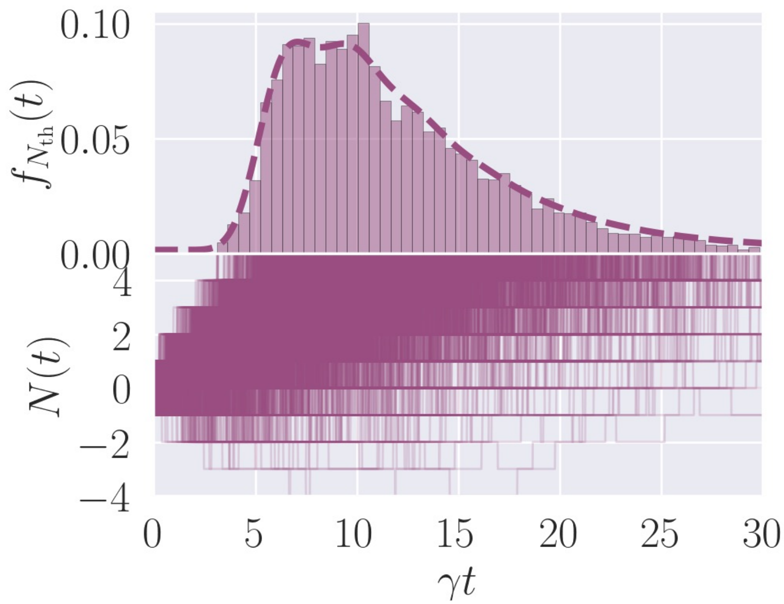

- Michael J. Kewming, Anthony Kiely, Steve Campbell, GTL, "**First Passage Times for Continuous Quantum Measurement Currents**," 2308.07810
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- Luis Felipe Santos, GTL, "**Waiting time statistics of a double quantum dot-based single-photon detector.**" *In preparation.*

First Passage Times for Continuous Measurement Currents

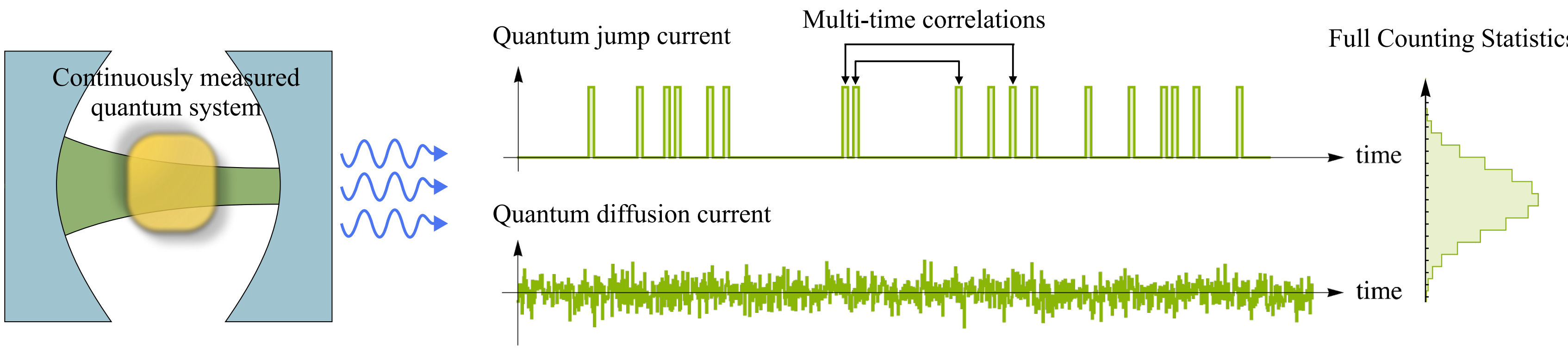
Gabriel T. Landi

University of Rochester

Time in Quantum Theory - Westport, Ireland - 04 Sep 2023

https://www.pas.rochester.edu/~gtlandi/

Statistics of continuously monitored quantum systems



Quantum Trajectories

$d\rho_c$

Classical (stochastic) measurement current $I(t)$

or net charge $N(t) = \int_0^t dt' \; I(t')$

FCS

$P(N(t) = n)$

Relevant questions:

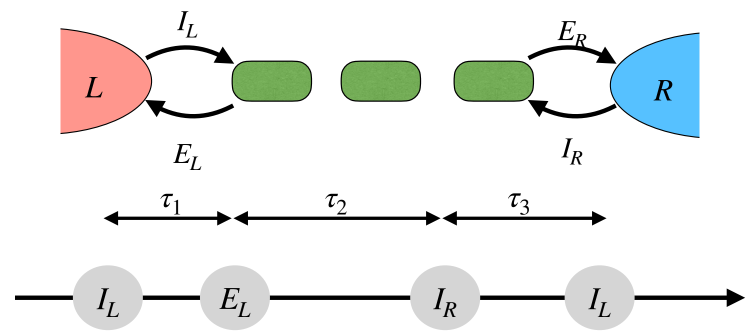
- **Waiting time between 2 jumps?**
- **2-point function:** $E\big(I(t)I(t')\big)$
- **FCS:** $P(N(t) = n)$

