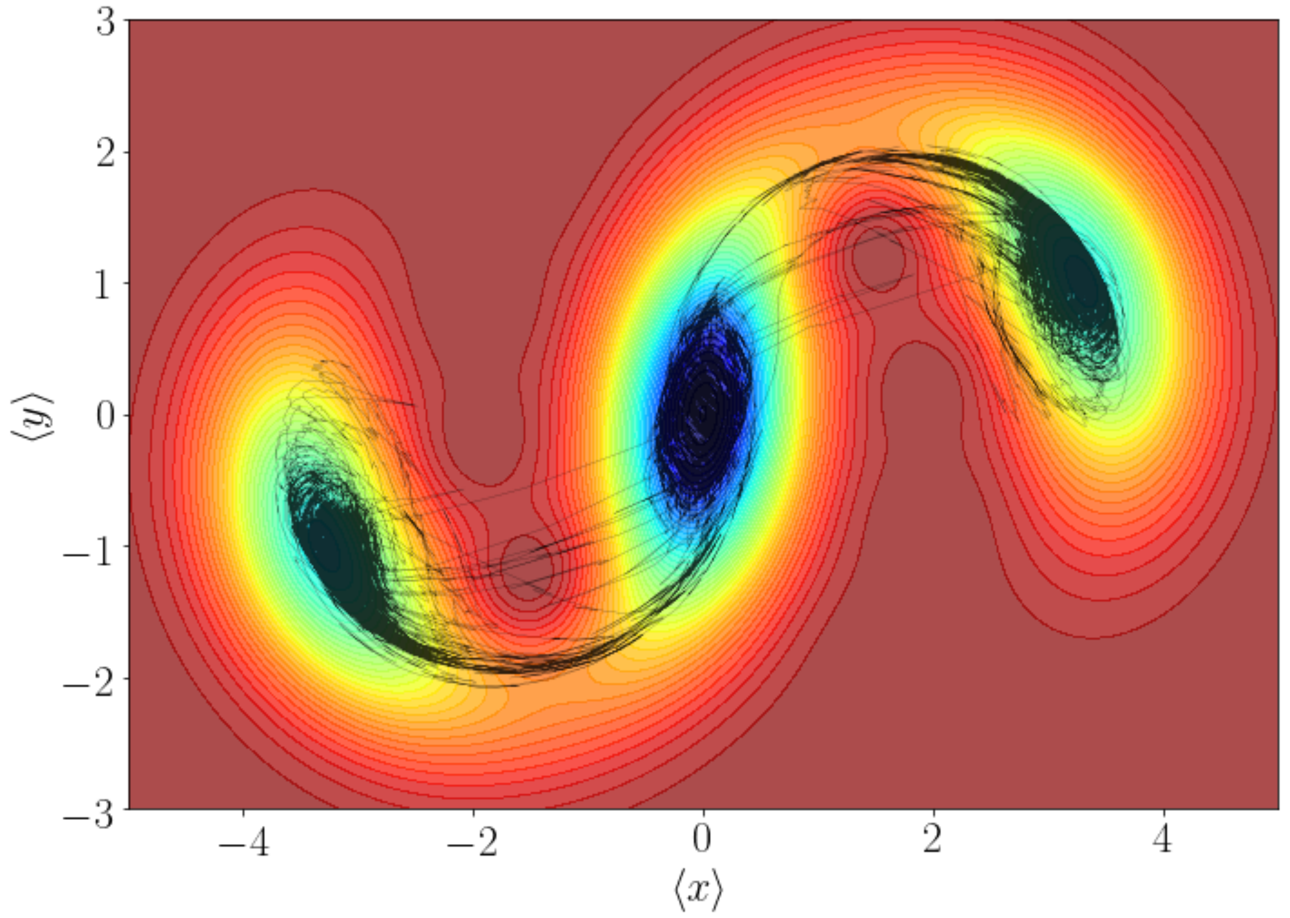




Time-domain fluctuations in quantum non-equilibrium systems

Gabriel T. Landi - Universidade de São Paulo

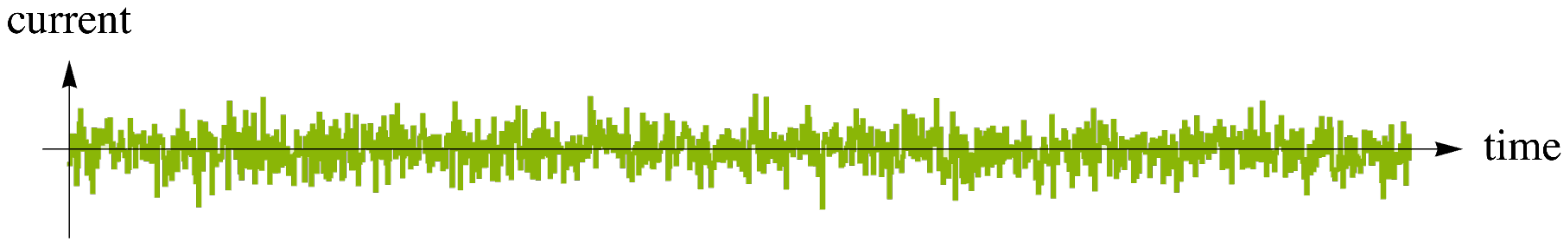
In collaboration with Mark Mitchison & Michael Kewming

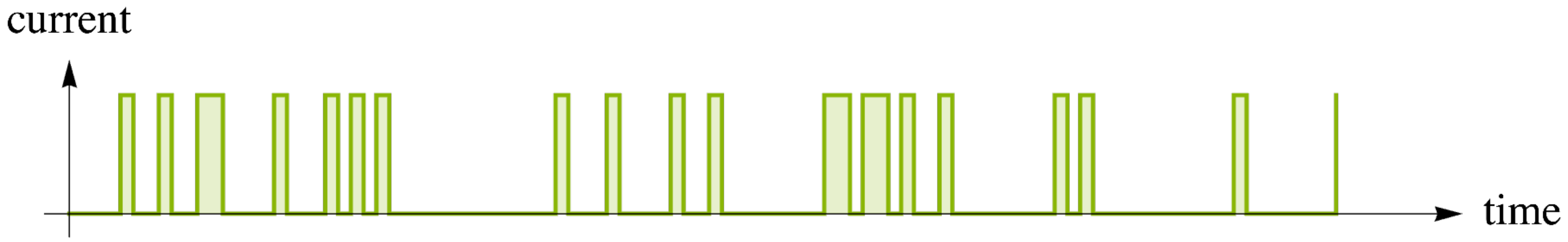
02/05/2022 - University of Basel

www.fmt.if.usp.br/~gtlandi

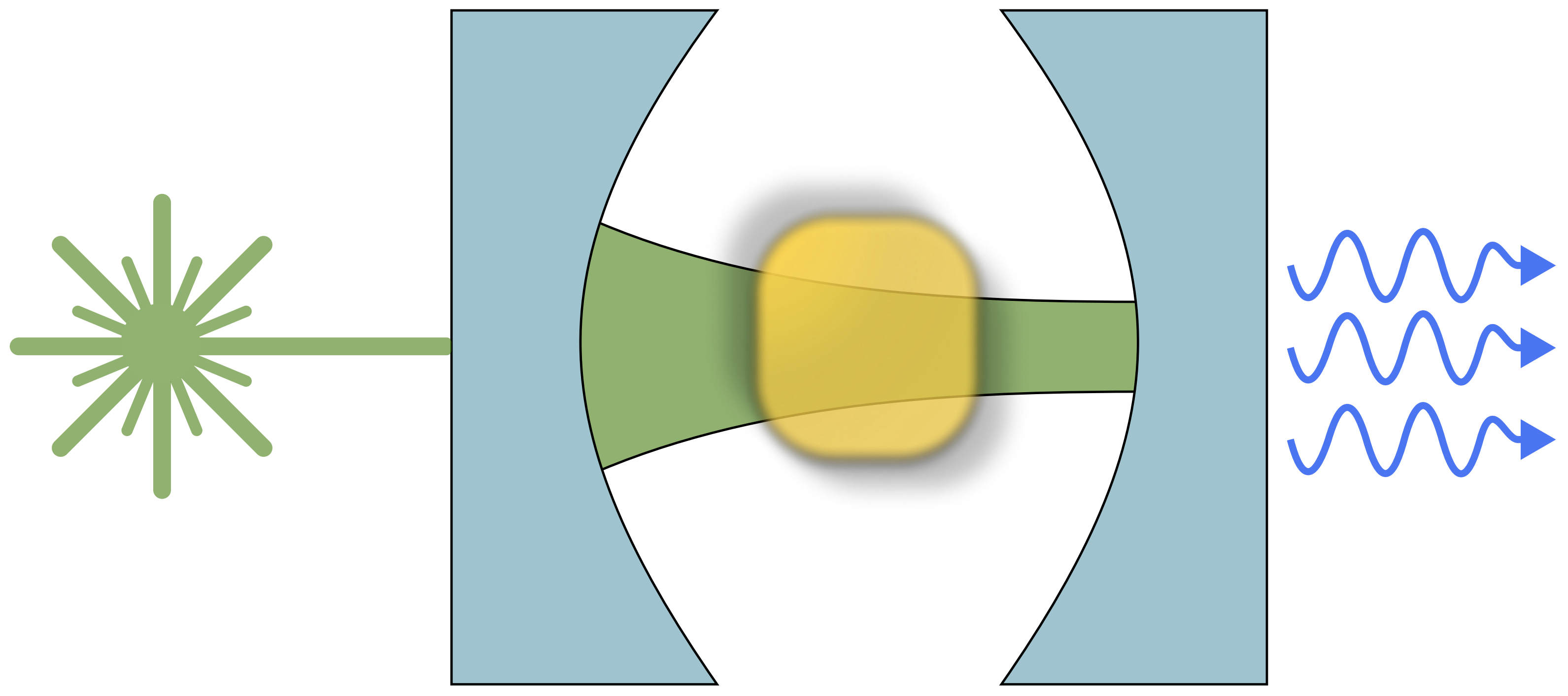
The story

- Open Quantum Systems generate **classical (*stochastic*) currents.**
- Experimental characterization via continuous measurements.
  - Discrete like emissions: photo-detection, electron counting.
  - Continuous emissions: homodyne, heterodyne.
- **Motivation:** to go beyond the average, and understand the **fluctuations** Δ_I^2 of the output current.

