


Waiting time statistics in boundary-driven free fermion chains

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We study the waiting time distributions of quantum chains coupled to two Lindblad baths at each end. Our focus is on free fermion chains, where we derive closed-form expressions in terms of single-particle matrices, allowing one to study arbitrarily large chain sizes. In doing so, we also derive formulas for 2-point correlation functions involving non-Hermitian propagators.

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I. INTRODUCTION

Transport in quantum chains constitutes a major research direction in nonequilibrium physics. The interplay between quantum coherent interactions and dissipative elements is known to produce a wide variety of physical phenomena. The basic example is the tuning of the ensuing transport regimes (e.g., ballistic, diffusive, etc.), which can be accomplished, e.g., by modifying the internal system interaction [1–6]. Further tuning the dissipation can also lead to noise-enhanced transport [7–12]. These developments open the prospect for numerous potential applications, such as quantum thermoelectricity [13–17] and thermal rectifiers [18–30].

As far as transport is concerned, most studies in quantum chains focus on either one of two scenarios [1]. The first is unitary time evolution, where the system is prepared in a localized wave packet and is then allowed to evolve unitarily. And the second is the steady state that is obtained when the system is placed in contact with two baths at different temperatures and/or chemical potentials. This is further divided into systems described in terms of coherent transport, e.g., the Landauer-Büttiker formalism [13,31], or systems described in terms of a quantum master equation, often referred to as boundary-driven systems [32].

In the case of steady states, even though the density matrix is no longer changing in time, the underlying process is still stochastic: At any given time, an excitation may enter from one of the baths and then travel through the system (possibly interacting with other excitations) until it eventually leaves to either bath. The quantum nature of the system makes this description much richer, as interference effects abound. But if one only looks at the steady-state density matrix, these effects are completely ignored.

The problem can be viewed pictorially as a detector with four colors, representing an excitation entering/leaving the left/right baths (Fig. 1). Each time an event occurs, a certain color clicks. The complete statistics of the detection events, including the times between clicks, as well as the colors of the clicks, is captured by the theory of full counting statistics

(FCS) [33–36]. The toolbox of FCS is extremely powerful, but usually difficult to apply, especially on many-body systems. For this reason, most practical studies on FCS have focused on the long-time statistics, i.e., on the accumulated number of clicks after a very long time, which satisfies a large-deviation principle [37,38].

A particularly interesting aspect of FCS concerns the waiting time distribution (WTD) between successive clicks [39,40]. There has been significant work on the study of WTDs in coherent conductors [36,41–52], such as double quantum dots or point contacts. However, WTDs are also useful in many other problems, where they have not yet been thoroughly explored. This paper will be concerned with boundary-driven systems, composed of a one-dimensional quantum chain coupled to two baths at each end, as described by a Lindblad master equation. The theory of WTDs in this case was laid down in [36], and subsequently applied to double quantum dot systems [41,49], Cooper pair splitters [53], and synchronized charge oscillations [54].

Here we develop formulas for the waiting time distribution of free fermion chains. As with most noninteracting problems, this allows the WTD to be written in terms of matrix elements and determinants of $L \times L$ matrices (where L is the number of sites in the chain), hence allowing one to study chains of arbitrary size. Despite being a noninteracting problem, the analysis turns out to be nontrivial since the time evolution between quantum jumps is non-Hermitian [55]. For this reason, we proceed by first casting the WTDs in terms of 2-point correlation functions involving non-Hermitian unitary evolution operators. We then develop general formulas for such propagators, which could find use beyond the present context. As an application, we study a simple tight-binding chain.