Applications:

- **Learn about the quantum system.**
- **Metrology.**
- **Time-resolved thermodynamics.**
- **Timekeeping.**

GTL, Michael J. Kewming, Mark T. Mitchison, Patrick P. Potts "**Current fluctuations in open quantum systems: Bridging the gap between quantum continuous measurements and full counting statistics**," 2303.04270

Quantum jump unravelling



- Quantum Master Equation with multiple jump channels:

$$\frac{d\rho}{dt} = \mathscr{L}\rho = -i[H(t),\rho] + \left[L_k\rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k,\rho\}\right]$$

- Each L_k represents a *channel.* Physical meaning depends on the problem.

Dynamics in the QJU:

- With probability $p_k = dt\,\mathrm{tr}(L_k^\dagger L_k \rho)$ jump to channel k: $\rho \rightarrow \dfrac{L_k\rho L_k^\dagger}{\mathrm{tr}(L_k^\dagger L_k \rho)}$
- Otherwise, evolve as $d\rho = \mathscr{L}_0\rho dt$ where $\mathscr{L}_0\rho = \mathscr{L}\rho - \sum_k L_k\rho L_k^\dagger$

Quantum trajectory: $\{dN_k(t)\}$ or $\omega_{1:N} = \{(k_1,\tau_1),(k_2,\tau_2),\ldots,(k_N,\tau_N)\}$

- τ_i = time between jumps.
- k_i = channel (runs over finite alphabet).

Integrated current (net charge):

$dN_k(t) = 0{,}1$ if jump occurs in k at t.

$$N(t) = \int_0^t dt' I(t') = \sum_k \nu_k \int_0^\infty dN_k(t')$$

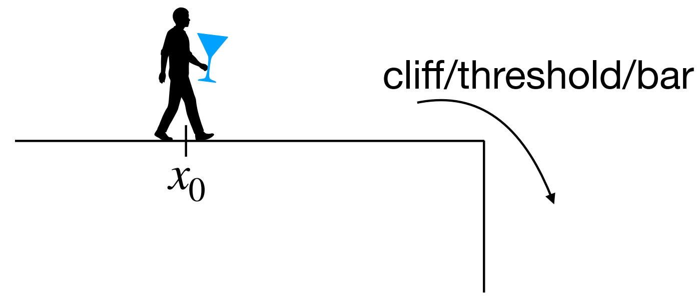
Choice of weights defines a current:

- $\nu_k = 1$: dynamical activity
- $\nu_{I(E)} = \pm 1$: excitation current

First Passage Time

Michael J. Kewming, Anthony Kiely, Steve Campbell, GTL,
”**First Passage Times for Continuous Quantum Measurement Currents**,"
2308.07810

First Passage Time



- Given a certain stochastic process $X(t)$, what is the first time τ when $X(t) > b$ or $X(t) < a$?
  - Region $\mathscr{R} = [a, b]$ with $a < 0, \quad b > 0$.
- Not the same as $P(a \leqslant X(t) \leqslant b)$ because there is the possibility that $X(t)$ leaves $\mathscr{R}$ and then comes back.
- Can be computed using **absorbing boundaries**.
  - Force $P(x, t) = 0$ for all $x \notin \mathscr{R}$.
    - Produces a new evolution $P_{\mathscr{R}}(x, t)$.

- Survival probability:

$$G_{\mathscr{R}}(t) = \int_a^b dx\ P_{\mathscr{R}}(x, t) = P_{\mathscr{R}}(a \leqslant X(t) \leqslant b)$$

- First passage time (FPT) distribution

$$f_{\mathscr{R}}(t) = -\frac{dG_{\mathscr{R}}}{dt} = \frac{G_{\mathscr{R}}(t) - G_{\mathscr{R}}(t + dt)}{dt}$$

- If $G_{\mathscr{R}}(\infty) = 0$ the boundary is always eventually reached and $\int_0^\infty dt f_{\mathscr{R}}(t) = 1$.

