Quantum jump patterns in Hilbert space and the stochastic operation of quantum thermal machines

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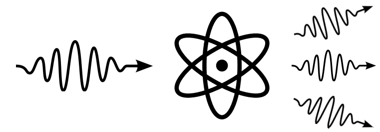
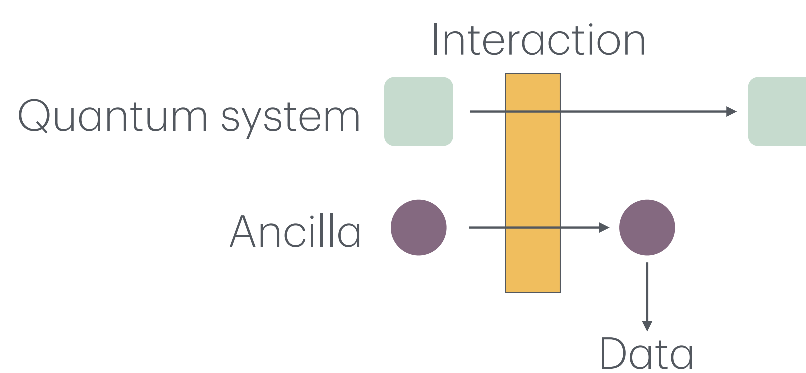
- We cannot see quantum systems...



- All we can do is perform *measurements* and analyze the resulting outcomes.

$$|\psi\rangle \rightarrow \text{classical outcome } x$$

- Thus, all we see are bit strings: ...1110000100010011100111101100...
- *What can we learn about a quantum system just from a bitstring?*
- To measure a system we must send in a **probe** (or **ancilla**).
  - S+A interaction encodes information about S on A.
  - Extract information by measuring A.
- **Information-back action trade-off:** the more information we want, the more we disturb the system.



Continuous measurements

- My interest is in systems that are constantly being measured, at a **stroboscopic** fashion.
- Looks a bit like a Hidden Markov Model (HMM):
  - Quantum system is hidden.
  - Measurement outcomes (what we see) = **emitted symbols**
- Data record = bitstring = $x_{1:n} = (x_1, \ldots, x_n)$.
- But one fundamental difference: quantum systems live in Hilbert space.

Similarity to Hidden Markov Models

- HMM is specified by a transition probability

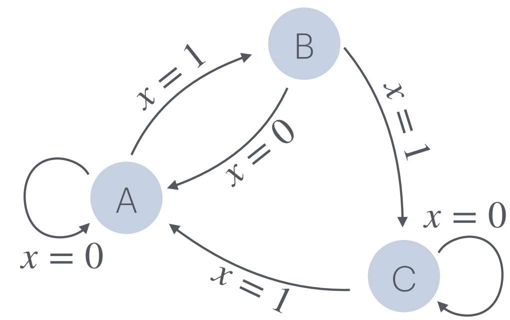
  $P(x, \sigma | \sigma')$ = prob. that system goes from $\sigma' \to \sigma$ while emitting a symbol x.

- Stochastic dynamics proceeds in 2 steps:
  - If HMM state is $\pi(\sigma')$ the prob. that we observe symbol x is

$$p(x) = \sum_{\sigma,\sigma'} P(x, \sigma | \sigma')\pi(\sigma')$$

  - If outcome was x, bayesian update the state of the hidden layer:

$$\pi(\sigma | x) = \frac{P(x, \sigma)}{p(x)} = \frac{\sum_{\sigma'} P(x, \sigma | \sigma')\pi(\sigma')}{p(x)}$$



Mixed state representation

Substochastic matrices:

$$(M_x)_{\sigma,\sigma'} = P(x, \sigma | \sigma')$$

and $\langle 1 | = (1, \ldots, 1)$

Then

$$p(x) = \langle 1 | M_x | \pi \rangle$$

and

$$| \pi_x \rangle = \frac{M_x | \pi \rangle}{p(x)}$$

Quantum instruments

- For any physical model of a system-ancilla interaction + measurement in the ancilla, we can always define an *instrument*, which is a superoperator acting on the system's density matrix:

$$p_x = \mathrm{tr}\{M_x \rho\} \qquad \text{and} \qquad \rho'_x = \frac{M_x \rho}{p_x}$$

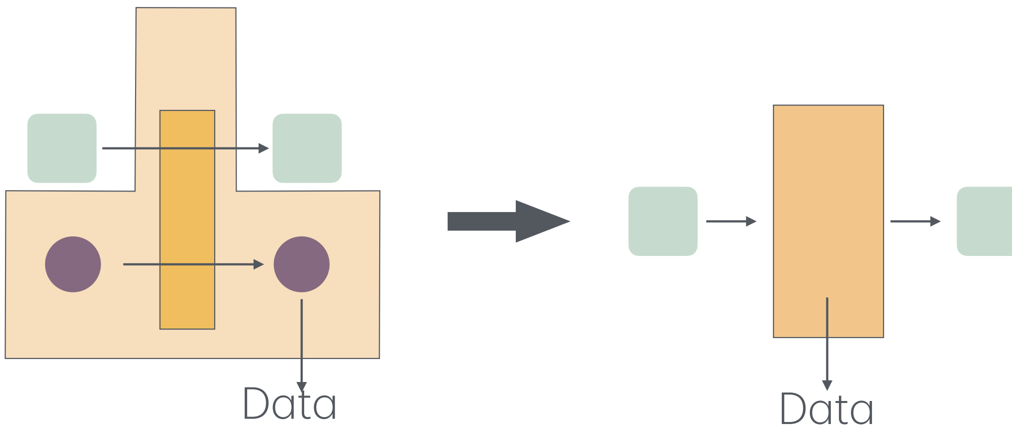
- Gives us the Bayesian update of the system's density matrix.

- If we measure but don't record the outcome the state of the system still changes (measurement back action)

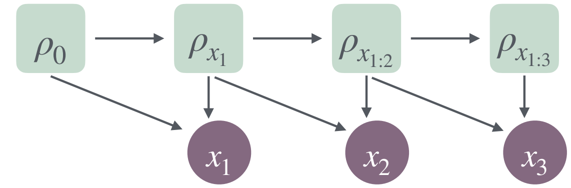
$$\rho' = \sum_x p_x \rho'_x = \sum_x M_x \rho = \mathscr{M} \rho$$

$$\mathscr{M} = \sum_x M_x$$

Unconditional dynamics



Stochastic dynamics



- Start at ρ_0.
- Step 1:
  - Draw a random number x_1 from $p(x_1) = \mathrm{tr}\{M_{x_1}\rho_0\}$.
  - Update the system to $\rho_{x_1} = M_{x_1}\rho_S^0/p(x_1)$.
- Step 2:
  - Draw a random number x_2 from $p(x_2|x_1) = \mathrm{tr}\{M_{x_2}\rho_{x_1}\}$
  - Update the system to $\rho_{x_{1:2}} = M_{x_2}\rho_{x_1}/p(x_2|x_1)$.
- Continue to generate a data string $x_{1:n}$.