II. FORMAL FRAMEWORK

We consider a one-dimensional fermionic chain with L sites, each represented by an annihilation operator c_i . The system Hamiltonian is assumed to be quadratic, of the form

$$H = \sum_{i,j} h_{ij} c_i^\dagger c_j, \quad (1)$$

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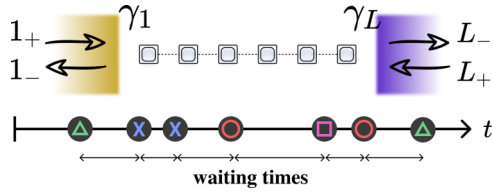


FIG. 1. Top: A quantum chain of length L , with four possible dissipation channels, associated with the injection/extraction of an excitation at the first/last sites, with coupling strengths $\gamma_{1(L)}$. Bottom: The interest in this work is on the waiting time distribution between clicks in each channel (here represented by buttons of a video game controller).

with a coefficient matrix h . The WTDs of free fermion chains were studied in [46], but only in the case of unitary dynamics. Instead, here we assume the system evolves connected to two local baths, coupled at sites 1 and L , and kept at Fermi-Dirac distributions f_1 and f_L . The dynamics is thus assumed to be governed by the local master equation

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = -i[H, \rho] + \sum_{i=1,L} \{\gamma_i^- D[c_i] + \gamma_i^+ D[c_i^\dagger]\}, \quad (2)$$

where $\gamma_i^- = \gamma_i(1 - f_i)$ and $\gamma_i^+ = \gamma_i f_i$, with γ_i being the coupling strengths to each bath. Here $D[A] = A\rho A^\dagger - \frac{1}{2}\{A^\dagger A, \rho\}$ is a Lindblad dissipator with arbitrary operator A .

The WTD in this case is defined in the context of full counting statistics. We split the Liouvillian in Eq. (2) as

$$\mathcal{L} = \mathcal{L}_0 + \sum_k \mathcal{J}_k, \quad (3)$$

where \mathcal{J}_k represent the four possible jump channels (“four colors in the detector”), which we label as 1_- , 1_+ , L_- , L_+ :

$$\begin{aligned} \mathcal{J}_{1_-}(\rho) &= \gamma_1^- c_1 \rho c_1^\dagger, & \mathcal{J}_{1_+}(\rho) &= \gamma_1^+ c_1^\dagger \rho c_1, \\ \mathcal{J}_{L_-}(\rho) &= \gamma_L^- c_L \rho c_L^\dagger, & \mathcal{J}_{L_+}(\rho) &= \gamma_L^+ c_L^\dagger \rho c_L. \end{aligned} \quad (4)$$

For instance, channel L_- means an excitation was absorbed by the right bath (at site L), and so on.

Starting from an arbitrary state ρ , the WTD between a jump in channel q at time 0 and a jump in channel k at time t is then given by [36]

$$P(t, k|q) = \frac{\text{tr} \mathcal{J}_k e^{\mathcal{L}_0 t} \mathcal{J}_q(\rho)}{\text{tr} \mathcal{J}_q(\rho)}, \quad (5)$$

which is normalized as

$$\sum_k \int_0^\infty P(t, k|q) dt = 1, \quad \forall q. \quad (6)$$

Equation (5) is a (conditional) joint distribution representing both the time between clicks and the channel of the click (note that clicks from different channels are usually statistically correlated [48]).

The marginal probability that jump q is followed by jump k , irrespective of when it occurs, is

$$p(k|q) = \int_0^\infty P(t, k|q) dt. \quad (7)$$

We can also filter the WTD to consider only the statistics conditioned on the sequence of jumps being $q \rightarrow k$. From Bayes’s rule one has

$$P(t|k, q) = P(t, k|q)/p(k|q). \quad (8)$$

This is now a properly normalized WTD, and so is more suitable for computing expectation values. We denote by T the random waiting time between any two events. The average $E(T|k, q)$, conditioned on the sequence of channels $q \rightarrow k$, is

$$E(T|k, q) = \int_0^\infty t P(t|k, q) dt. \quad (9)$$

Similarly, the variance of the waiting time reads

$$\text{var}(T|k, q) = E(T^2|k, q) - E(T|k, q)^2, \quad (10)$$

where $E(T^2|k, q)$ is defined similarly to $E(T|k, q)$.

We call attention to the fact that the WTDs defined above assume that all four channels are constantly being monitored (called “exclusive” WTDs in [53]). One could also study a situation where only channel k is being monitored (“inclusive” WTD). Unfortunately, this is not related to (5) in a simple way, since the inclusive distribution must account for all possible jumps to the other channels before a click in k is detected.