FPT for Quantum Jumps

- It does not make sense to talk about FPT of a quantum system: they live in superpositions.
  - But we can talk about the FPT of the *classical* measurement record.
- Consider a specific current $I(t) = \sum_k \nu_k \frac{dN_k}{dt}$ and the net charge $N(t) = \int_0^t dt' \; I(t')$.
  - What is the FPT for $N(t)$ to first cross a region $\mathscr{R} = [a, b]$? ($a < 0,\ b > 0$)
- Define the charge resolved state $\rho_n(t) = E\Big[\rho_c(t)\delta_{N(t),n}\Big]$.
  - We show that ρ_n satisfies a charge-resolved master equation

$$\frac{\partial \rho_n}{\partial t} = \mathscr{L}_0 \rho_n + \sum_k L_k \rho_{n-\nu_k} L_k^\dagger$$

$\mathrm{tr}(\rho_n(t)) = P(n,t) = P\big(N(t) = n\big)$

→ **Full Counting Statistics probability**

- FPT is now easy to implement with absorbing boundaries: set $\rho_n(t) \equiv 0$ for $n \notin \mathscr{R}$.
  - Produces new evolution $\rho_n^{\mathscr{R}}(t)$ and new $P_{\mathscr{R}}(n,t) = \mathrm{tr}\{\rho_n^{\mathscr{R}}(t)\}$

Michael J. Kewming, Anthony Kiely, Steve Campbell, GTL, "**First Passage Times for Continuous Quantum Measurement Currents**," 2308.07810

- Example: system with 1 injection and one extraction channel:

$$\nu_- = 1 \qquad \text{and} \qquad \nu_+ = -1$$

- The charge resolved equation will look like

$$\frac{\partial \rho_n}{\partial t} = \mathscr{L}_0 \rho_n + L_- \rho_{n-1} L_-^\dagger + L_+ \rho_{n+1} L_+^\dagger$$

- This is just a system of coupled equations

$$\frac{d}{dt}\begin{pmatrix} \rho_a^{\mathscr{R}} \\ \rho_{a+1}^{\mathscr{R}} \\ \vdots \\ \rho_{b-1}^{\mathscr{R}} \\ \rho_b^{\mathscr{R}} \end{pmatrix} = \begin{pmatrix} \mathscr{L}_0 & \mathscr{J}_+ & 0 & \dots & 0 \\ \mathscr{J}_- & \mathscr{L}_0 & \mathscr{J}_+ & \dots & 0 \\ 0 & \mathscr{J}_- & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & \mathscr{L}_0 & \mathscr{J}_+ \\ 0 & 0 & \dots & \mathscr{J}_- & \mathscr{L}_0 \end{pmatrix} \begin{pmatrix} \rho_a^{\mathscr{R}} \\ \rho_{a+1}^{\mathscr{R}} \\ \vdots \\ \rho_{b-1}^{\mathscr{R}} \\ \rho_b^{\mathscr{R}} \end{pmatrix}$$

- Then we can show that

$$f_{\mathscr{R}}(t) = \mathrm{tr}\left\{L_+^\dagger L_+ \rho_b^{\mathscr{R}}(t)\right\} + \mathrm{tr}\left\{L_-^\dagger L_- \rho_a^{\mathscr{R}}(t)\right\}$$

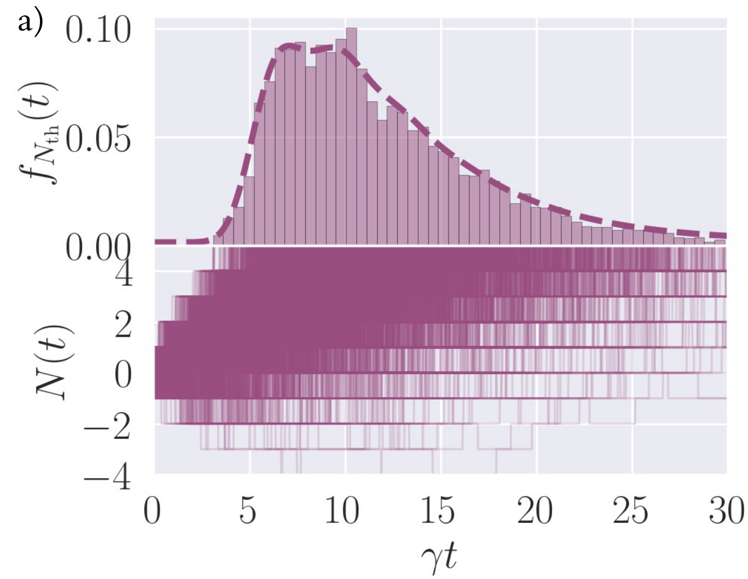
- FPT therefore depends only on the boundary states.