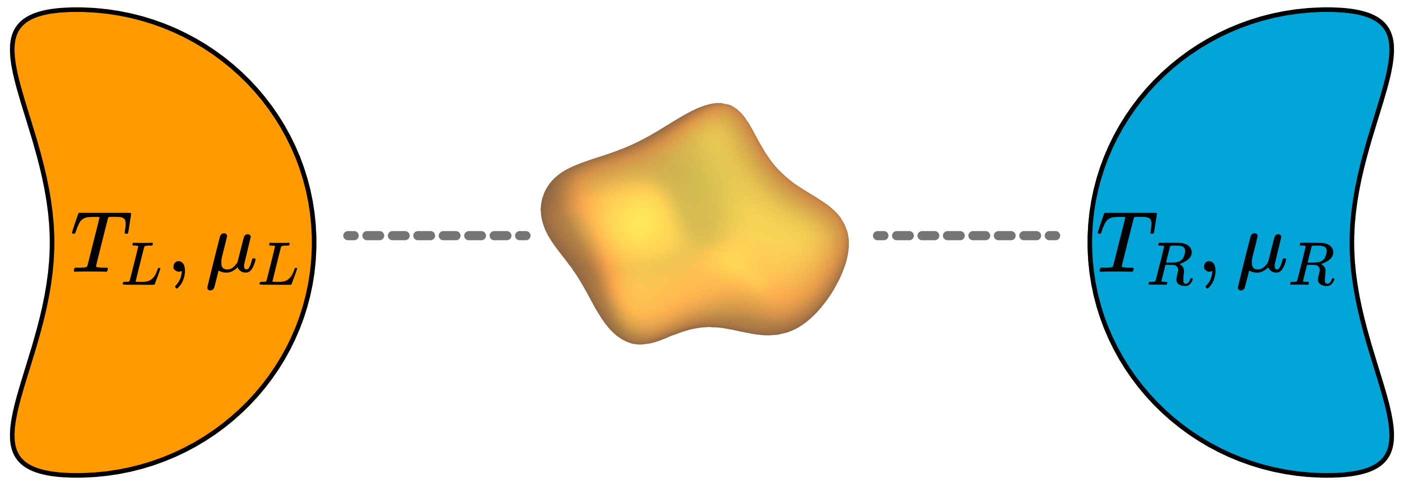




Quantum optics paradigm



Mesoscopics paradigm



Detection record

$$\zeta_t = (0,0,1,0,0,0,0,1,1,0,1,\ldots)$$

Thermodynamic Uncertainty Relations (TUR) & Kinetic Uncertainty Relations (KUR):

- Fluctuations in classical systems are bounded by dissipation:

$$\frac{\Delta_I^2}{\langle I \rangle^2} \geqslant \frac{2}{\dot{\sigma}} \qquad \text{and} \qquad \frac{\Delta_I^2}{\langle I \rangle^2} \geqslant \frac{1}{K}$$

$K =$ dynamical activity (jumps/second)
$\dot{\sigma} =$ entropy production rate
(measure of dissipation and the 2nd law)

- Simple, elegant and counter-intuitive.
  - But can be violated in the quantum regime!

Metrology:

- Fluctuations determine the precision (Cramér-Rao bound): $\Delta_I^2 \geqslant 1/F(\zeta_t)$
- But fluctuations also contain information (because output is correlated in time).

A. C. Barato, U. Seifert, *PRL*, **114**, 158101 (2015)
S. Gammelmark, K. Mølmer, *PRL*, **112**, 170401 (2014)

Summary

Unpublished work.

Part 1: tutorial paper in PRX Quantum.

- Bridge the gap between these 2 fields.
    - Develop methods/formulas to efficiently compute fluctuations numerically.

Part 2: applications to Kerr non-linearity.

- Study fluctuations in critical Kerr resonators.
- Continuous & discontinuous transitions.
- Exponential divergence of fluctuations.

Toolbox

Cond. Mat: Full Counting Statistics.
Quantum optics: input-output, power spectrum, etc.

Michael Kewming

Mark Mitchison

Setup: quantum master equation

- We are going to consider systems described by a Quantum Master equation

$$\frac{d\rho}{dt} = \mathscr{L}(\rho) = -i[H(t),\rho] + \sum_k L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}$$

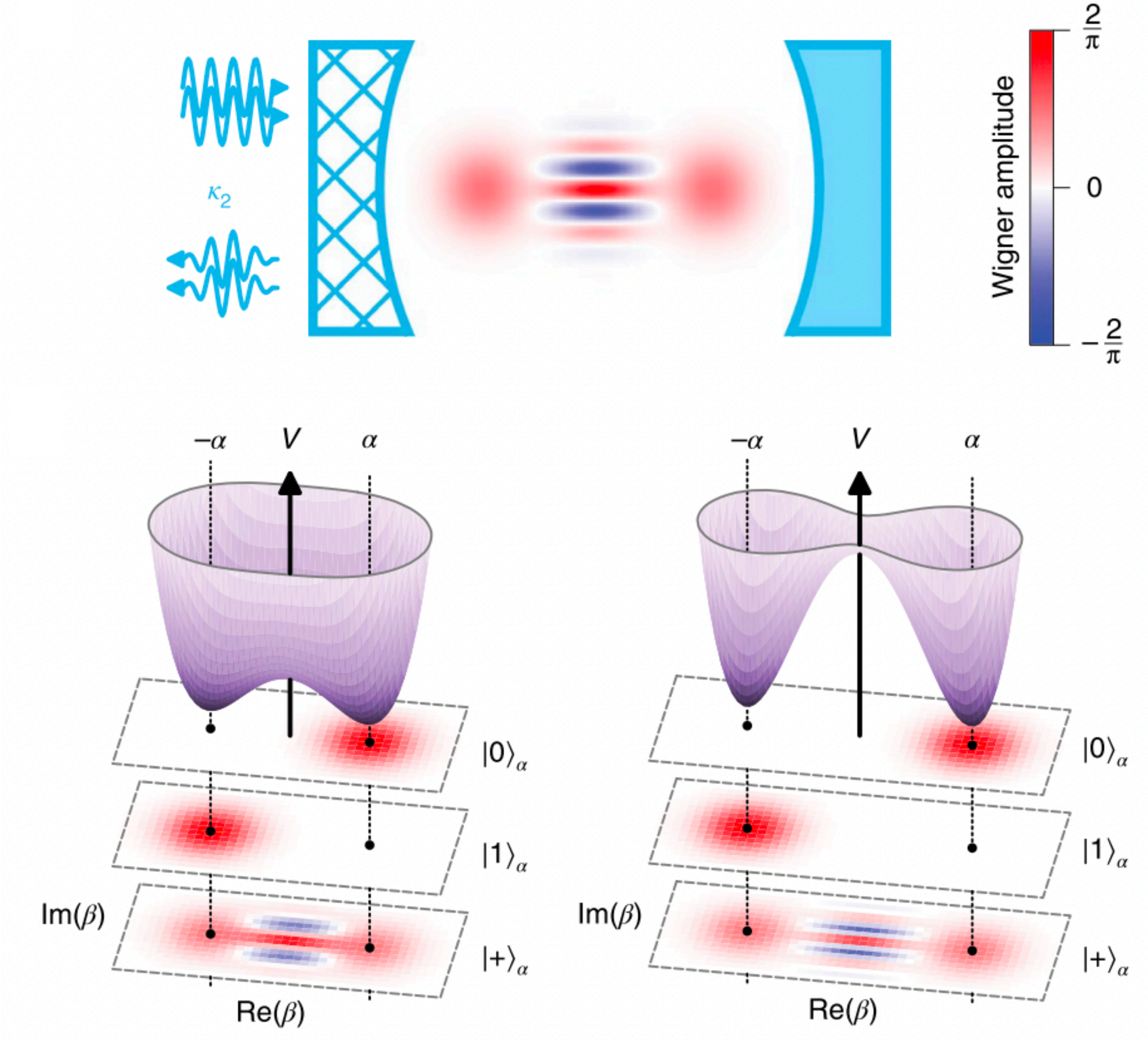
- **Ex. 1:** optical cavity with leaky photons.

$$D[a] = \kappa\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right]$$

  - e.g. the Parametric Kerr model.

$$H = -\Delta a^\dagger a + \frac{U}{2} a^\dagger a^\dagger a a + \frac{G}{2}(a^{\dagger 2} + a^2)$$

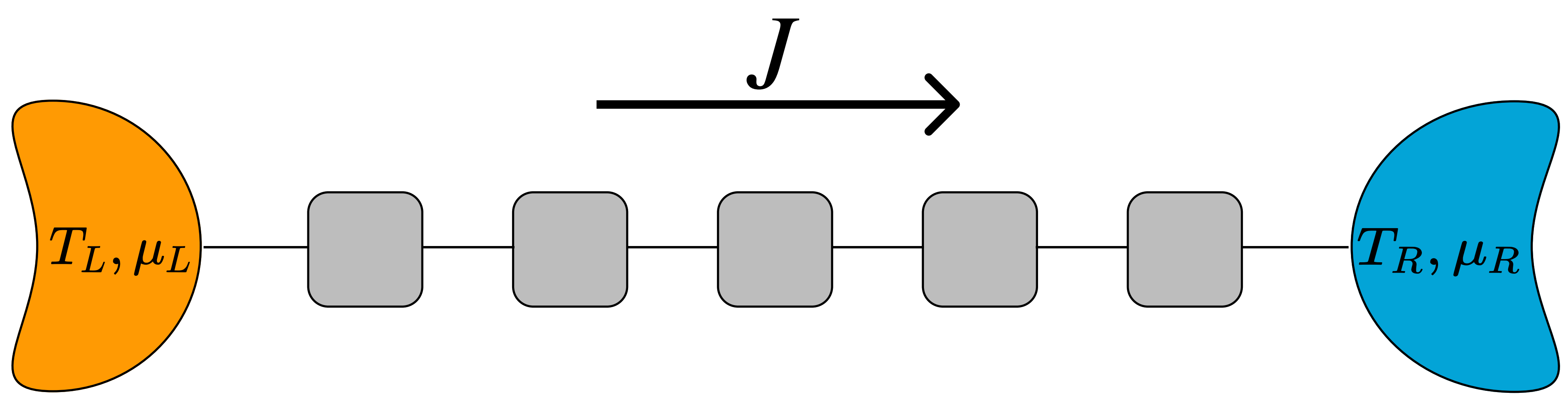
Cat qubits:
useful for quantum error correction



Lescanne, *et. al.*, Nature, **16**, 509-513 (2020)

- **Ex. 2:** fermionic transport. A chain of fermonic sites (e.g. quantum dots) modeled by

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_i \gamma(1-f_i)\, D_{c_i}(\rho) + \gamma f_i\, D_{c_i^\dagger}(\rho)$$