Prob. of a string:

$$P(x_{1:n}) = \mathrm{tr}\{M_{x_N}\ldots M_{x_1}\rho_0\}$$

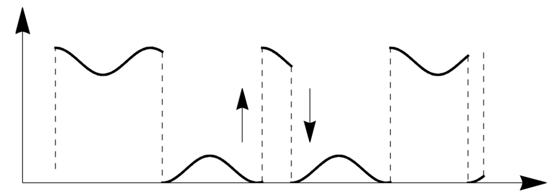
Conditional state

$$\rho_{x_{1:n}} = M_{x_N}\ldots M_{x_1}\rho_0/P(x_{1:n})$$
$$= M_{x_n}\rho_{x_{1:n-1}}/P(x_n|x_{1:n-1})$$

Unconditional state

$$\rho_n = \mathscr{M}^n\rho_0$$

Quantum jumps $\frac{d\rho}{dt} = -i[H,\rho] + L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}$



- Particular case: continuous *weak* measurements:
  - Measure at each time step, but effect of the measurement is small.

- One example of such a weak measurement is quantum jumps:
  Two outcomes: 1 (jump) and 0 (no-jump)

$$M_1\rho = dtL\rho L^\dagger \quad \text{and} \quad M_0\rho = \rho - idt(H_e\rho - \rho H_e^\dagger) = \rho + dt\ \mathscr{L}_0\rho$$

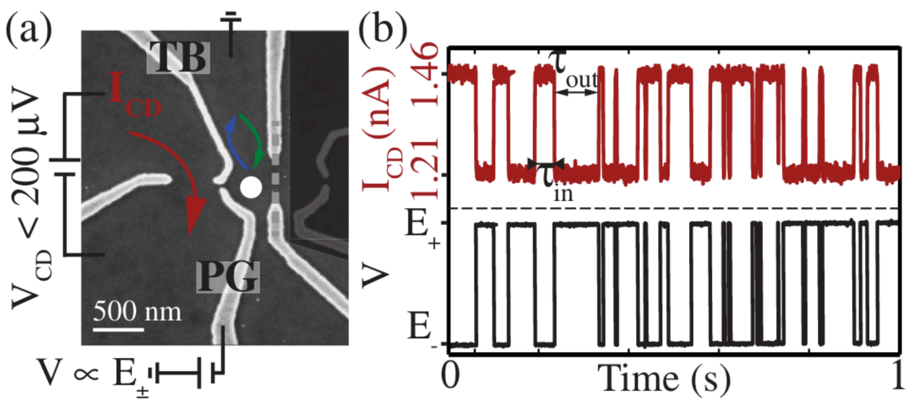
where $H_e = H - \frac{i}{2}L^\dagger L$

- $p_{\mathtt{jump}} = dt\ \mathrm{tr}(L\rho L^\dagger)$ is very small: most of the time the system evolves with no jump.

GTL, Michael J. Kewming, Mark T. Mitchison, Patrick P. Potts "**Current fluctuations in open quantum systems: Bridging the gap between quantum continuous measurements and full counting statistics**," PRX Quantum 5, 020201 (2024)

Jumps with multiple channels

- Quantum dot ≃ something which fits either 0 or 1 electron.
- Dot is close to a metallic wire (lead): electrons can tunnel in or out.
- Population in the dot can be measured with a Quantum Point Contact (QPC).
- Two "channels": injection or extraction.



Quantum trajectory = list of channels and their corresponding time-tags:

$$\{(x_1, t_1), (x_2, t_2), \dots, \}$$

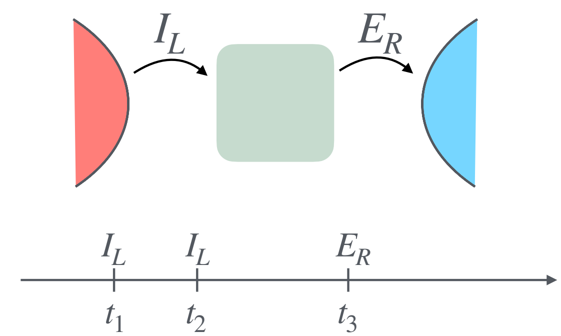
Hofmann, *et. al.* "**Measuring the Degeneracy of Discrete Energy Levels Using a GaAs / AlGaAs Quantum Dot**," Phys Rev. Lett **117**, 206803 (2016)
GTL, Dario Poletti, Gernot Schaller, "**Nonequilibrium boundary-driven quantum systems: Models, methods, and properties.**" Reviews of Modern Physics, 94, (2022)

The t and the N ensembles

- t-ensemble: final time is fixed; number of jumps might fluctuate.
  - Instruments: $M_0\rho = (1 + dt\mathscr{L}_0)\rho$ and $M_x\rho = dt\ L_x\rho L_x^\dagger$ for $x = 1,2,\ldots r$
- N-ensemble: work with a fixed number of jumps. But final time when last jump occurs is random.
  - Instruments: $M_{x,\tau}\rho = \mathscr{J}_x e^{\mathscr{L}_0\tau}\rho$ where $\mathscr{J}_x\rho = L_x\rho L_x^\dagger$.
    - Instrument has two indices: the jump channel and the jump interval τ
- Quantum jumps without time tags: we know a jump happened, but do not know when
  - Instruments: $M_x = -\mathscr{J}_x\mathscr{L}_0^{-1}$.

Quantum jumps without time tags

- Quantum jumps without time tags: we know a jump happened, but do not know when they happened.
  - Instruments: $M_x = -\mathcal{J}_x \mathcal{L}_0^{-1}$.