The WTD (5) refers to specific channels $q \rightarrow k$. One may also be interested in what shall be referred to as the *net activity time distribution* (NATD), which is the WTD between any two events, irrespective of the channel. In the steady state, it can be defined as

$$P(t) = \sum_{k,q} P(t, k|q)p(q), \quad (11)$$

where $p(q)$ is the relative frequency of occurrence for a jump of type q (in the steady state) and is given, up to a normalization, by $p(q) = \text{tr} \mathcal{J}_q \rho$. Expectation values for NATDs may be defined similarly to, e.g., Eqs. (9) and (10), and will be denoted by $E(T)$, $\text{var}(T)$, etc.

Computing the waiting time distribution is generally hard, as it involves studying the evolution under the map \mathcal{L}_0 , which is generally not completely positive and trace preserving. In fact, \mathcal{L}_0 can be decomposed as $\mathcal{L}_0 = -i(H_e \rho - \rho H_e^\dagger)$, where

$$\begin{aligned} H_e &= H - \frac{i}{2} [\gamma_1(1 - f_1)c_1^\dagger c_1 + \gamma_1 f_1 c_1 c_1^\dagger \\ &\quad + \gamma_L(1 - f_L)c_L^\dagger c_L + \gamma_L f_L c_L c_L^\dagger]. \end{aligned} \quad (12)$$

Hence, the action of \mathcal{L}_0 is tantamount to a non-Hermitian Hamiltonian evolution. Given the four possible channels in Eq. (4), there can be in total 16 WTDs (5). They can be more compactly written as

$$P(t, i_-|j_+) = \frac{\gamma_i^-}{\langle c_j^\dagger c_j^\dagger \rangle} \text{tr} \{c_i^\dagger c_i e^{-iH_e t} c_j^\dagger \rho c_j e^{iH_e^\dagger t}\}, \quad (13)$$

$$P(t, i_+|j_+) = \frac{\gamma_i^+}{\langle c_j c_j^\dagger \rangle} \text{tr} \{c_i c_i^\dagger e^{-iH_e t} c_j^\dagger \rho c_j e^{iH_e^\dagger t}\}, \quad (14)$$

$$P(t, i_-|j_-) = \frac{\gamma_i^-}{\langle c_j^\dagger c_j \rangle} \text{tr} \{c_i^\dagger c_i e^{-iH_e t} c_j \rho c_j^\dagger e^{iH_e^\dagger t}\}, \quad (15)$$

$$P(t, i_+|j_-) = \frac{\gamma_i^+}{\langle c_j^\dagger c_j \rangle} \text{tr} \{c_i c_i^\dagger e^{-iH_e t} c_j \rho c_j^\dagger e^{iH_e^\dagger t}\}, \quad (16)$$

with $i, j = 1, L$.

III. TRACE-DET FORMULAS FOR NON-HERMITIAN FERMIONIC FORMS

The traces in the WTDs (13)–(16) resemble 2-time correlation functions. However, the biggest difference is that the time propagator is H_e , which is non-Hermitian. This makes the direct computation of the WTDs more difficult than they may seem at first. For instance, one cannot use the usual Baker-Campbell-Hausdorff formulas [56], since the quantities in question here are of the form $e^{-iH_e t} O e^{iH_e^\dagger t}$, instead of $e^{-iH_e t} O e^{iH_e t}$. Instead, to compute these traces, we first develop a series of formulas which hold even for non-Hermitian operators. They are all based on variations of the so-called Blankenbecler-Scalapino-Sugar (BSS) “trace-det” relations [57,58], which are widely used in quantum Monte Carlo. Below, we only provide an overview of the main results. The actual derivations are given in the Appendix.