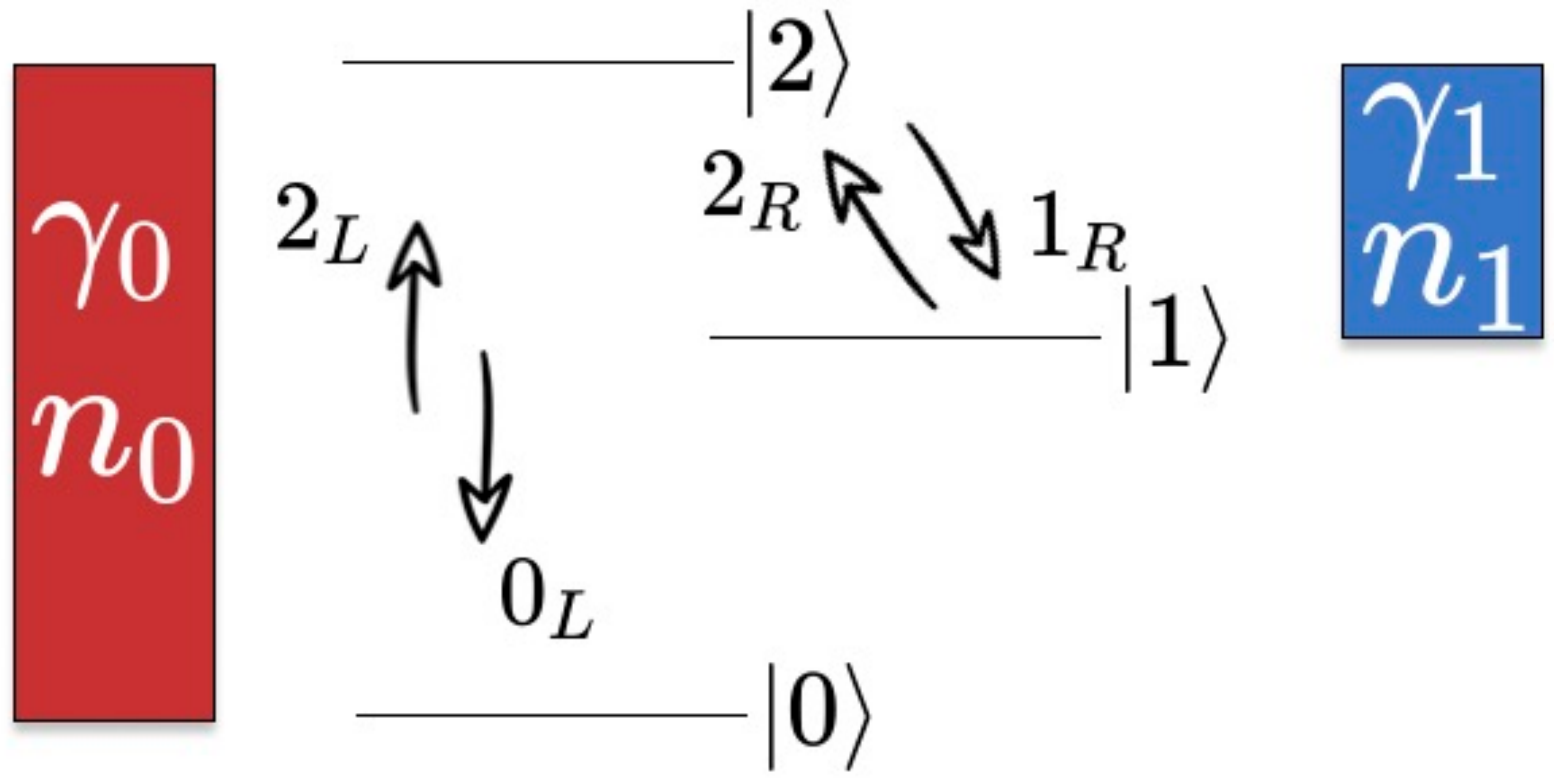
with

$$H = \sum_{i=1}^{L} \epsilon_i c_i^\dagger c_i - J \sum_{i=1}^{L-1} (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i)$$

- **Ex. 3:** 3-level maser

$$\frac{d\rho}{dt} = -i[H,\rho] + \gamma_0(n_0+1)D[\sigma_{02}] + \gamma_0 n_0 D[\sigma_{20}]$$
$$+\gamma_1(n_1+1)D[\sigma_{12}] + \gamma_1 n_1 D[\sigma_{21}]$$

- $\sigma_{ij} = |i\rangle\langle j|$



Conditional evolution

$$\frac{d\rho}{dt} = \mathscr{L}(\rho) = -i[H(t), \rho] + \sum_k L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}$$

- We associate each $L_k \rho L_k^\dagger$ with a **quantum jump:**

$$\mathscr{L}_1(\rho) := \sum_k \mathscr{J}_k(\rho) = \sum_k L_k \rho L_k^\dagger$$

- Decompose:

$$\mathscr{L}(\rho) = \mathscr{L}_0(\rho) + \mathscr{L}_1(\rho)$$ where $\mathscr{L}_0 = \mathscr{L} - \mathscr{L}_1$ is the **no-jump operator.**

- **Dyson series:**

$$e^{\mathscr{L}t}\rho_0 = e^{\mathscr{L}_0 t}\rho_0 + \int_0^t dt_1 e^{\mathscr{L}_0(t-t_1)} \mathscr{L}_1 e^{\mathscr{L}_0 t_1}\rho_0 + \int_0^t dt_1 \int_0^{t_1} dt_2 e^{\mathscr{L}_0(t-t_1)} \mathscr{L}_1 e^{\mathscr{L}_0(t_1-t_2)} \mathscr{L}_1 e^{\mathscr{L}_0 t_2}\rho_0 + \dots$$

$$e^{\mathscr{L}t}\rho_0 = e^{\mathscr{L}_0 t}\rho_0 + \int_0^t dt_1 e^{\mathscr{L}_0(t-t_1)}\mathscr{L}_1 e^{\mathscr{L}_0 t_1}\rho_0 + \int_0^t dt_1 \int_0^{t_1} dt_2 e^{\mathscr{L}_0(t-t_1)}\mathscr{L}_1 e^{\mathscr{L}_0(t_1-t_2)}\mathscr{L}_1 e^{\mathscr{L}_0 t_2}\rho_0 \; + \ldots$$

- This allows us to break the evolution into pieces **conditioned** on specific numbers of jumps.

- **Ex:** probability of no jump up to time t

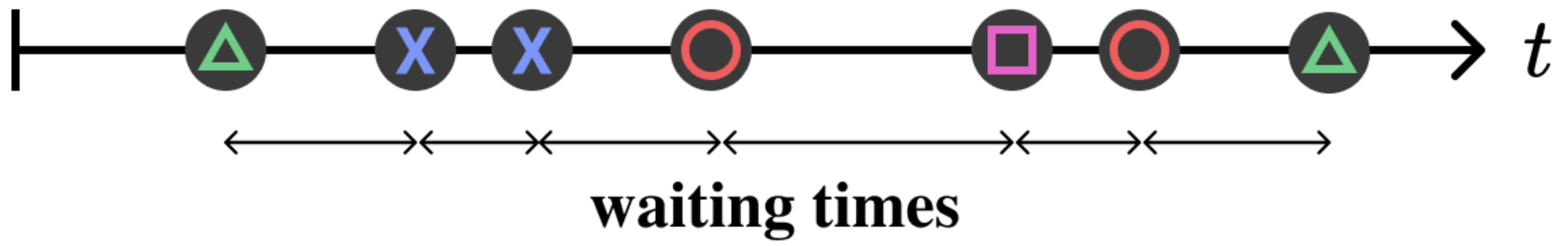
$$P_{\text{no jump}}(t) = \text{tr}(e^{\mathscr{L}_0 t}\rho_0)$$

- **Ex:** probability that first jump occurs exactly at time t (**waiting-time distribution**)

$$W(t) = \text{tr}\left\{\mathscr{L}_1 e^{\mathscr{L}_0 t}\rho_0\right\}$$

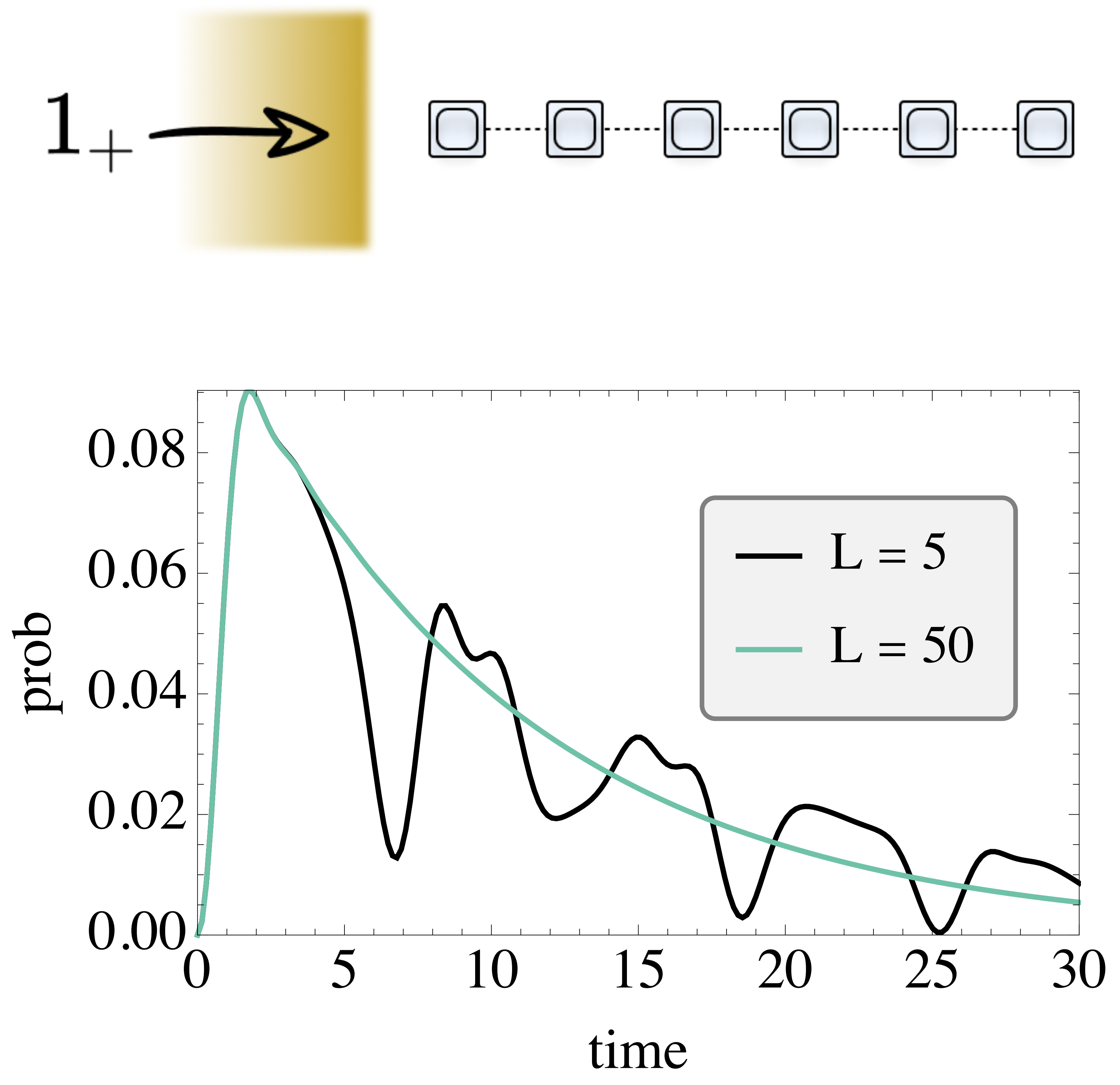
  - Can also be resolved over individual channels