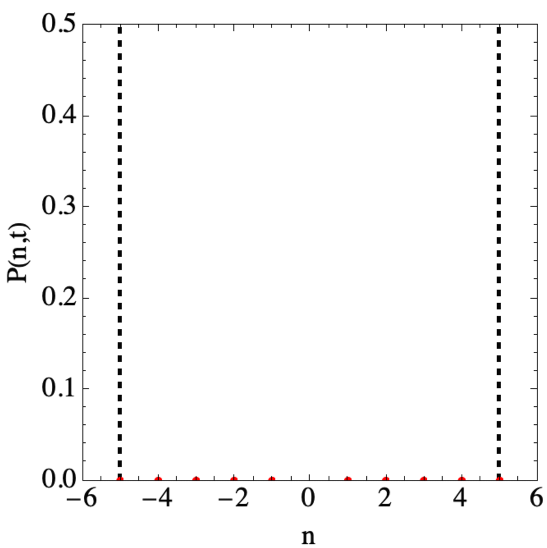
Ex: driven qubit

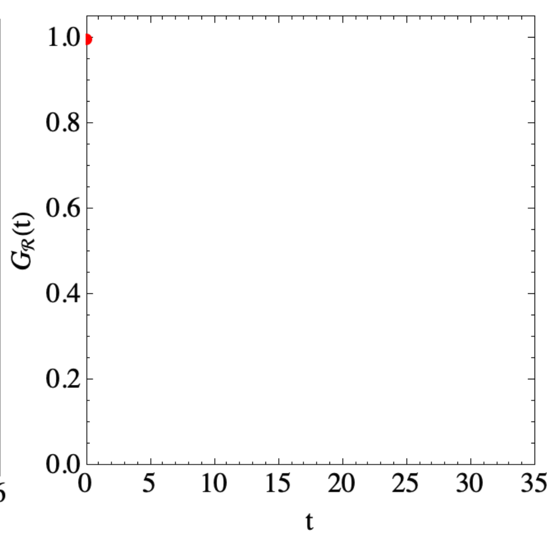
- Consider

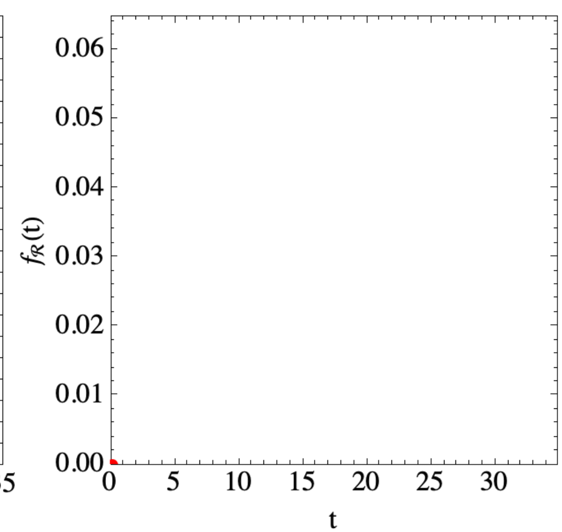
$$\frac{d\rho}{dt} = -i[\Omega\sigma_x, \rho] + \gamma(\bar{n}+1)D[\sigma_-] + \gamma\bar{n}D[\sigma_+]$$

- We choose boundaries $\mathscr{R} = [-5,5]$









Kinetic Uncertainty Relations

- In classical stochastic processes (incoherent master equations) the signal-to-noise ratio of the FPT is bounded as

$$\mathrm{SNR} := \frac{E(\tau)^2}{\mathrm{Var}(\tau)} \leqslant E(\tau)K$$

where $K = \sum_k \mathrm{tr}\left\{L_k^\dagger L_k \rho_{ss}\right\}$ = dynamical activity

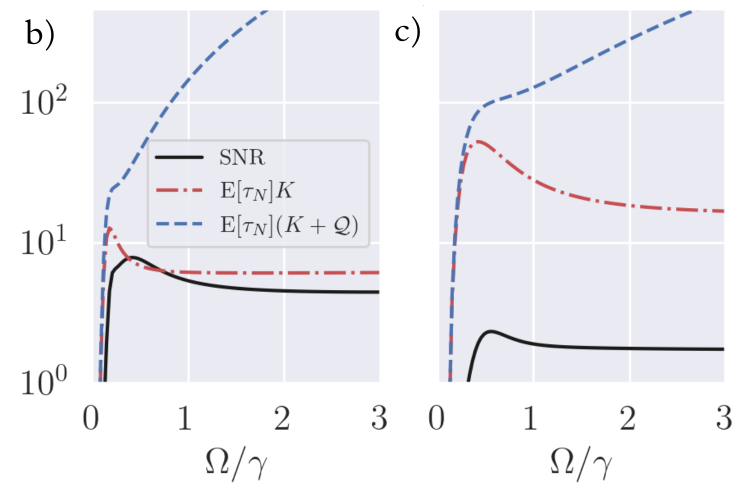
(average number of jumps per unit time).

- This KUR can be violated for coherent dynamics.