Prob. of a string:

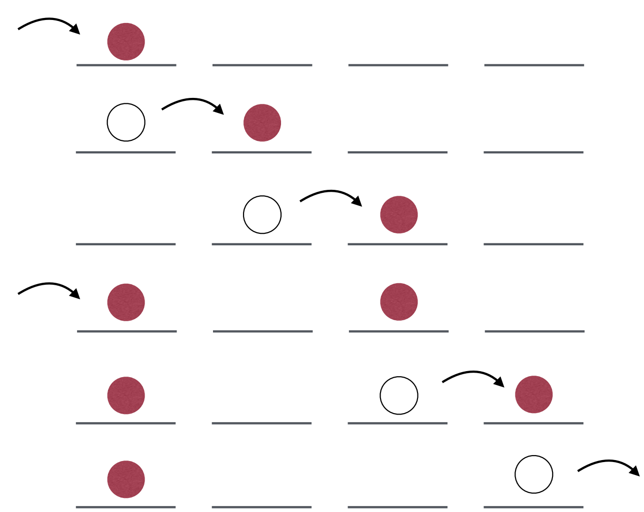
$$P(x_{1:n}) = \mathrm{tr}\{M_{x_N} \dots M_{x_1} \rho_0\}$$

Conditional state

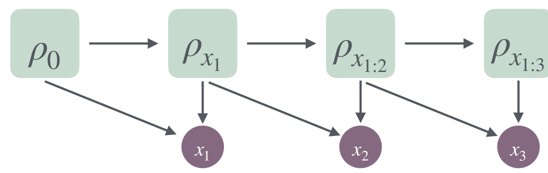
$$\rho_{x_{1:n}} = M_{x_N} \dots M_{x_1} \rho_0 / P(x_{1:n})$$

Classical analogy:

- Lattice with L sites, each of which can have 0 or 1 particles.
  - Excitations can be injected on the left, or extracted on the right.
  - And they can tunnel back and forth through the chain.
- All we would observe are symbols;
  - e.g. $I_L I_L E_R$. for the figure.



- While we see symbols $x_1, x_2, x_3, \ldots$ (visible layer), the system is evolving in Hilbert space (hidden layer).
    - *Which "states" does it traverse through?*



- Knowing this would make the dynamics more predictable:

$$P(x_n \,|\, x_1, x_2, \ldots, x_{n-1}) = P(x_n \,|\, \rho_{x_{1:n-1}})$$

    - No need to keep track of the entire history.

- For HMMs this is called the **mixed state representation**.
    - **Unifilar model:** if we know ρ and we observe x we know with certainty that the system evolved to $M_x \rho / \mathrm{tr}(M_x \rho)$.

Prob. of a string:

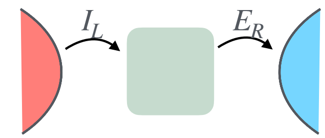
$$P(x_{1:n}) = \mathrm{tr}\{M_{x_N} \ldots M_{x_1} \rho_0\}$$

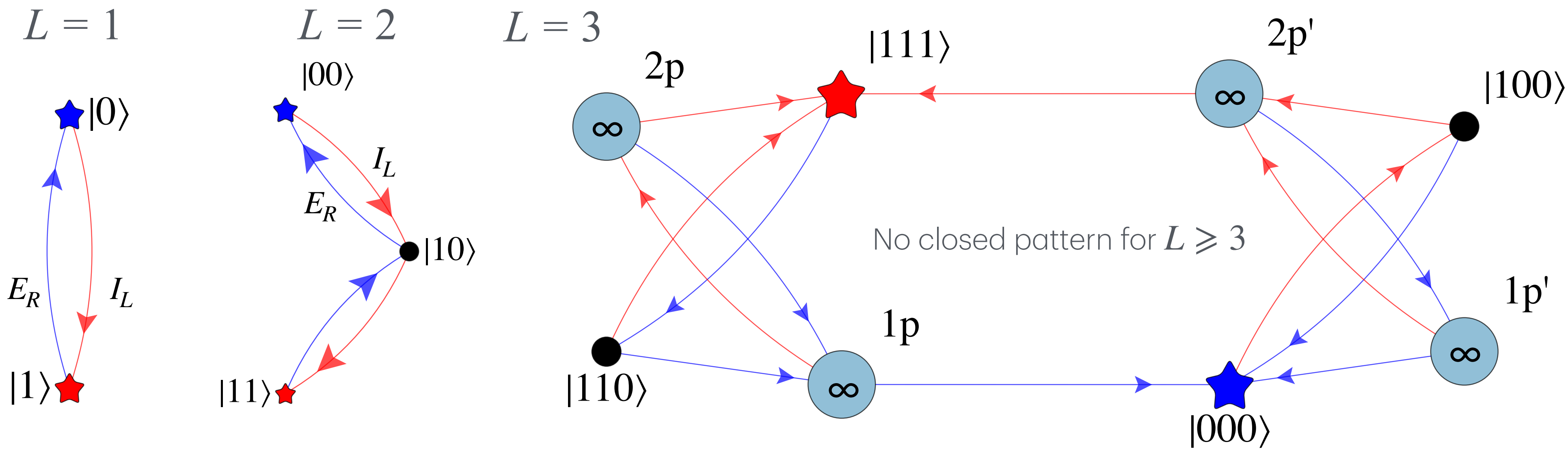
Conditional state

$$\rho_{x_{1:n}} = M_{x_N} \ldots M_{x_1} \rho_0 / P(x_{1:n})$$

If we can know the internal state, we can make statistical predictions of future outcomes.

Quantum model: fermions hopping on a lattice





- System will follow a *closed pattern* if there exists a finite set of states $\mathbb{B} = \{\sigma_1, \sigma_2, \ldots\}$ such that

$$\frac{M_x \sigma_i}{\mathrm{tr}(M_x \sigma_i)} = \sigma_{f(i,x)} \in \mathbb{B}$$

I

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

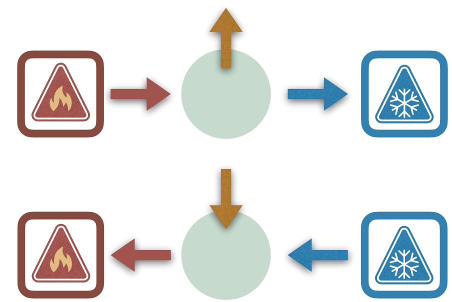
GTL "**Patterns in the jump-channel statistics of open quantum systems**," arXiv 2305.07957