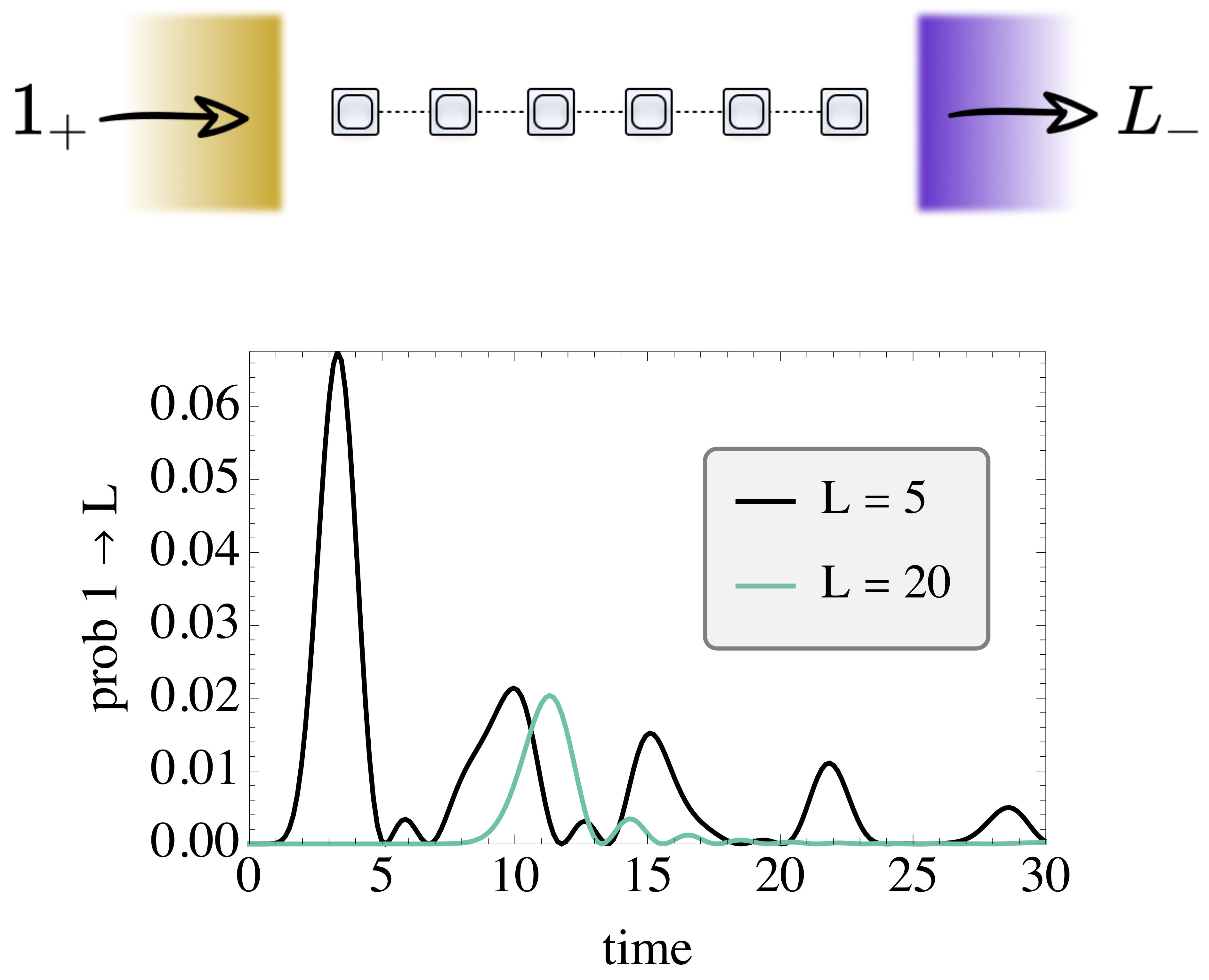
$$W(t,k) = \text{tr}\left\{\mathscr{J}_k e^{\mathscr{L}_0 t}\rho_0\right\}$$



T. Brandes, "**Waiting times and noise in single particle transport,"** *Annalen Der Physik*, **17**(7), 477–496 (2008).

Ex: tight-binding chain





- 2nd injection is affected by the 1st.
  - And also by finite-size interference effects.

Output currents

- To each jump we associate a counting variable $N_k(t)$, such that $dN_k = 1$ when $L_k \rho L_k^\dagger$ occurs.
- The probability that a jump happens in a time interval dt is

$$P\Big(dN_k(t) = 1\Big) = dt \mathrm{tr}\Big(\mathcal{J}_k \rho_t\Big) = dt \mathrm{tr}(L_k \rho_t L_k^\dagger) = dt \langle L_k^\dagger L_k \rangle$$

- The joint probability for two jumps at different times is

$$P\Big(dN_k(t) = 1,\, dN_q(t+\tau) = 1\Big) = dt^2 \mathrm{tr}\Big\{\mathcal{J}_q e^{\mathcal{L}\tau} \mathcal{J}_k \rho_t\Big\}$$

  - Note that this assumes anything can happen in the middle (unlike the WTD).

- The counting operators now define **physical (classical) currents:**

$$I(t) = \sum_k \mu_k \frac{dN_k}{dt} \qquad\qquad N(t) = \int_0^t dt' I(t')$$

- The μ_k are parameters that depend on the current in question; e.g.,
  - Particle current: $\mu_k = 1$ for $c\rho c^\dagger$ and $\mu_k = -1$ for $c^\dagger \rho c$
  - Energy current : $\mu_k = \pm \epsilon$ (tight-coupling).
  - Dynamical activity: $\mu_k = 1$ for all channels.
- Very general framework. Not widely known/appreciated.
- Can also be extended to multiple currents: $I_\alpha = \sum_k \mu_{\alpha,k} \frac{dN_k}{dt}$
  - Interesting for studying current-current correlations.

- **Average current:**

$$J(t) = E\big(I(t)\big) = \frac{1}{dt}\sum_k \mu_k E\big(dN_k\big)$$

But

$$E\big(dN_k\big) = 1 \times P\big(dN_t = 1\big) = dt\langle L_k^\dagger L_k\rangle$$

$$\therefore \qquad J(t) = \sum_k \mu_k \langle L_k^\dagger L_k\rangle$$

- **Fluctuations: 2-point correlation function.**

$$F(t, t+\tau) = E\Big(\delta I_t \delta I_{t+\tau}\Big), \qquad \delta I_t = I_t - J_t$$

- A similar & simple calculation leads to

$$F(t, t+\tau) = K_t\ \delta(\tau) + \mathrm{tr}\Big\{\mathscr{L}_I e^{\mathscr{L}\tau}\mathscr{L}_I \rho_t\Big\} - J_t^2$$

where

$$K_t = \sum_k \mu_k^2 \langle L_k^\dagger L_k\rangle$$

and

$$\mathscr{L}_I(\rho) = \sum_k \mu_k L_k \rho L_k^\dagger$$

$$F(t, t+\tau) = K_t\ \delta(\tau) + \mathrm{tr}\left\{ \mathscr{L}_I e^{\mathscr{L}\tau} \mathscr{L}_I \rho_t \right\} - J_t^2$$

- The second term is the generalization of Glauber's 2nd order coherence function $g^{(2)}$ in quantum optics:
  - Indeed, assuming the quantum regression theorem:

$$\mathrm{tr}\left\{ \mathscr{L}_I e^{\mathscr{L}\tau} \mathscr{L}_I \rho_t \right\} = \sum_{k,q} \mu_k \mu_q \left\langle L_q(t) L_k^\dagger(t+\tau) L_k(t+\tau) L_q(t) \right\rangle$$

  - Reduces to $g^{(2)}$ when $L_k = a$.
- At steady-state $F(t, t+\tau) = F(\tau)$ and one usually studies the **power spectrum** (Fourier transform)

$$S(\omega) = \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} F(\tau)$$

Connection with full counting statistics

- Probability distribution of the integrated current $N(t) = \int_0^t dt' I(t')$.
  - Append a counting field χ to each jump operator

$$\mathscr{L}_\chi(\rho) = -i[H(t), \rho] + \sum_k e^{i\chi\mu_k} L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}$$

  The weights of the counting fields are related to the current coefficients μ_k.

- Cumulant generating function & Probability distribution

$$C(\chi) = \ln \mathrm{tr}\left\{ e^{\mathscr{L}_\chi t} \rho_0 \right\} \qquad \text{and} \qquad P(N(t) = n) = \int_{-\pi}^{\pi} \frac{d\chi}{\pi} e^{-in\chi} \mathrm{tr}\left\{ e^{\mathscr{L}_\chi t} \rho_0 \right\}$$

- **Famous FCS result:** $C(\chi) \sim t \ln \lambda_\chi$

G. Schaller, **Open quantum systems far from equilibrium,** Springer.
M.Esposito, U.Harbola, S.Mukamel, *Rev. Mod. Phys*. **81**, 1665 (2009)

Diffusion coefficient

- The **diffusion coefficient,** or **noise** or **scaled variance**, is defined as

$$\mathscr{D} = \lim_{t\to\infty} \frac{d}{dt}\left[E\left(N^2(t)\right) - E\left(N(t)\right)^2\right]$$