- Van Vu and Saito showed instead that in this case there is another bound

$$\mathrm{SNR} \leqslant E(\tau)(K + \mathcal{Q})$$

where $\mathcal{Q}$ is a quantum correction.



For the qubit model:

$$K = \frac{2\gamma(2\bar{n}+1)\left[\gamma^2\bar{n}(\bar{n}+1) + 2\Omega^2\right]}{\gamma^2(2\bar{n}+1)^2 + 8\Omega^2}$$

$$\mathcal{Q} = \frac{32\Omega^2}{\gamma^2(2\bar{n}+1)^2}K$$

J. P. Garrahan, Phys. Rev. E 95, 032134 (2017)
T. Van Vu and K. Saito, Phys. Rev. Lett. 128, 140602 (2022).

Stochastic clocks

- Recall that quantum jumps can be described in the:
  - "N-ensemble:" total number of jumps are fixed. Final time t_f fluctuates.
  - "t_f-ensemble:" final time is fixed. Total number of jumps fluctuates.
- If you want to use your system as a clock, we have to work in the N-ensemble.
  - Time standard = first passage time when $N(t) > b$ or $N(t) < a$.
- Simple choice: take the current = dynamical activity; i.e. $\nu_k = 1$. Then

$$\frac{\partial \rho_n^{\mathscr{R}}}{\partial t} = \mathscr{L}_0 \rho_n^{\mathscr{R}} + \mathscr{J} \rho_{n-1}^{\mathscr{R}}, \qquad \mathscr{J}\rho = \sum_k L_k \rho L_k^\dagger$$

$$\frac{d}{dt}\begin{pmatrix} \rho_0^{\mathscr{R}} \\ \rho_1^{\mathscr{R}} \\ \vdots \\ \rho_{b-1}^{\mathscr{R}} \\ \rho_b^{\mathscr{R}} \end{pmatrix} = \begin{pmatrix} \mathscr{L}_0 & 0 & 0 & \dots & 0 \\ \mathscr{J} & \mathscr{L}_0 & 0 & \dots & 0 \\ 0 & \mathscr{J} & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & \mathscr{L}_0 & 0 \\ 0 & 0 & \dots & \mathscr{J} & \mathscr{L}_0 \end{pmatrix} \begin{pmatrix} \rho_0^{\mathscr{R}} \\ \rho_1^{\mathscr{R}} \\ \vdots \\ \rho_{b-1}^{\mathscr{R}} \\ \rho_b^{\mathscr{R}} \end{pmatrix}$$

- We just need to solve this for $n = 0, 1, \dots, b$. The FPT will be

$$f_b(t) = \mathrm{tr}\big\{\mathscr{J}\rho_b(t)\big\}$$

Diffusion unravelling

- In this case the system follows a stochastic master equation

$$d\rho_c = \mathscr{L}\rho dt + \sum_k \left(\mathscr{K}[L_k e^{-i\phi_k}]\rho_c - \langle x_k \rangle_c \rho_c\right) dW_k(t)$$

where $\mathscr{K}[A]\rho = A\rho + \rho A^\dagger$, $x_k = L_k e^{-i\phi_k} + L_k^\dagger e^{i\phi_k}$ and $dW_k(t)$ are independent Wiener increments.

- The current and the charge in this case is given by

$$I(t) = \sum_k \nu_k \left(\langle x_k \rangle_c + \frac{dW}{dt}\right), \qquad N(t) = \int_0^t dt' \; I(t')$$

- The charge is now a continuous stochastic process. Our main result in this case is that the charge-resolved master equation becomes a Fokker-Planck like equation:

$$\frac{\partial \rho_n}{\partial t} = \mathscr{L}\rho_n - \sum_k \nu_k \mathscr{K}[L_k e^{-i\phi_k}] \frac{\partial \rho_n}{\partial n} + \frac{K_{\text{diff}}}{2} \frac{\partial^2 \rho_n}{\partial n^2} \qquad\qquad K_{\text{diff}} = \sum_k \nu_k^2$$

Similar equations appear in reaction-diffusion systems.

- Absorbing boundary conditions can now be easily implemented, as before.