Let $X = \sum_{ij} X_{ij} c_i^\dagger c_j$, $Y = \sum_{ij} Y_{ij} c_i^\dagger c_j$, $Z = \sum_{ij} Z_{ij} c_i^\dagger c_j$ be quadratic forms in fermionic operators, with arbitrary coefficient matrices X , Y , and Z . The BSS trace-det formula states that [57,58]

$$\text{tr}\{e^X e^Y e^Z\} = \det(1 + e^X e^Y e^Z). \quad (17)$$

Here and henceforth we will not distinguish between the number 1 and the identity matrix 1. Equation (17) extends identically to more than three operators, but for our purposes 3 will suffice. This formula provides a huge simplification since the right-hand side is a determinant on the space of $L \times L$ matrices, X , Y , Z . This is to be contrasted with the left-hand side, which is a trace of a $2^L \times 2^L$ dimensional operator.

Using Eq. (17), we show in the Appendix that

$$\text{tr}\{c_i^\dagger c_i e^X e^Y e^Z\} = \mathbb{D} \mathcal{T}_{ii}, \quad (18)$$

where

$$\mathbb{D} = \det(1 + e^X e^Y e^Z), \quad (19)$$

and

$$\mathcal{T} = (e^{-Z} e^{-Y} e^{-X} + 1)^{-1} = e^X e^Y e^Z (1 + e^X e^Y e^Z)^{-1}. \quad (20)$$

Equation (18) again holds for more than 3 exponentials, provided the order of the exponentials is preserved.

Similarly, using both (17) and (18), we show in the Appendix that

$$\begin{aligned} \text{tr}\{c_i^\dagger c_i e^X c_j^\dagger c_j e^Y e^Z\} &= \mathbb{D} [(e^{-X} \mathcal{T} e^X)_{jj} \mathcal{T}_{ii} \\ &+ (\mathcal{T} e^{-Z} e^{-Y})_{ij} (e^{-X} \mathcal{T})_{ji}]. \end{aligned} \quad (21)$$

Compared with, e.g., Eq. (13), the main difference is that here there is a term $c_j^\dagger c_j e^Y$ while in (13) it reads $c_j^\dagger e^Y c_j$. Using the fact that $e^Y c_i e^{-Y} = \sum_j (e^Y)_{ij} c_j$, together with the fermionic algebra, one finds that

$$\begin{aligned} \text{tr}\{c_i^\dagger c_i e^X c_j^\dagger e^Y c_j e^Z\} &= \mathbb{D} [(e^{-Y} e^{-X} \mathcal{T} e^X)_{jj} \mathcal{T}_{ii} \\ &+ (\mathcal{T} e^{-Z} e^{-Y})_{ij} (e^{-Y} e^{-X} \mathcal{T})_{ji}]. \end{aligned} \quad (22)$$

This is of the same form as the trace appearing in Eq. (13), provided we take $i' = i$ and $j' = j$. Proceeding similarly, we can also compute expressions for the other 3 traces in

Eqs. (14)–(16):

$$\begin{aligned} \text{tr}\{c_i^\dagger c_i e^X c_j^\dagger e^Y c_j e^Z\} &= \mathbb{D} [(e^{-Y} e^{-X} \mathcal{T} e^X)_{jj} (\delta_{ii'} - \mathcal{T}_{ii'}) \\ &- (\mathcal{T} e^{-Z} e^{-Y})_{ij} (e^{-Y} e^{-X} \mathcal{T})_{ji}], \end{aligned} \quad (23)$$

$$\begin{aligned} \text{tr}\{c_i^\dagger c_i e^X c_j^\dagger e^Y c_j e^Z\} &= \mathbb{D} [(e^Y)_{jj'} - (e^{-X} \mathcal{T} e^X e^Y)_{jj'}] \mathcal{T}_{ii} \\ &- (\mathcal{T} e^{-Z})_{ij'} (e^{-X} \mathcal{T})_{ji}], \end{aligned} \quad (24)$$

$$\begin{aligned} \text{tr}\{c_i^\dagger c_i e^X c_j e^Y c_j^\dagger e^Z\} &= \mathbb{D} [(e^Y)_{jj'} - (e^{-X} \mathcal{T} e^X e^Y)_{jj'}] (\delta_{ii'} - \mathcal{T}_{ii'}) \\ &+ (\mathcal{T} e^{-Z})_{ij'} (e^{-X} \mathcal{T})_{ji'}. \end{aligned} \quad (25)$$

All formulas hold for *arbitrary* matrices X , Y , Z . But before we can apply them to the WTDs, some adaptations are still required.