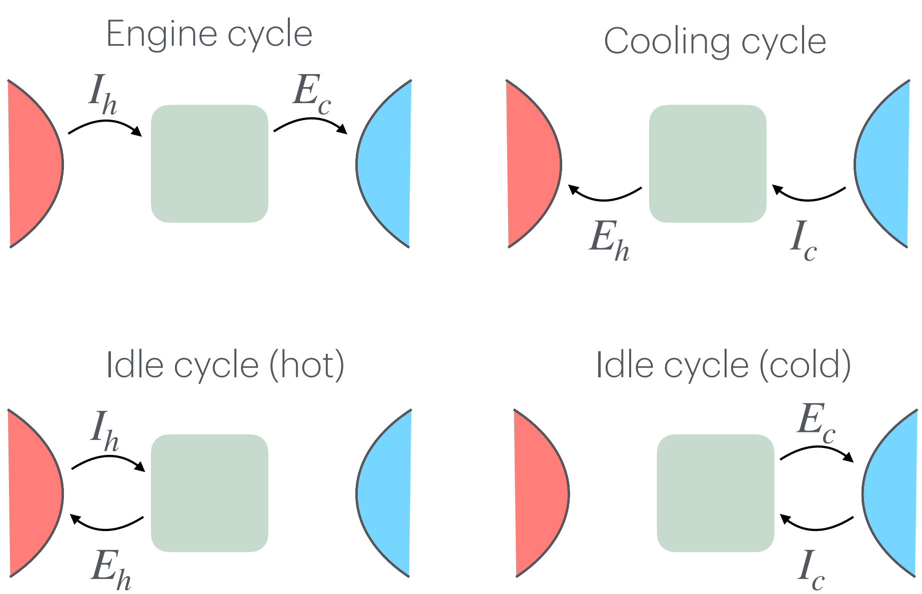
Patterns in a thermal machines

Abhaya Hegde

Abhaya S. Hegde, Patrick P. Potts, GTL, "**Time-resolved Stochastic Dynamics of Quantum Thermal Machines**," arXiv:2408.00694

- A heat engine extracts work by allowing heat to flow from hot to cold.
- A refrigerator consumes work to allow heat to flow from cold to hot.
- At the mesoscopic level injection/extraction of heat are described by individual quantum (or classical!) jumps.

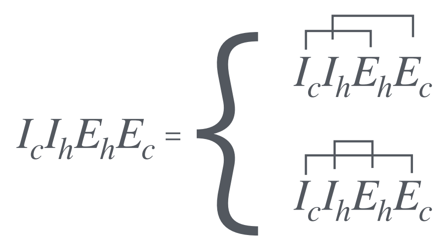




How to identify cycles from a bitstring?

$$\ldots I_h E_c I_c I_h E_h E_c I_h I_c E_h I_c \ldots$$

Impossible in general, if excitations are indistinguishable



Master equation model

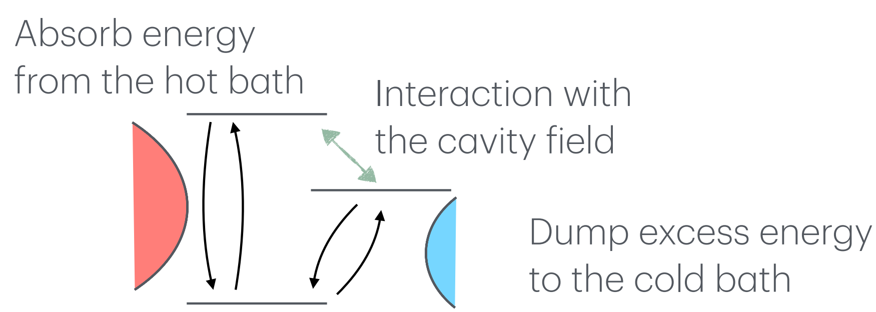
$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_n D[K_n]\rho + \sum_{\alpha\in\{h,c\}} \sum_j \gamma^-_{\alpha j} D[L_{\alpha j}]\rho + \gamma^+_{\alpha j} D[L^\dagger_{\alpha j}]\rho$$

- $-i[H,\rho]$: Unitary work
- $\sum_n D[K_n]\rho$: Work reservoirs
- $\gamma^-_{\alpha j} D[L_{\alpha j}]\rho$: Extraction to bath α
- $\gamma^+_{\alpha j} D[L^\dagger_{\alpha j}]\rho$: Injection from bath α

- To categorize cycles, we need to work with models that can host just a single excitation: $I_h E_c I_c E_h I_c E_c I_h E_c I_c E_c$.
- Result: *for a model to have only a single excitation, the Hilbert space must be split into two subspaces, P_E and P_I,* such that:
  1. Injection (extraction) quantum jumps always take the system to P_I (P_E).
  2. Work extraction can only take place *within* each subspace.

Hilbert space =

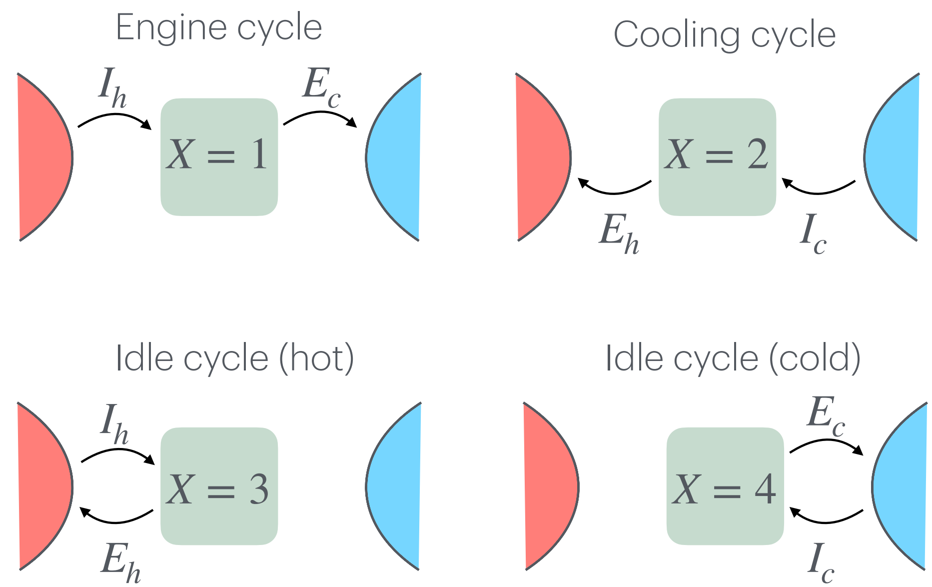
Example: 3-level maser or DQD with C. Blockade



- bitstring of jumps → bitstring of cycles

$$I_{\bullet}E_{\bullet}I_{\bullet}E_{\bullet}I_{\bullet}E_{\bullet}I_{\bullet}E_{\bullet}\ldots = X_1X_2X_3X_4\ldots$$

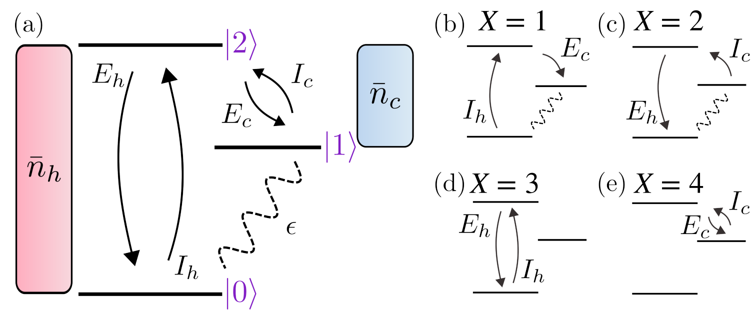
- We used this to answer the following questions:
  - What is the probability that the next cycle is of type X and takes a time τ?
  - How are cycles correlated with each other?
  - What is the average time required to complete each cycle?
  - How many idle cycles happen between two useful cycles?