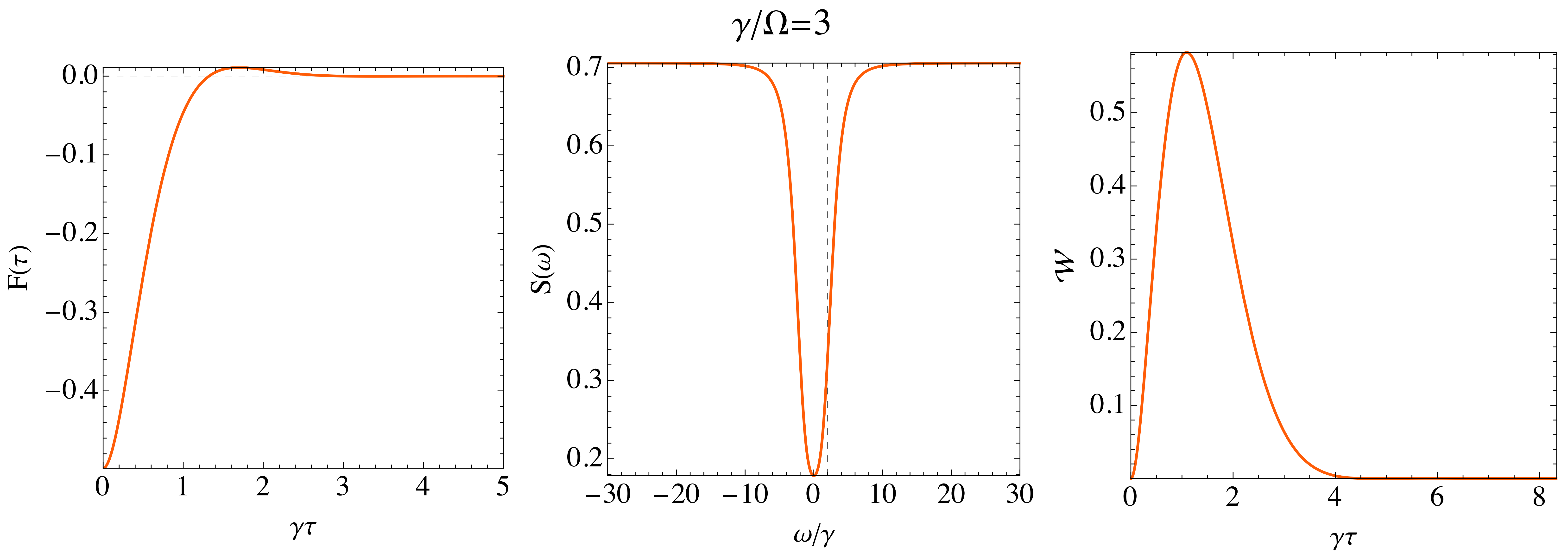
- ***This is the quantity Δ_I^2 studied in any TUR paper.***

- Connection with previous results:

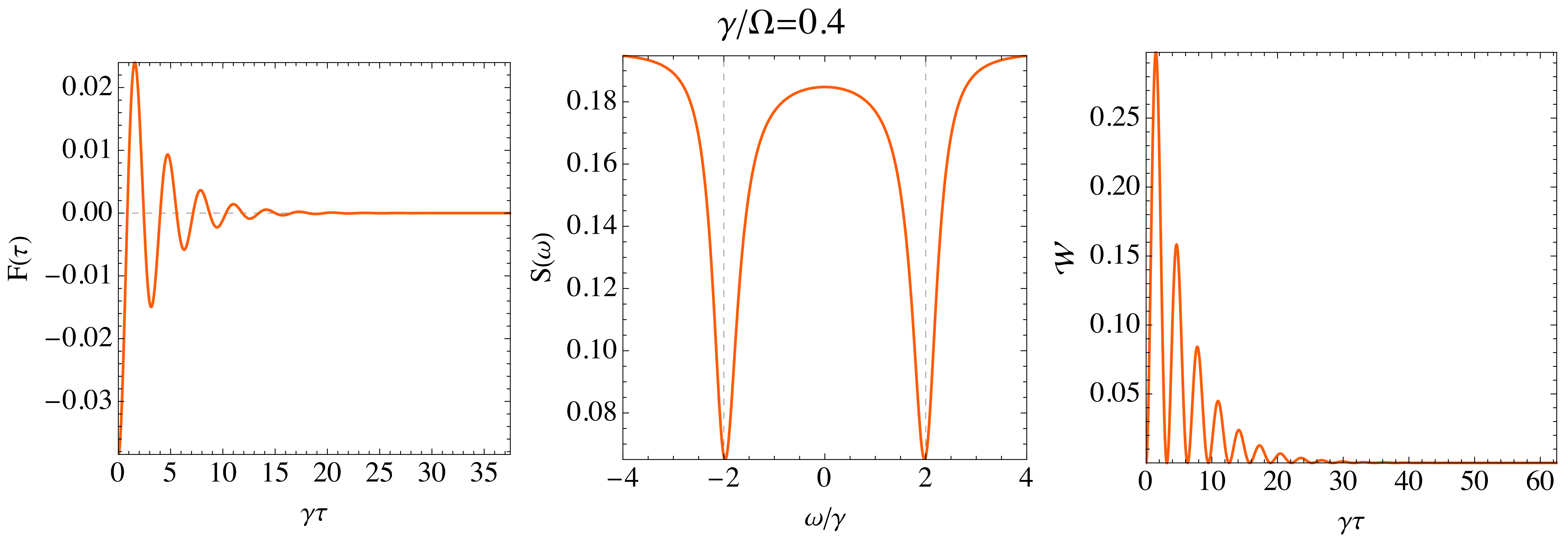
$$\mathscr{D} = S(0) = \int_{-\infty}^{\infty} F(\tau)$$

- Can also be generalized to multiple currents: *noise covariance matrix* $\mathscr{D}_{\alpha\beta}$.
  - Describes statistical correlations between different currents.

- **Ex:** driven qubit $H = \Omega\sigma_x$ and $\dfrac{d\rho}{dt} = -i[H,\rho] + \gamma D(\sigma_-)$



Mollow Triplet



Efficient computation of $\mathscr{D}$ and $S(\omega)$

- Focusing on steady-state $\mathscr{L}(\rho) = 0$: $F(\tau) = K\,\delta(\tau) + \mathrm{tr}\left\{\mathscr{L}_I e^{\mathscr{L}\tau}\mathscr{L}_I\rho\right\} - J^2$

  where $K = \sum_k \mu_k^2 \langle L_k^\dagger L_k \rangle$ and $\mathscr{L}_I(\rho) = \sum_k \mu_k L_k \rho L_k^\dagger$

- Then

$$S(\omega) = \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} F(\tau) = K - \mathrm{tr}\left\{\mathscr{L}_I \left(\frac{\mathscr{L}}{\mathscr{L}^2 + \omega^2}\right)\mathscr{L}_I\rho\right\}$$

- and

$$\mathscr{D} = K - \mathrm{tr}\left\{\mathscr{L}_I \mathscr{L}^+ \mathscr{L}_I \rho\right\}$$

  where $\mathscr{L}^+$ is the Drazin inverse.

Only require solving a linear system of equations.

qulib.nb

75%

Qulib

Simple, yet useful, functions for dealing with Quantum Information and Open Quantum Systems.
Gabriel T. Landi

QT^2 group

Contributors: Jader P. dos Santos, Artur Lacerda, Anthony Kiely.

General purpose functions

Quantum theory, information & thermodynamics

Open quantum systems

Vectorization-based routines

Collision models

Full counting statistics

DrazinApply: Applies the Drazi inverse of an operator

Average current

Diffusion matrix

Power spectrum

$g^{(2)}$ function

Two-point function $F_{\alpha\beta} = M_{\alpha\beta}\,\delta(\tau) + J_\alpha J_\beta\left(g^{(2)}_{\alpha\beta} - 1\right)$

Examples

Example usage

Example: Reproducing Fig. 2(a) of arXiv 2103.07791

Example: “Mollow Triplet”; comparison with the emission & absorption spectra

Ex: Mollow Triplet, exact formulas

FCS example usage.nb 100%

```
H = Δ/2 σz + Ω/2 σx;
c = √γ σm;
ℒ = Liouvillian[H, {c}] // cf;
ρ = AnalyticalSteadyState[ℒ];

S = FCSPowerSpectrumLinearSys[{ω}, ρ, ℒ, c][[1, 2]] // cf
```

Out[12798]=

$$\frac{\gamma\,\Omega^2\left(1+\frac{\gamma\,(i\,\gamma+2\,\omega)\,\Omega^2}{\left(4\,\Delta^2+(\gamma-2\,i\,\omega)^2\right)\,(\gamma-i\,\omega)\,\omega+2\,(\gamma-2\,i\,\omega)\,\omega\,\Omega^2}+\frac{\gamma\,(-i\,\gamma+2\,\omega)\,\Omega^2}{\left(4\,\Delta^2+(\gamma+2\,i\,\omega)^2\right)\,(\gamma+i\,\omega)\,\omega+2\,(\gamma+2\,i\,\omega)\,\omega\,\Omega^2}\right)}{\gamma^2+4\,\Delta^2+2\,\Omega^2}$$

Homodyne detection (quantum diffusion)

- The Lindblad equation is invariant under the gauge transformation

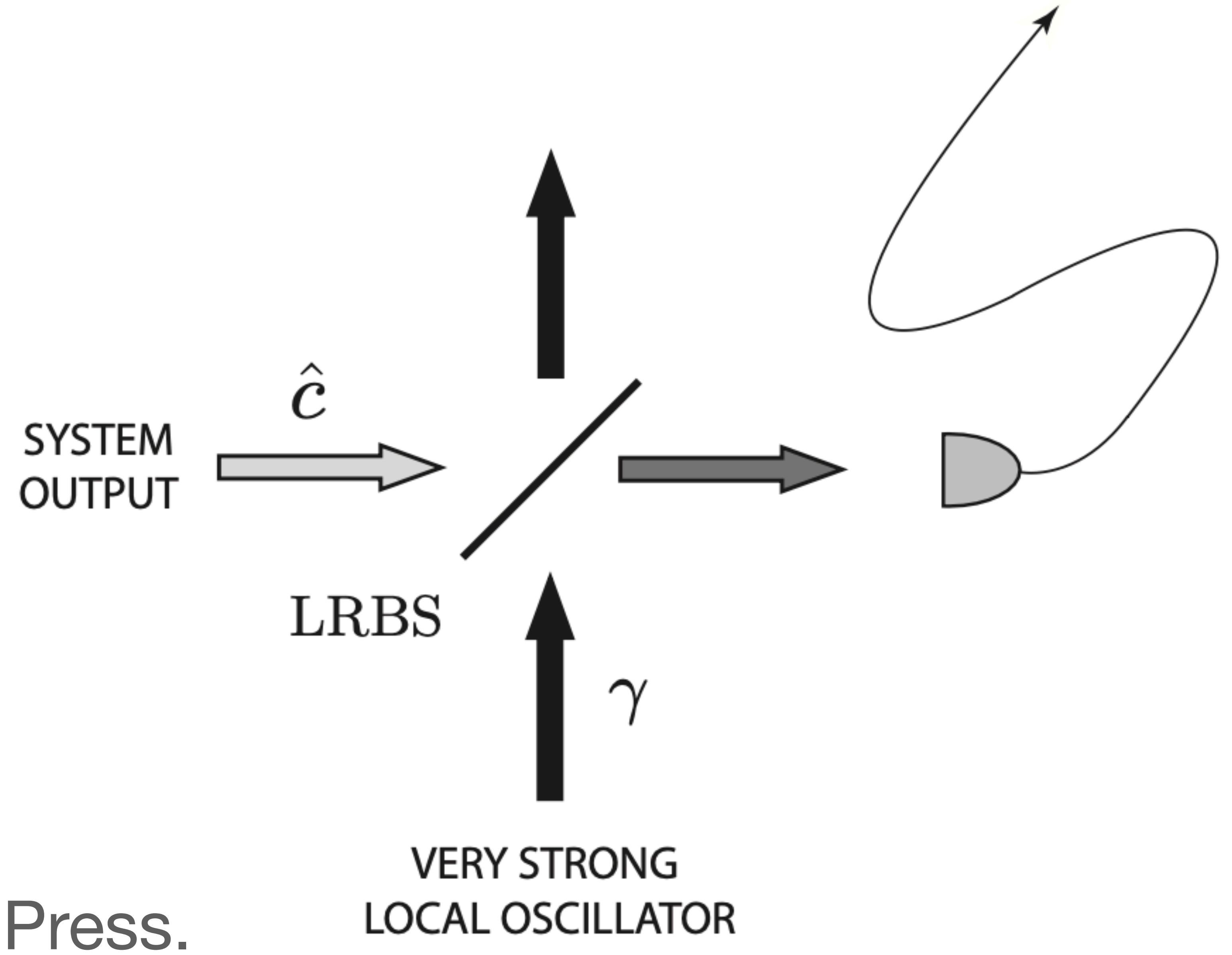
$$L_k \to L_k + \alpha_k, \qquad H \to H - \frac{i}{2}(\alpha_k^* L_k - \alpha_k L_k^\dagger)$$

- In optical cavities this can be done by mixing the signal with a classical (high intensity) laser field.