B. Annby-Andersson, F. Bakhshinezhad, D. Bhattacharyya, G. D. Sousa, C. Jarzynski, P. Samuelsson, and P. P. Potts, Physical Review Letters 129 (2022),

Homodyne detection of σ_y

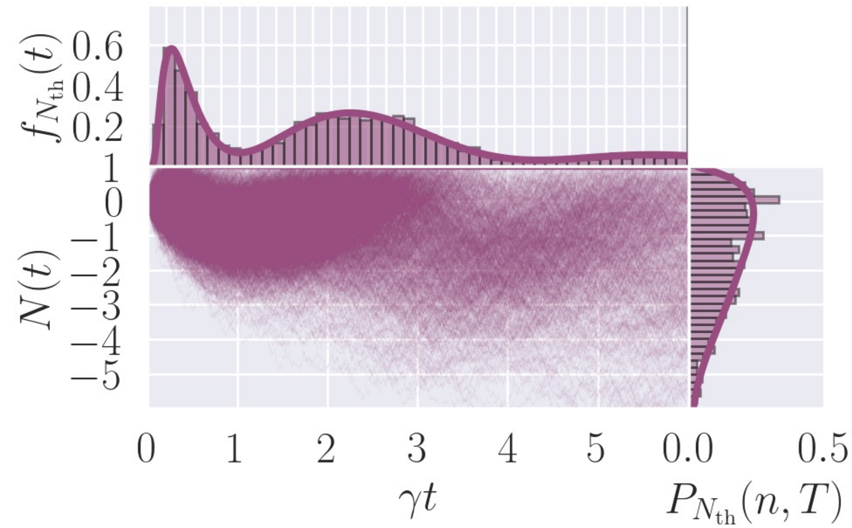
- We consider again

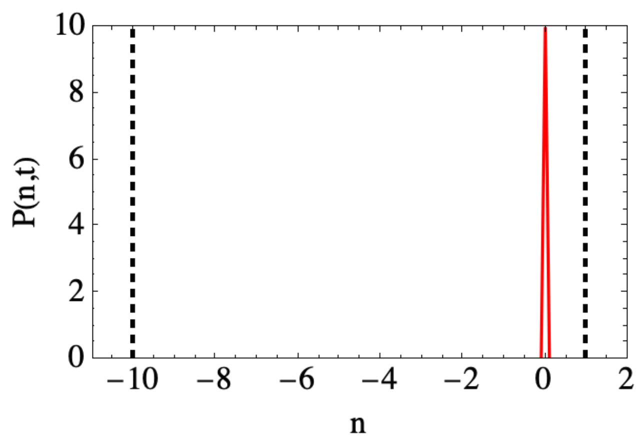
$$\frac{d\rho}{dt} = -i[\Omega\sigma_x, \rho] + \gamma D[\sigma_-]\rho, \qquad \nu = 1,$$

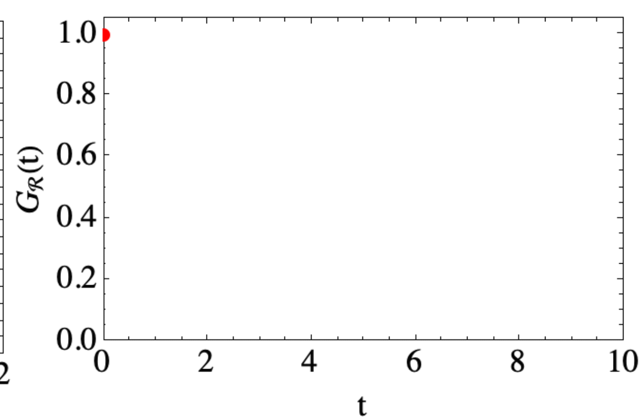
- Choose $\phi = -\pi/2 \quad \rightarrow \quad x = i(L - L^\dagger) = \sqrt{\gamma}\sigma_y$

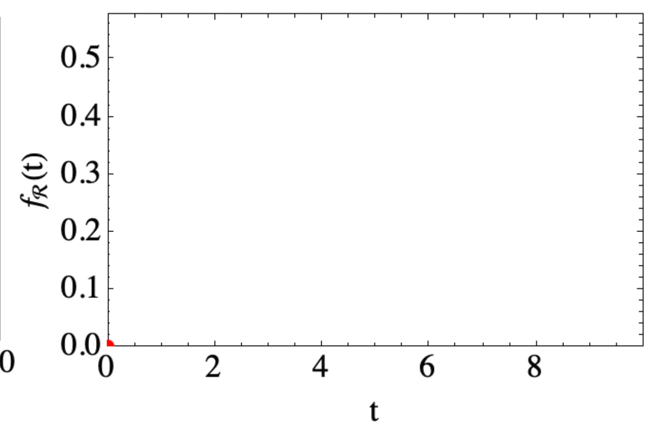
- $I(t) = \langle x \rangle_c + \dfrac{dW}{dt}.$

- For the FPT, we set the region to be $\mathscr{R} = (-\infty, N_{\rm th}]$ with $N_{\rm th} = 1$









FPT, feedback and other conditionings

- Quantum continuous measurements are associated with a *classical & detectable* stochastic process:

$$X_0, X_1, X_2, \ldots, \qquad\qquad X_j = X(jdt)$$

  - Quantum jumps: $X_j = 0,1,\ldots,\mathscr{M}$ (either the jump channel or no jump)

  - Quantum diffusion: $X_j = I(jdt) = \mathrm{tr}(x\rho_{c,j}) + \dfrac{dW}{dt}$

- If these outcomes are detectable, we can consider dynamics which use this data to do something.