Relation to steady-state currents:

$$I_{\mathrm{exc}} = \frac{p_1 - p_2}{E(\tau)}$$

Results for the 3-level maser



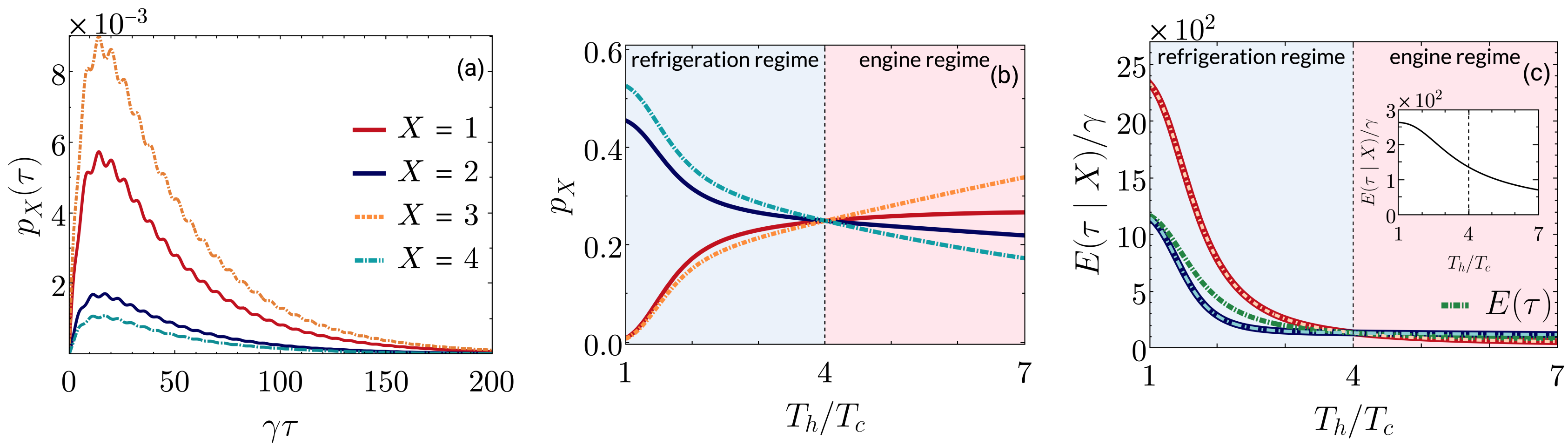


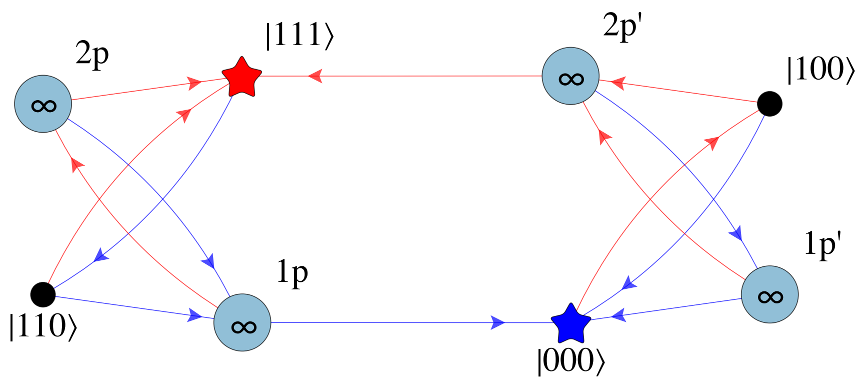
FIG. 3. **(a)** Probability of observing a cycle X within a duration τ [Eq. (9)] at resonance $\Delta = 0$ and $T_h/T_c = 10$. **(b)** Total probability of observing a cycle X [Eq. (10)] and **(c)** expectation values for cycle duration [Eqs. (11), (12)] as a function of the ratio of bath temperatures. A vertical line at $T_h/T_c = \omega_h/\omega_c$ separates the refrigerator and engine regimes. The inset shows all expectation values nearly converge at resonance. The parameters are fixed (in units of $T_c = 1$) at $\gamma_h = \gamma_c \equiv \gamma = 0.05$, $\omega_h = 8$, $\omega_c = 2$, $\omega_d = 4$, $\epsilon = 0.5$ unless mentioned otherwise.

Conclusions

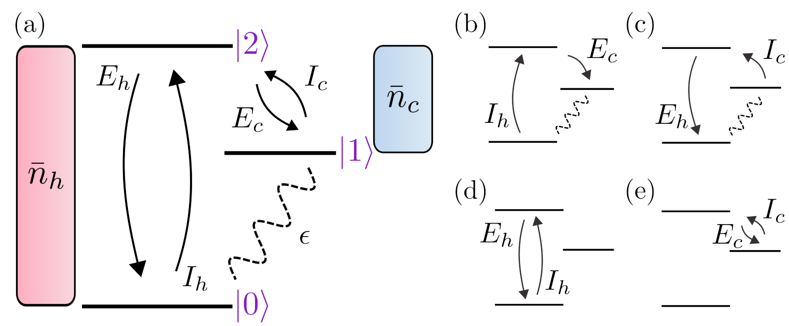
Thank you!

- Continuous quantum measurements yield a **time-series** of correlated stochastic outcomes which encodes information about the underlying quantum system.