IV. COMPUTATION OF THE WTDs

Since H_e in Eq. (12) is a quadratic form, we can use Eqs. (22)–(25) to compute the WTDs (13)–(16) provided the initial state ρ is Gaussian. We will focus on two main choices of initial states: the steady state ρ_{ss} of the master equation (2) and the vacuum state $\rho_{\text{vac}} = |0\rangle\langle 0|$. We can consider both together, by taking a generic Gaussian initial state of the form

$$\rho = \frac{1}{\mathbb{Z}} e^{-\sum_{ij} M_{ij} c_i^\dagger c_j}, \quad (26)$$

with some $L \times L$ matrix M . The partition function \mathbb{Z} is, in light of Eq. (17),

$$\mathbb{Z} = \det(1 + e^{-M}). \quad (27)$$

Alternatively, one can also characterize the Gaussian state by the covariance matrix $C_{ij} = \langle c_j^\dagger c_i \rangle$. The relation between C and M reads

$$e^M = \frac{1 - C}{C}, \quad C = \frac{1}{e^M + 1}. \quad (28)$$

The quadratic nature of the master equation (2) implies that the standard time evolution of C will be given by a Lyapunov equation

$$\frac{dC}{dt} = -(WC + CW^\dagger) + F, \quad (29)$$

where

$$W = ih + \frac{1}{2} \text{diag}(\gamma_1, 0, \dots, 0, \gamma_L), \quad (30)$$

$$F = \text{diag}(\gamma_1 f_1, 0, \dots, 0, \gamma_L f_L). \quad (31)$$

The steady state is thus the long-time solution of Eq. (29), viz.,

$$WC_{\text{ss}} + C_{\text{ss}} W^\dagger = F. \quad (32)$$

Similarly, the vacuum state is simply $C_{\text{vac}} = 0$. In practice, it may be more convenient to set C_{vac} to be proportional to the identity, with some small constant that is ultimately taken to zero. This approach will actually be used below, around Eq. (41). One should also bear in mind that the conditional

evolution which appears in the WTDs is *not* Gaussian because, e.g., $c_q \rho c_q^\dagger$ is not a Gaussian state. Notwithstanding, as we will show, it is still possible (and convenient) to express most results in terms of the matrices C (or M), W , F .

The operator H_e in Eq. (12) is not yet in a canonical quadratic form due to the terms $c_1 c_1^\dagger$ and $c_L c_L^\dagger$. In fact, writing $c_k c_k^\dagger = 1 - c_k^\dagger c_k$ turns out to yield a nontrivial constant. The resulting Hamiltonian can be conveniently written as

$$H_e = -i \sum_{i,j} Q_{ij} c_i^\dagger c_j - \frac{i}{2} \Gamma \equiv \tilde{H}_e - \frac{i}{2} \Gamma, \quad (33)$$

where $\Gamma = \gamma_1 f_1 + \gamma_L f_L$ is a constant and

$$Q = W - F. \quad (34)$$

A trace such as that in Eq. (13) can thus finally be written as

$$\begin{aligned} & \text{tr}\{c_i^\dagger c_i e^{-iH_e t} c_j^\dagger \rho c_j e^{iH_e^\dagger t}\} \\ &= \frac{e^{-\Gamma t}}{\mathbb{Z}} \text{tr}\{c_i^\dagger c_i e^{-i\tilde{H}_e t} c_j^\dagger e^{-\sum_{kl} M_{kl} c_k^\dagger c_l} c_j e^{i\tilde{H}_e^\dagger t}\}, \end{aligned} \quad (35)$$