  ➡ First Passage Times uses this data as a stopping criterion.

  ➡ Feedback uses this data to adaptively change the Hamiltonian.

- At the level of the stochastic master equation both have the general form

$$d\rho_c(jdt) = \mathbb{W}_{X_{1:j}}(\rho_c)$$

with some superoperator that may depend on the entire history up to time t.

- Implementing arbitrary maps like this is easy at the level of quantum trajectories.
  - But stochastic averages are expensive.
    - Having ways to implement this deterministically is very valuable.
      - Our formalism allows us to implement arbitrary conditioning on the integrated current $N(t)$.
- We can easily adapt our formalism to include feedback:

$$H \to H(N(t)) \quad \text{and} \quad L_k \to L_k(N(t))$$

  - Charge-resolved master equations then become:

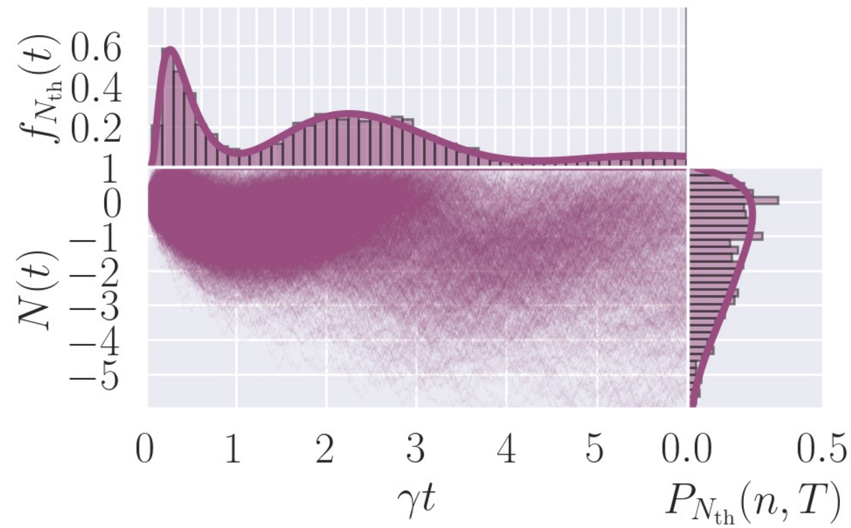
$$\frac{\partial \rho_n}{\partial t} = \mathscr{L}_0(n)\rho_n + \sum_k L_k(n)\rho_{n-\nu_k}L_k(n)^\dagger$$

$$\frac{\partial \rho_n}{\partial t} = \mathscr{L}(n)\rho_n - \sum_k \nu_k \mathscr{K}[L_k(n)e^{-i\phi_k}]\frac{\partial \rho_n}{\partial n} + \frac{K_{\text{diff}}}{2}\frac{\partial^2 \rho_n}{\partial n^2}$$

H. M. Wiseman and G. J. Milburn, Physical Review Letters 72, 4054 (1994).

Conclusions

- New results on how to compute first passage times in continuously measured quantum systems.
    - FPT until the integrated current $N(t)$ first crosses a region $\mathscr{R} = [a, b]$
- Basic idea: charge-resolved master equation + absorbing boundary conditions.
- Potential future applications:
    - Thermodynamics: stopping times in a gambling strategy.
    - Feedback.
    - Time-keeping.



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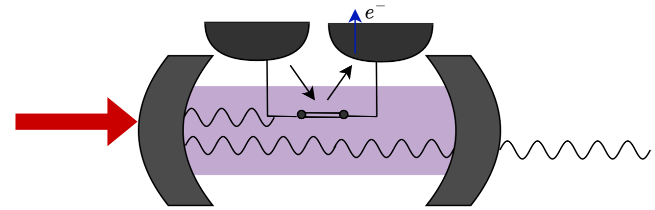
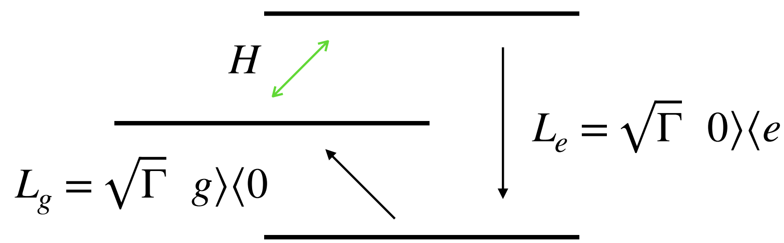
Ex: DQD photon detector



- Optical cavity interacting with a DQD:

$$\frac{d\rho}{dt} = -i[H,\rho] + \kappa D[a]\rho + \Gamma D[\ 0\rangle\langle e\ ]\rho + \Gamma D[\ g\rangle\langle 0\ ]\rho$$

with $H = g\left(a^\dagger\ g\rangle\langle e\ + a\ e\rangle\langle g\ \right)$



- DQD has 3 levels: $0\rangle,\ g\rangle,\ e\rangle$

- An absorbed photon takes the DQD from $g\rangle$ to $e\rangle$.