Patterns in Hilbert space



Time-resolving the stochastic operation of quantum thermal machines



GTL, Michael J. Kewming, Mark T. Mitchison, Patrick P. Potts "**Current fluctuations in open quantum systems: Bridging the gap between quantum continuous measurements and full counting statistics**," PRX Quantum 5, 020201 (2024)

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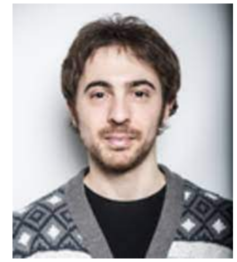
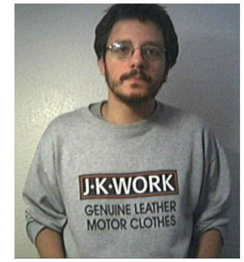


Alessio Belenchia



Mauro Paternostro

Informational steady-states

Alessio Belenchia, Luca Mancino, GTL and Mauro Paternostro, "**Entropy production in continuously measured quantum systems**", npj Quantum Information, **6**, 97 (2020).

GTL, Mauro Paternostro and Alessio Belenchia, "**Informational steady-states and conditional entropy production in continuously monitored systems**", PRX Quantum **3**, 010303, (2020).

The Holevo Information

- The Holevo information measures the information learned about the system from the data record $x_{1:n}$:

$$I(x_{1:n}) = D(\rho_{x_{1:n}} \| \rho_n) \geqslant 0$$

$$D(\rho \| \sigma) = \mathrm{tr}\left\{\rho \log \rho - \rho \log \sigma\right\}$$

- Reflects total gain of information from all data points.

- When averaged over all trajectories:

$$E(I) = \sum_{x_{1:n}} P(x_{1:n}) D(\rho_{x_{1:n}} \| \rho_n) = S(\rho_n) - S(\rho_n \,|\, x_{1:n})$$

$$S(\rho_n) = -\,\mathrm{tr}(\rho_n \log \rho_n)$$

$$S(\rho_n \,|\, x_{1:n}) = \sum_{x_{1:n}} P(x_{1:n}) S(\rho_{x_{1:n}})$$

Holevo gain & loss rate

- At each measurement:
  - Gain some information from the data point.
  - Loose some information due to noise.

- Can split the information rate into a gain and a loss (both individually non-negative):

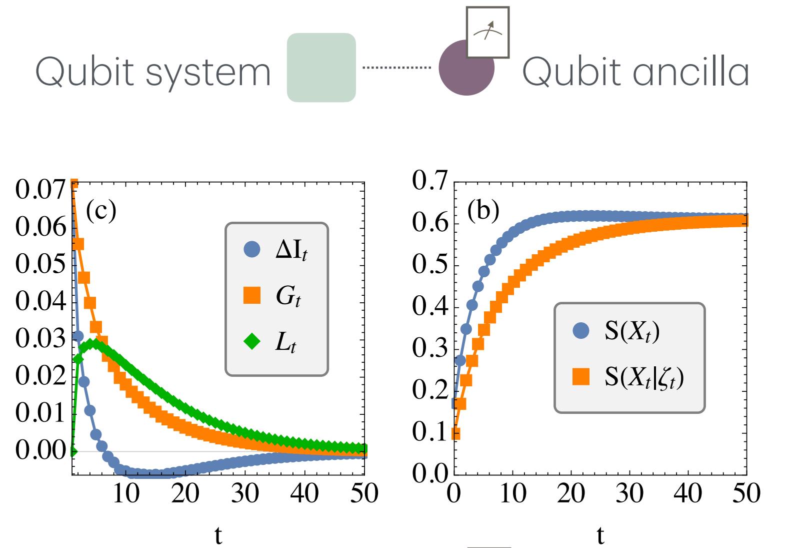
$$\Delta I = G - L$$

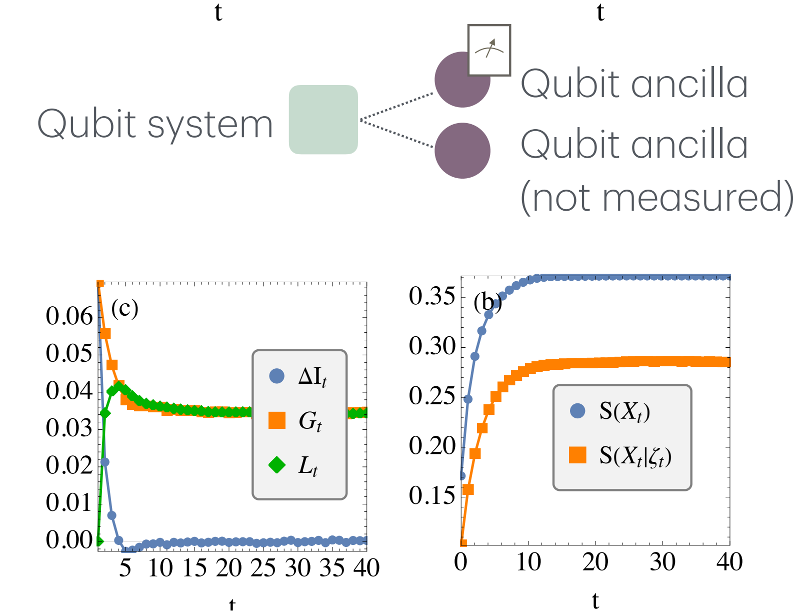
- Information gain:

$$G(x_{1:n}) = D(\rho_{x_{1:n}} \| \rho_n) - D(\mathscr{M}\rho_{x_{1:n-1}} \| \rho_n)$$

- Information loss:

$$L(x_{1:n}) = D(\rho_{x_{1:n-1}} \| \rho_{n-1}) - D(\mathscr{M}\rho_{x_{1:n-1}} \| \mathscr{M}\rho_{n-1})$$





Informational steady-state

- After a long time has passed $E(\Delta I_\infty) = 0$ (average change in Holevo information vanishes)
- But this does not mean that nothing is happening.
  - All it means is that the information gained is balanced by the information lost.

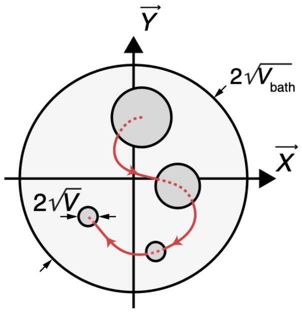
$$E(\Delta I_\infty) = E(G_\infty) - E(L_\infty) = 0$$

- In an informational steady-state (ISS)

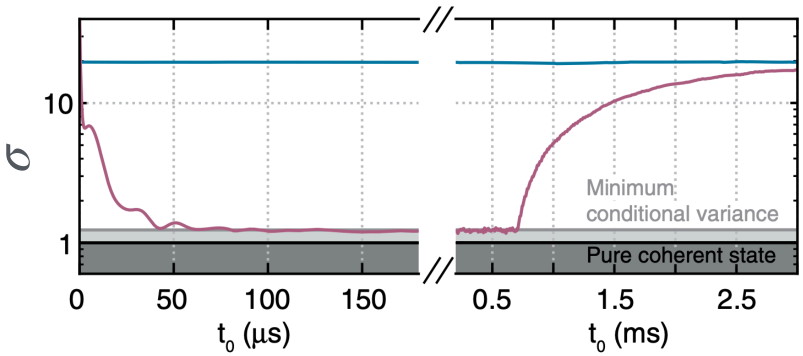
$$E(G_\infty) = E(L_\infty) \neq 0$$

- Constantly gathering; constantly loosing.