- This leads to stochastic currents

$$J_{\text{hom}} = \sum_k \mu_k \big(\langle x_k \rangle + \xi_k(t) \big), \qquad x_k = L_k + L_k^\dagger$$

where $\xi_k(t)$ are Gaussian white noises.



H. Wiseman and G. Milburn, **Quantum Measurement and Control**, Cambridge University Press.

- Define $\mathscr{H}_I(\rho) = \sum_k \mu_k \big(L_k \rho + \rho L_k^\dagger \big)$.

- The average homodyne current reads

$$J_{\text{hom}} = \text{tr} \mathscr{H}_I(\rho) = \sum_k \mu_k \langle x_k \rangle$$

- The 2-point function reads

$$F_{\text{hom}}(\tau) = K_{\text{hom}} \delta(\tau) + \text{tr} \Big\{ \mathscr{H}_I e^{\mathscr{L}\tau} \mathscr{H}_I \rho \Big\} - J_{\text{hom}}^2$$

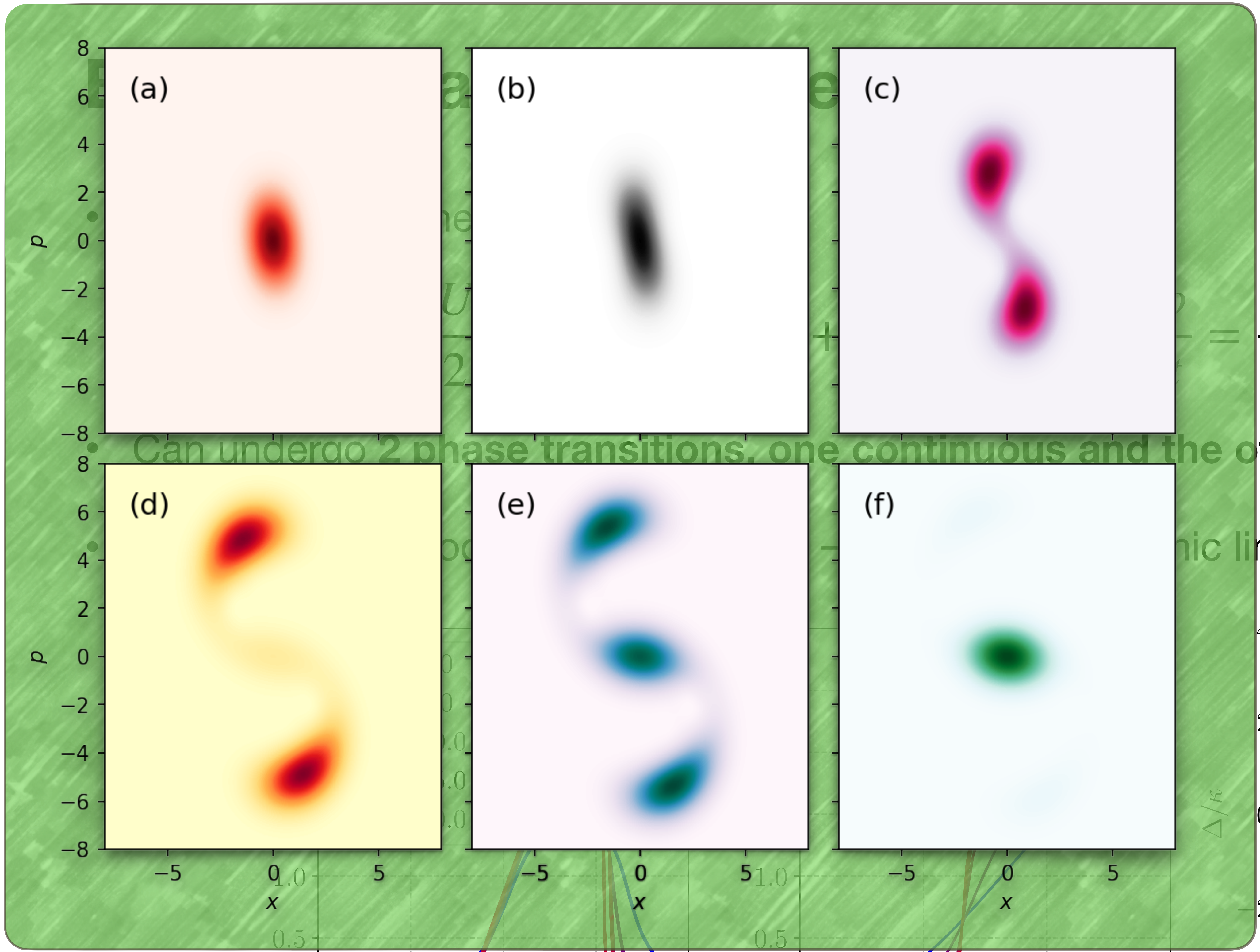
  where $K_{\text{hom}} = \sum_k \mu_k^2$

  - The delta term is now proportional to “1”: vacuum fluctuations (shot noise) of the local oscillator.

- Results for homodyne detection are almost identical to what we had before,

$$K \to K_{\text{hom}} \qquad\qquad \mathscr{L}_I \to \mathscr{H}_I$$

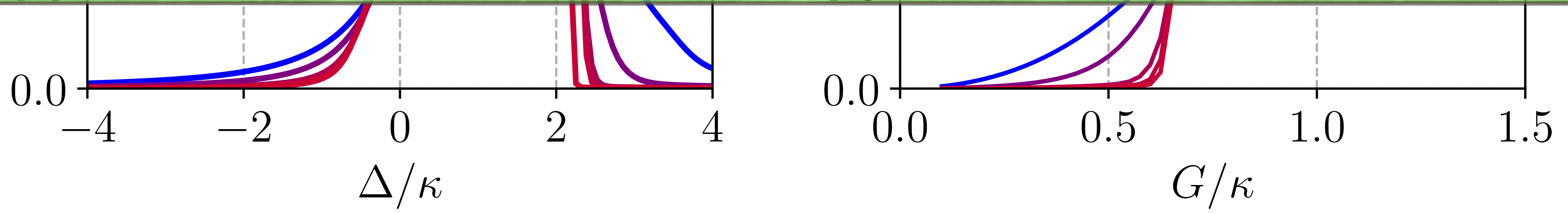
Parametric Kerr model



$$-\,i[H,\rho]+\kappa D[a]$$

- Can undergo 2 phase transitions, one continuous and the other discontinuous:

- …mic limit")