  - Photo-detection successful if jump occurs in channel $L_e = \sqrt{\Gamma}\ 0\rangle\langle e\ $.

  - Channel $L_g = \ g\rangle\langle 0$ (injection of an electron on the DQD) is assumed not monitorable.

  - Unsuccessful if it occurs at $L_c = \sqrt{\kappa a}$ (let us assume it is monitorable for now).

Convenient parameters:
$\alpha = \Gamma/\kappa$
$C = 4g^2/\Gamma\kappa$
$=$ cooperativity.

W. Khan, *et. al.*, "**Efficient and continuous microwave photoconversion in hybrid cavity-semiconductor nanowire double quantum dot diodes**." Nature Communications, **12,** 5130, (2021).

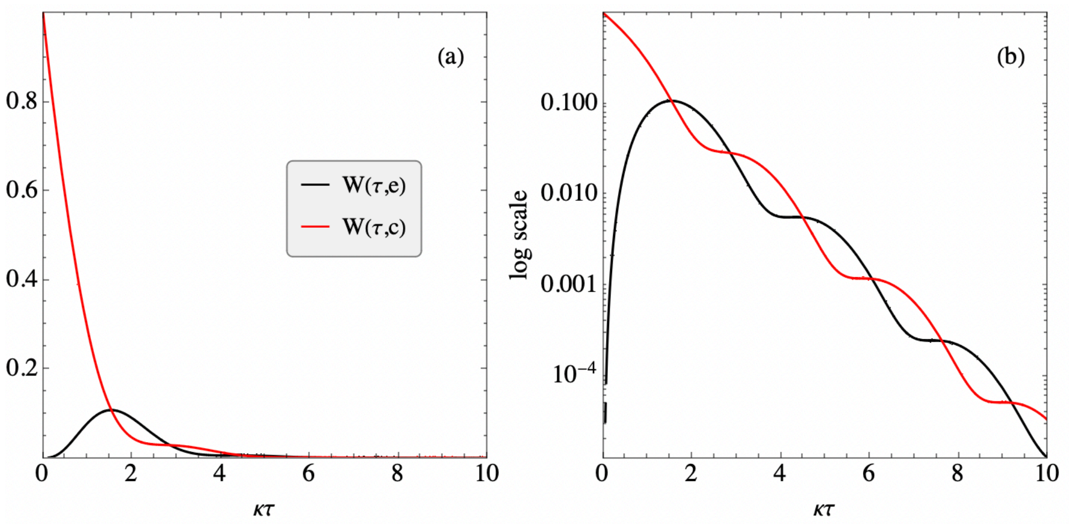
Drilon Zenelaj, Patrick P. Potts, Peter Samuelsson, "**Full counting statistics of the photocurrent through a double quantum dot embedded in a driven microwave resonator.**" Physical Review B, **106**, 205135, (2022).

- Suppose we start with exactly one photon inside the cavity: $\rho_0 = \ 0\rangle\langle 0\ _{\mathrm{DQD}} \otimes \ 1\rangle\langle 1\ _{\mathrm{cav}}$

- WTD: $W(\tau, k) = \mathrm{tr}\left\{\mathcal{J}_k e^{\mathcal{L}_0 \tau} \rho_0\right\}$.

  Alphabet $k = e, c$.

$\alpha = 1$ and $C = 4$



Oscillations occur when $4C > (\alpha - 1)^2/\alpha$

- $p_e = \int_0^\infty W(\tau, e) d\tau = \frac{C}{1 + C} \frac{\alpha^2}{(1 + \alpha)^2}$

- $\kappa E(\tau \ \ e) = \kappa \int_0^\infty d\tau \ \tau \frac{W(\tau, e)}{p_e}$

$$= \frac{3}{\alpha + 1} + \frac{1}{\alpha(C + 1)} + \frac{1}{C + 1}$$

- $\kappa E(\tau) = E(\tau \ \ e) p_e + E(\tau \ \ g) p_g$

$$= \frac{1}{C + 1} + \frac{C}{C + 1} \frac{3\alpha + 1}{(\alpha + 1)^2}$$

Luis Felipe Santos, GTL, “**Waiting time statistics of a double quantum dot-based single-photon detector.**” In preparation.