which is now in the form (22), provided we identify

$$X = -Qt, \quad Y = -M, \quad Z = -Q^\dagger t. \quad (36)$$

The final expression for all WTDs therefore reads

$$\begin{aligned} P(t, i_- | j_+) &= \frac{\gamma_i^-}{1 - C_{jj}} \frac{e^{-\Gamma t}}{\mathbb{Z}} \mathbb{D}[(e^M e^{Qt} \mathcal{T} e^{-Qt})_{jj} \mathcal{T}_{ii} \\ &+ (\mathcal{T} e^{Q^\dagger t} e^M)_{ij} (e^M e^{Qt} \mathcal{T})_{ji}], \end{aligned} \quad (37)$$

$$\begin{aligned} P(t, i_+ | j_+) &= \frac{\gamma_i^+}{1 - C_{jj}} \frac{e^{-\Gamma t}}{\mathbb{Z}} \mathbb{D}[(e^M e^{Qt} \mathcal{T} e^{-Qt})_{jj} (1 - \mathcal{T}_{ii}) \\ &- (\mathcal{T} e^{Q^\dagger t} e^M)_{ij} (e^M e^{Qt} \mathcal{T})_{ji}], \end{aligned} \quad (38)$$

$$\begin{aligned} P(t, i_- | j_-) &= \frac{\gamma_i^-}{C_{jj}} \frac{e^{-\Gamma t}}{\mathbb{Z}} \mathbb{D}\{[(e^{-M})_{jj} - (e^{Qt} \mathcal{T} e^{-Qt} e^{-M})_{jj}] \mathcal{T}_{ii} \\ &- (\mathcal{T} e^{Q^\dagger t})_{ij} (e^{Qt} \mathcal{T})_{ji}\}, \end{aligned} \quad (39)$$

$$\begin{aligned} P(t, i_+ | j_-) &= \frac{\gamma_i^+}{C_{jj}} \frac{e^{-\Gamma t}}{\mathbb{Z}} \mathbb{D}\{[(e^{-M})_{jj} \\ &- (e^{Qt} \mathcal{T} e^{-Qt} e^{-M})_{jj}] (1 - \mathcal{T}_{ii}) \\ &+ (\mathcal{T} e^{Q^\dagger t})_{ij} (e^{Qt} \mathcal{T})_{ji}\}, \end{aligned} \quad (40)$$

where

$$\mathbb{D} = \det(1 + e^{-Qt} e^{-M} e^{-Q^\dagger t}), \quad \mathcal{T} = (e^{Q^\dagger t} e^M e^{Qt} + 1)^{-1}. \quad (41)$$

In view of the fact that $e^M = (1 - C)/C$, we therefore see that everything is expressed in terms of the quantities C , W , F associated with the Lyapunov equation (29), which is nice.

Next we specialize these formulas to the case where the initial state is the vacuum, $C_{\text{vac}} = 0$. It is prudent to first assume C is proportional to the identity, $C = \lambda$, and then take $\lambda \rightarrow 0$. In light of Eq. (28), we have that $e^M = (1 - \lambda)/\lambda$, so that in the limit $\lambda \rightarrow 0$ we get $Z = \mathbb{D} = 1$, and $C_{jj} = 0$. Moreover, $\mathcal{T} = \frac{\lambda}{1-\lambda} e^{-Qt} e^{-Q^\dagger t}$. Terms containing products of

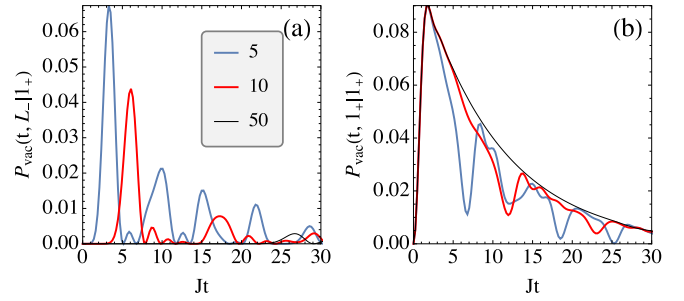


FIG. 2. Waiting time distributions (a) $P_{\text{vac}}(t, L_- | 1_+)$ and (b) $P_{\text{vac}}(t, 1_+ | 1_+)$, starting in the vacuum. Each curve is for a different system size $L = 5, 10, 50$. Parameters: $V = J = 1$, $\gamma_1 = \gamma_L = 0.1J$, $f_1 = 1$, $f_L = 0$.