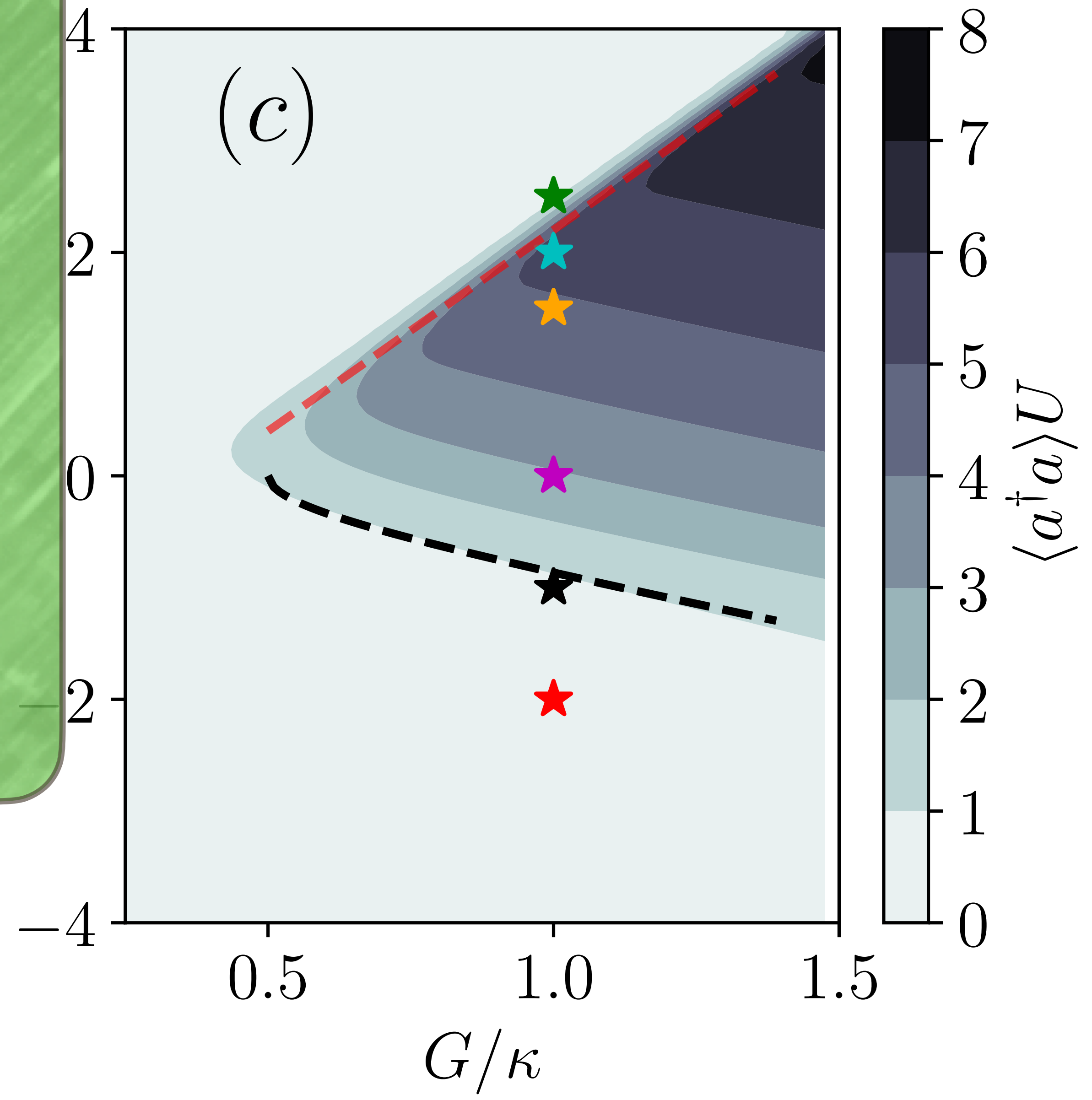
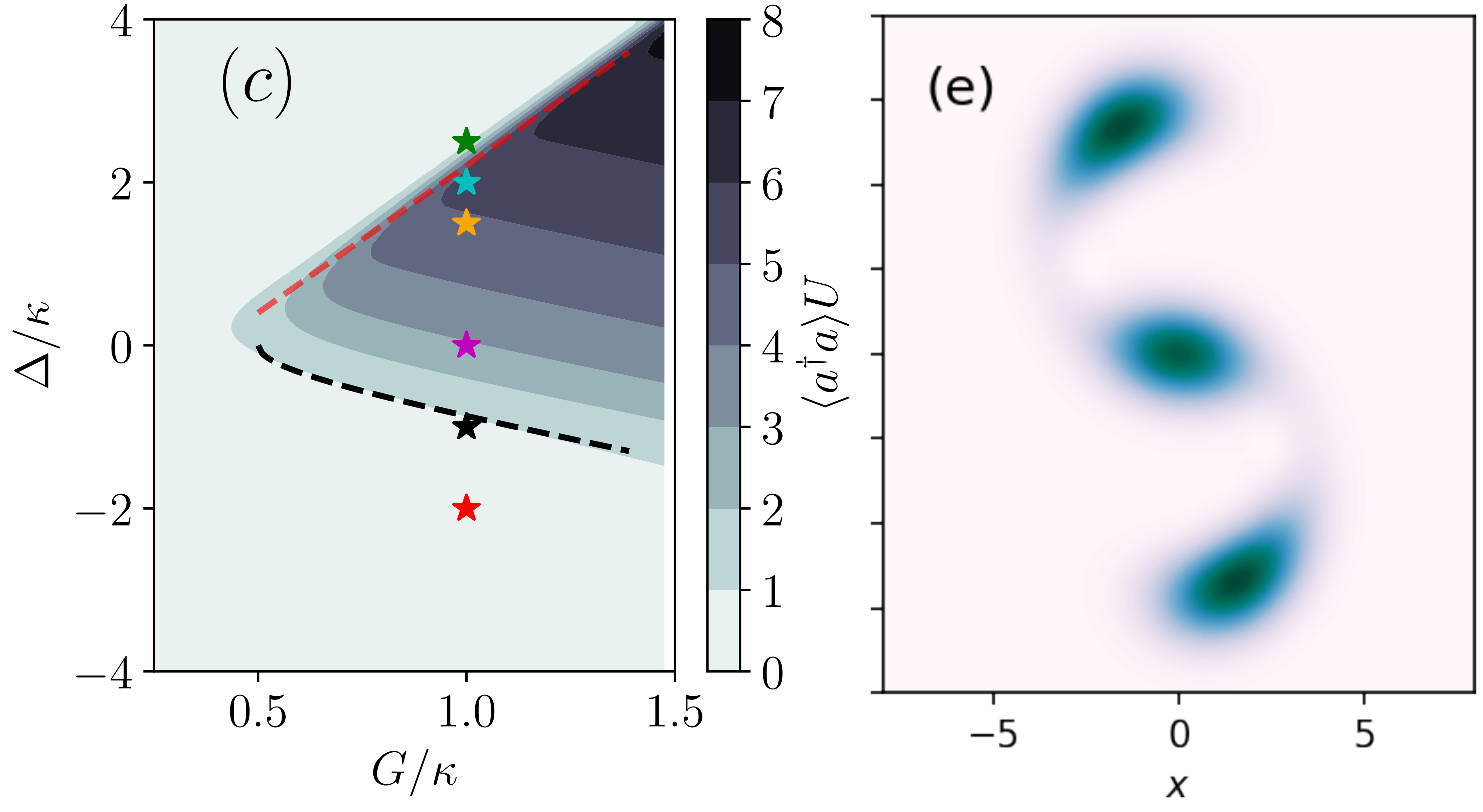
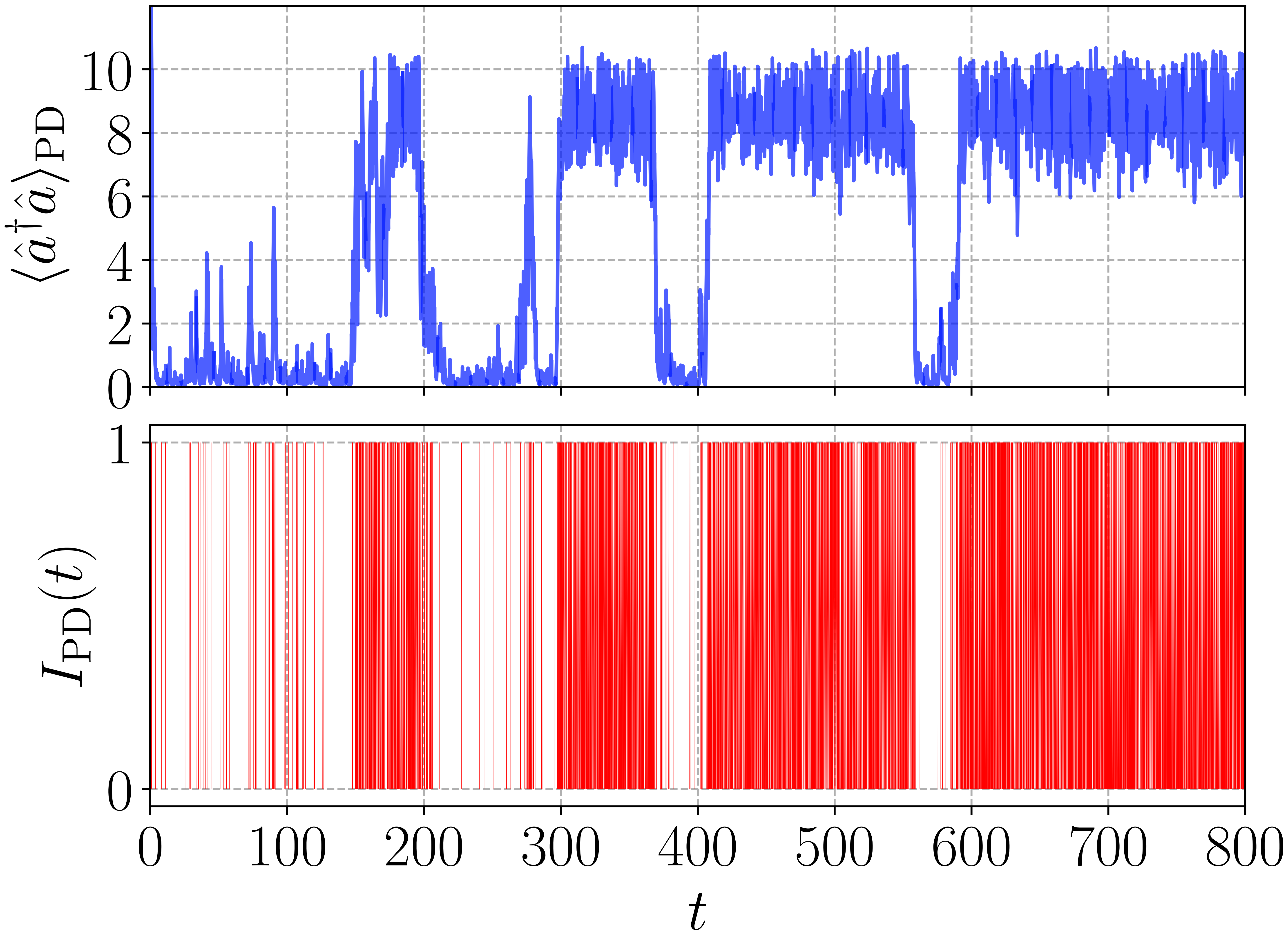


Photo-detection current

- @ discontinuous transition: on/off (telegraph) behavior of the current.
- Photo-detection cannot resolve upper or lower lobes.