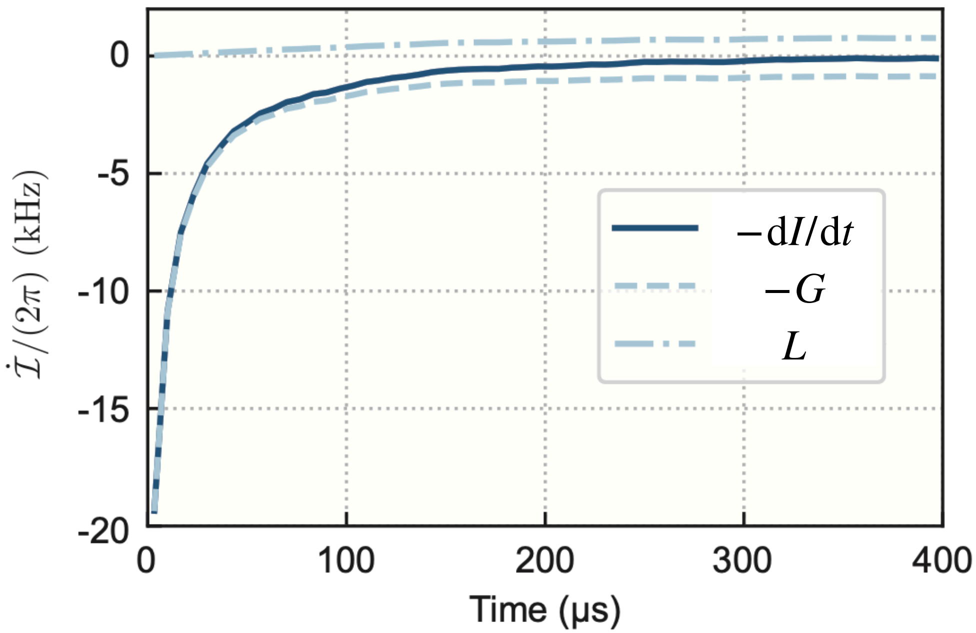
Optomechanical system



Phase space distribution becomes narrower as more data is gathered.



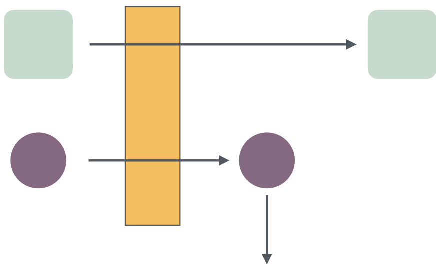
$$\sigma = \mathrm{var}(X)$$



Experimental assessment of the informational steady-state .
As long as we keep measuring, we effectively cool the system.

Massimiliano Rossi, Luca Mancino, GTL, Mauro Paternostro, Albert Schliesser, Alessio Belenchia, "**Experimental assessment of entropy production in a continuously measured mechanical resonator**", Phys. Rev. Lett. **125**, 080601 (2020)

Entropy production



- At each collision (step) the entropy of the system changes.
  - Part of this change is due to a flow of entropy from the ancilla (environment).
  - But part is due to the irreversible entropy production in the process.

$$\Delta S = \Delta\Sigma - \Delta\Phi$$

$\Delta\Phi$ = entropy flux = "Q/T"

$\Delta\Sigma \geq 0$ = entropy production.

- Fully information-theoretic formulation of the 2nd law:

$$\Delta\Sigma = I(S:A) + D(A' \| A)$$

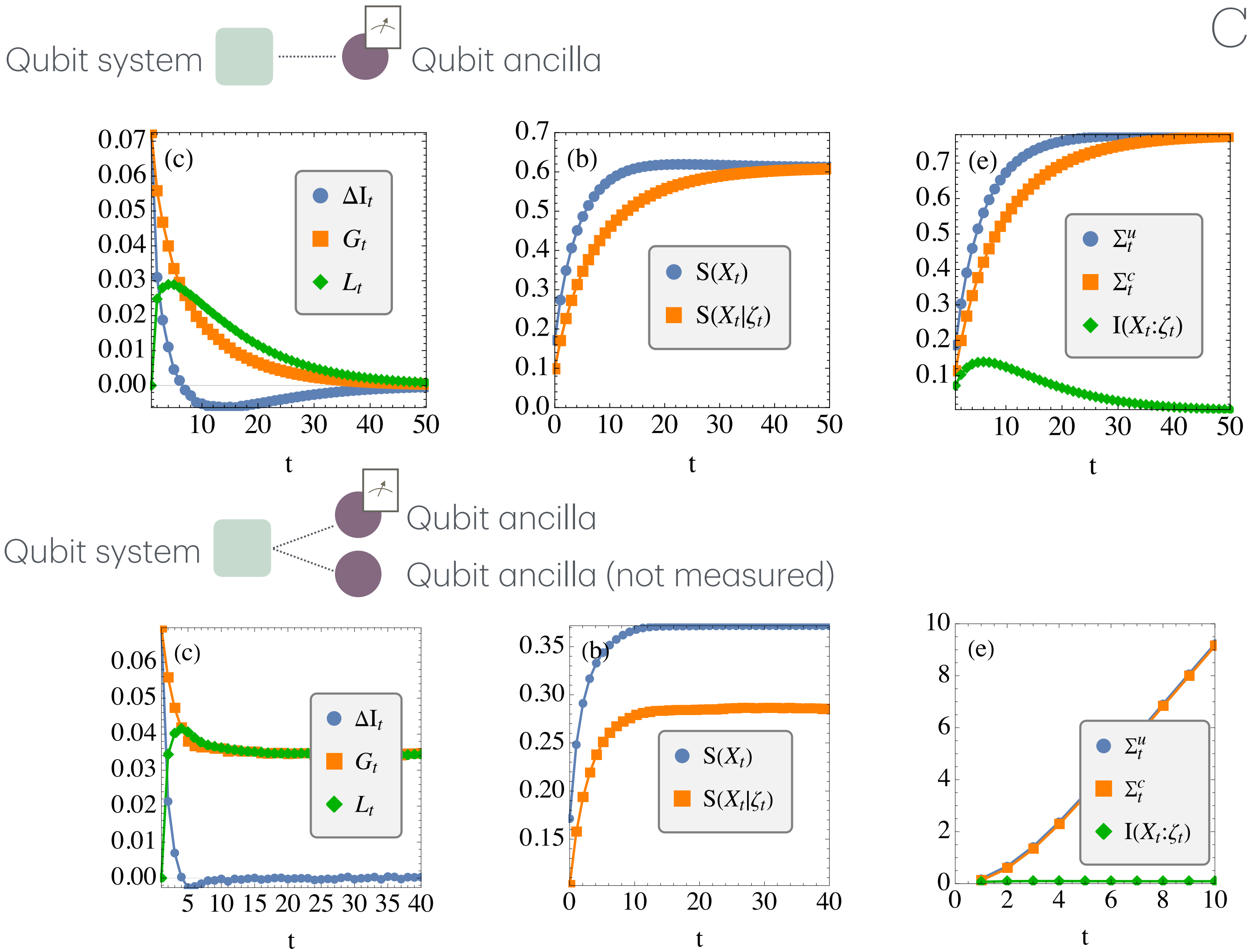
$I(S:A)$ = mutual information

$D(A' \| A)$ = rel. ent. of the ancilla

- Irreversibility occurs because:
  - system develops correlations with the ancilla, which are no longer accessible.
  - interaction pushes the ancillas away from initial state.

Massimiliano Esposito, Katja Lindenberg and Christian Van den Broeck, *New Journal of Physics*, 12, 013013 (2010).
GTL and Mauro Paternostro, "**Irreversible entropy production, from quantum to classical**", *Review of Modern Physics*, **93**, 035008 (2021)

Conditional 2nd law



We can similarly derive a 2nd law for the conditional dynamics.

It turns out that

$$\Delta\Sigma^c = \Delta\Sigma^u - \Delta I$$

Also follows for the integrated quantities

$$\Sigma^c = \Sigma^u - I < \Sigma^u$$

Reading the measurement outcomes can only reduce our understanding of irreversibility.

Hypothesis testing on words

System-ancilla interaction

- System-ancilla unitary interaction produces an entangled state:

$$\rho'_{SA} = U(\rho_S \otimes |0\rangle\langle 0|_A)U^\dagger$$

- Measure ancilla with projective measurements $\Pi_x = I_S \otimes |x\rangle\langle x|_A$. Data point = x.

- Prob. that outcome is x is $p_x = \mathrm{tr}\{\Pi_x \rho'_{SA} \Pi_x\}$.

- If outcome x is observed, state of the system must be updated to $\rho_S \to \rho'_{S|x} = \mathrm{tr}_A\{\Pi_x \rho'_{SA} \Pi_x\}$.

-

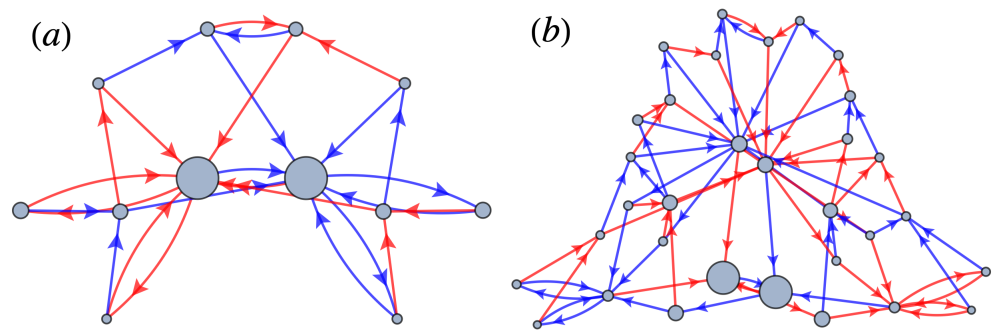
- Fix a cluster size N_c.

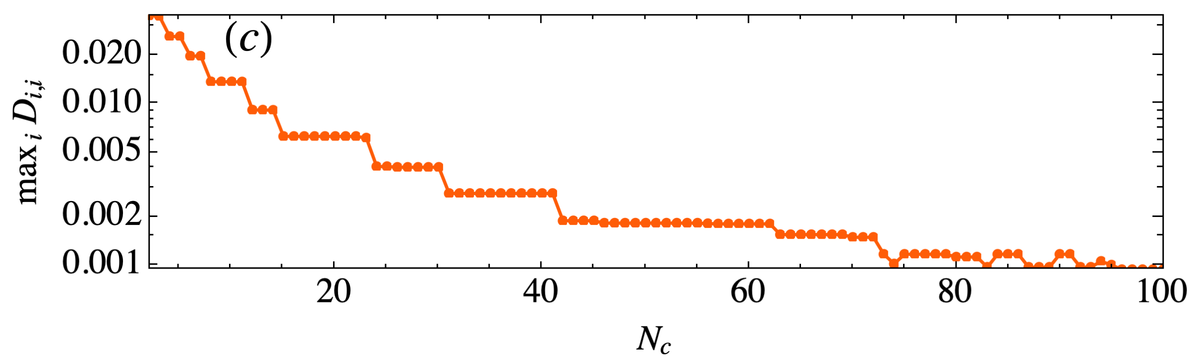
- employ the hierarchical agglomerative clustering algorithm with single-linkage and distance function $D(\rho_1, \rho_2)$.

- Average distance within each cluster

$$D_{ij} = \frac{1}{|S_i|^2} \sum_{k \in S_i, q \in S_i} D(\rho_k, \rho_q)$$

(S_i = set of states forming cluster i)





XY chain

- We now generalize this to a model where excitations can be spontaneously created or destroyed in pairs:

$$H = \sum_{i=1}^{L-1} (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+) + \kappa(\sigma_i^+ \sigma_{i+1}^+ + \sigma_i^- \sigma_{i+1}^-)$$

- In this case, no states of the system ever repeat.

Two states ρ_1 and ρ_2 are considered the same "causal states" if they predict the same future probabilities

$$P(x_{1:\infty} | \rho_1) = P(x_{1:\infty} | \rho_2)$$

In practice, for a fixed n define

$$D(\rho_1, \rho_2) = \sum_{x_{1:n}} | P(x_{1:n} | \rho_1) - P(x_{1:n} | \rho_2) |$$