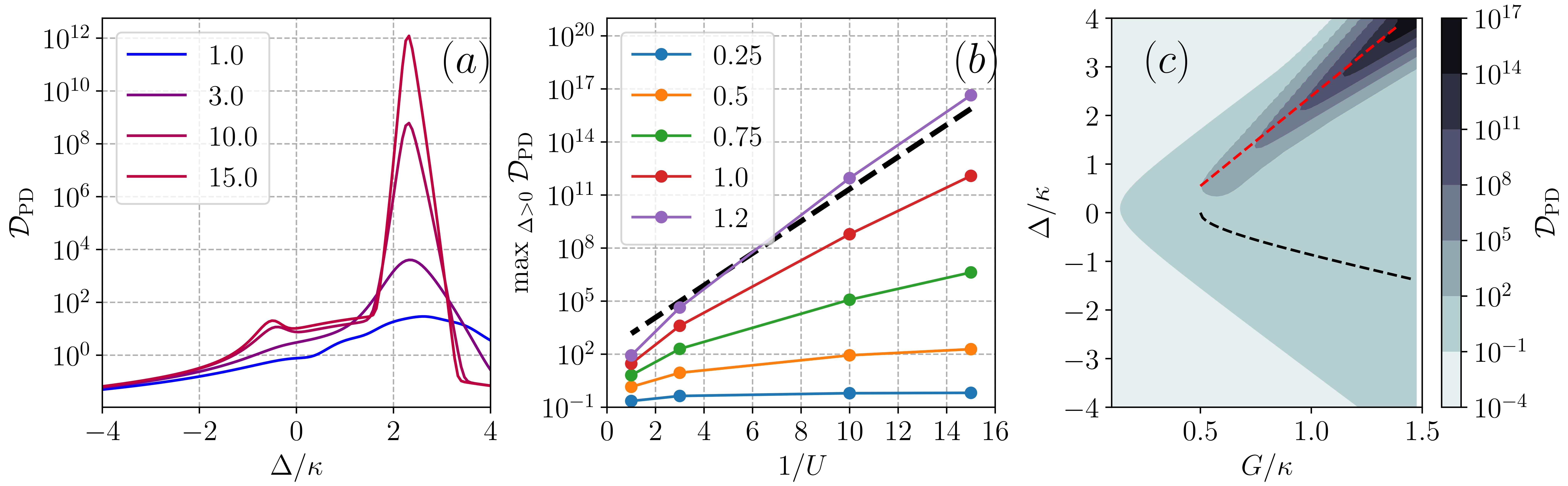
Divergence of the diffusion coefficient

In the continuous transition $(\Delta < 0)$

$$\mathscr{D} \sim (1/U)^2$$

In the discontinuous transition $(\Delta > 0)$

$$\mathscr{D} \sim e^{1/U}$$



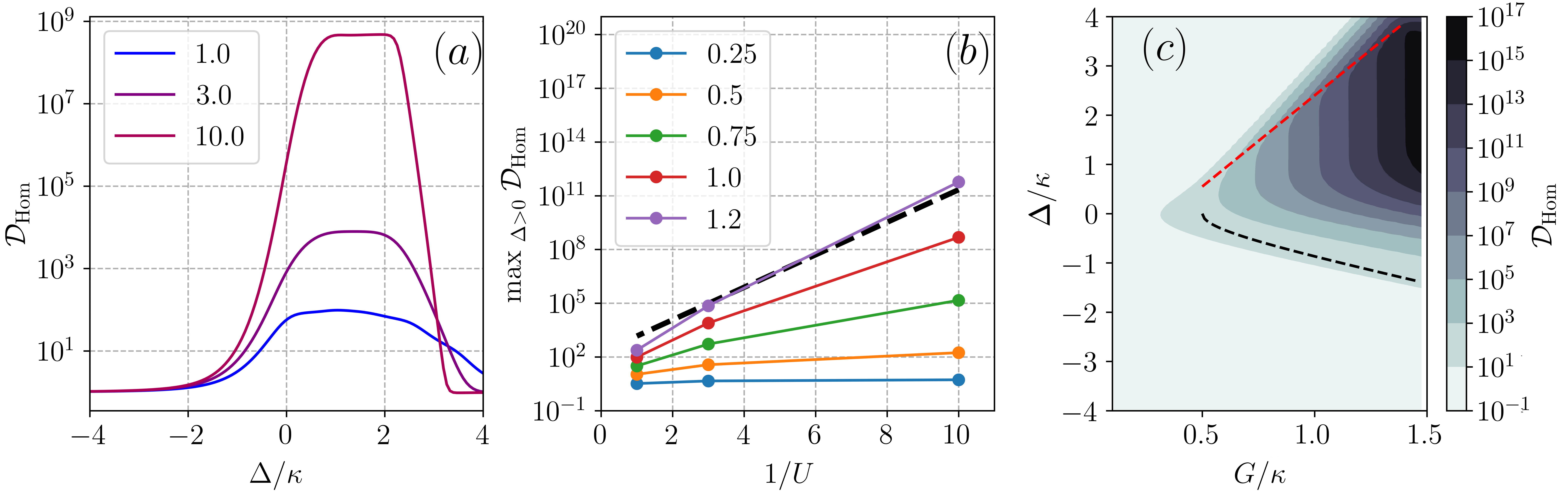
Homodyne current (in p quadrature)

- The homodyne current switches between 3 values (+,0,-).
- Captures the tunneling between the 3 blobs.

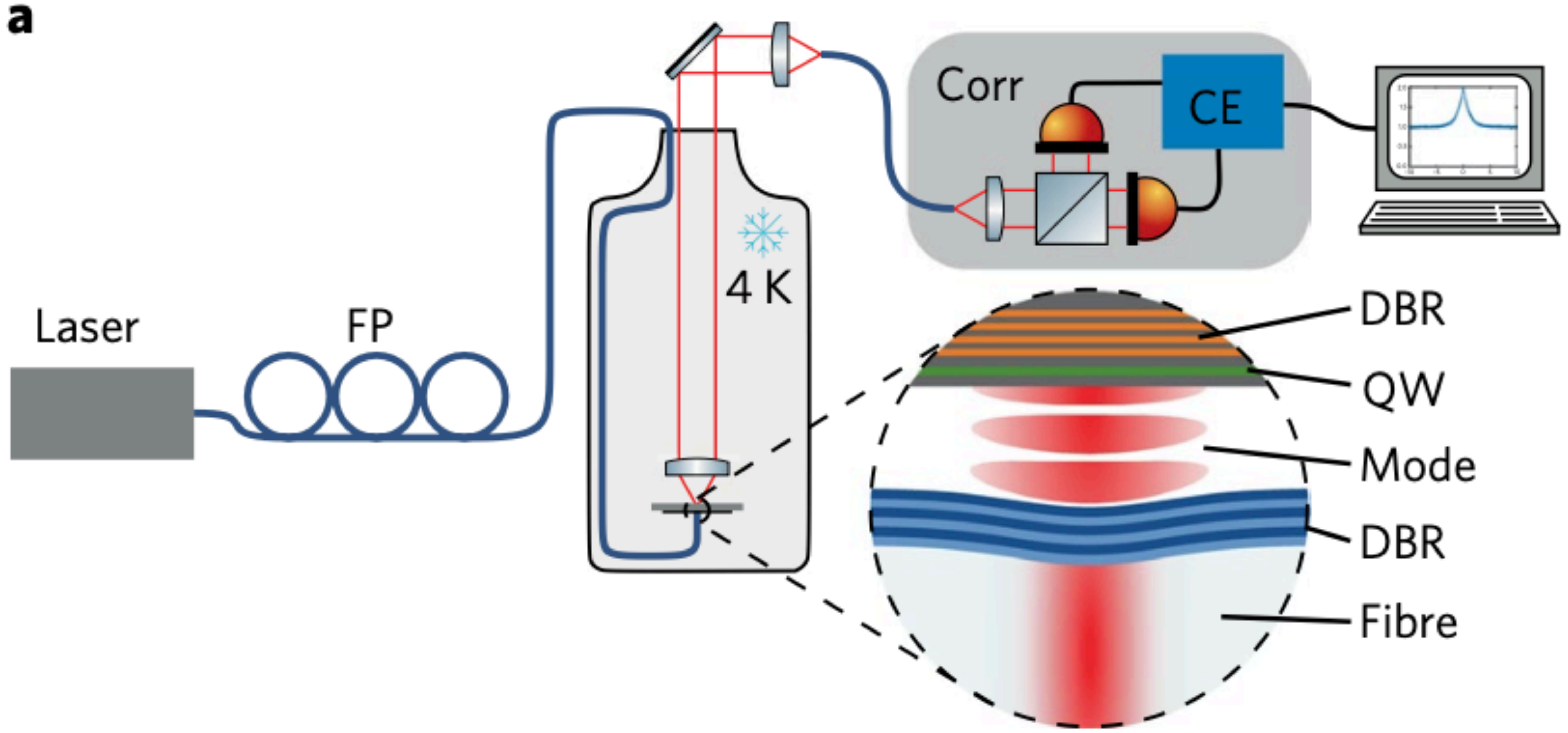
Divergence of the diffusion coefficient

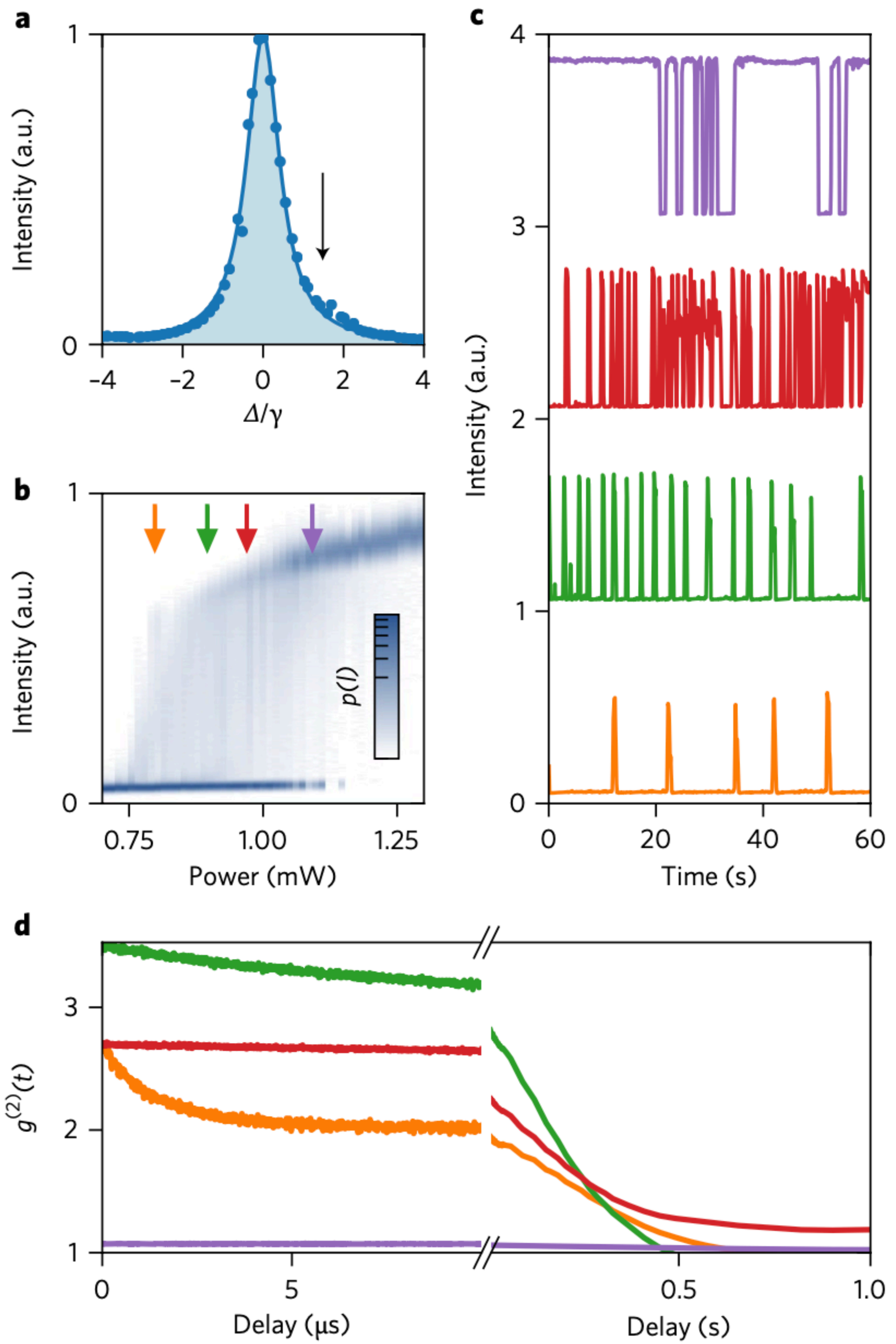
Homodyne current diverges exponentially in a much broader region

$$\mathscr{D} \sim e^{1/U}$$



GaAs cavity polaritons.





T. Fink, *et. al.*, *Nature Physics,* **14**, 365 (2018)

Conclusions

- Efforts to go beyond the NESS.
  - Analyze fluctuations in the time domain.
- Connection between quantum optics and full counting statistics.
  - Easy to use/compute formulas.
- Parametric Kerr model: exponential divergences of the diffusion in discontinuous transitions.

Thank you.

www.fmt.if.usp.br/~gtlandi