e^M and \mathcal{T} will thus be of order 1, while terms containing only \mathcal{T} will vanish. Equations (37) and (38) thus reduce to

$$P(t, i_- | j_+) = \gamma_i^- e^{-\Gamma t} (e^{-Qt})_{ij} (e^{-Q^\dagger t})_{ji}, \quad (42)$$

$$P(t, i_+ | j_+) = \gamma_i^+ e^{-\Gamma t} [(e^{-Q^\dagger t} e^{-Qt})_{jj} - (e^{-Qt})_{ij} (e^{-Q^\dagger t})_{ji}]. \quad (43)$$

The other two WTDs, Eqs. (39) and (40), vanish in this case because $c_j | 0 \rangle \langle 0 | c_j^\dagger \equiv 0$.

V. EXAMPLE: TIGHT-BINDING MODEL

As an application, we consider a tight-binding model with Hamiltonian

$$H = - \sum_{i=1}^L V c_i^\dagger c_i - J \sum_{i=1}^{L-1} (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i). \quad (44)$$

This is a prototypical example of ballistic transport [59–61]. We henceforth fix $V = J = 1$, $\gamma_1 = \gamma_L$, $f_1 = 1$, and $f_L = 0$. This means that excitations can only be *injected* in site 1 or *collected* on site L . This reduces the problem to four WTDs, $P(t, L_- | 1_+)$, $P(t, 1_+ | 1_+)$, $P(t, 1_+ | L_-)$, $P(t, L_- | L_-)$. Due to the symmetry $\gamma_1 = \gamma_L$, the first two equal the last two. Hence, we have to focus only on $P(t, L_- | 1_+)$ and $P(t, 1_+ | 1_+)$. It is also important to distinguish the fundamental physical difference between these two distributions. Namely, $P(t, 1_+ | 1_+)$ is a local quantity, associated with clicks on the same site, while $P(t, L_- | 1_+)$ is nonlocal, describing events at spatially distant points.

In analyzing these WTDs, we start by considering the case where the system is initially in the vacuum. The reason is that this more closely resembles standard unitary transport protocols, where a wave packet is inserted in an empty chain, and one watches how it propagates (cf. Ref. [46]). The corresponding WTDs are shown in Fig. 2. The most familiar scenario is that of Fig. 2(a): an excitation is created on the left and then propagates with time. As can be seen, the resulting WTD is initially zero since it takes a finite amount of time for the excitation to travel from one site to the other. It then presents a series of peaks, characteristic of quantum coherent processes. The first peak is the primary absorption, where the excitation leaves the chain (and hence a click is detected). The other peaks are secondary processes, related to the wavelike

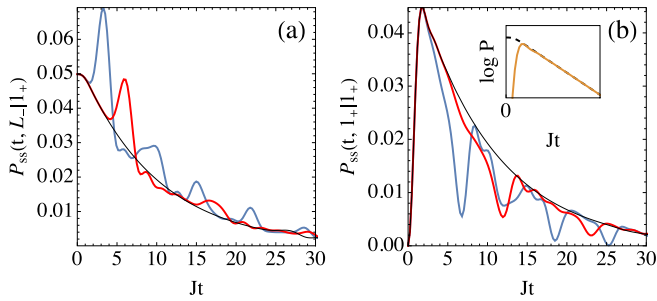


FIG. 3. Similar to Fig. 2, but for the system starting in the steady state. The inset is a log-scale plot of $P_{ss}(t, L_-|1_+)$ (black dashed) and $P_{ss}(t, 1_+|1_+)$ (orange) for $L = 50$.

nature of the particle's propagation in the chain, and the fact that, for finite sizes, the wave packets may move back and forth multiple times within the chain, until they are eventually removed. As L increases the position of the peaks tends to be pushed to longer times, which was found from numerics to scale as $t_{\text{peak}} \propto L$, exactly as expected for ballistic transport. Moreover, the relative magnitudes of the peaks also diminish [the curve for $L = 50$ in Fig. 2(a) is only barely visible, around $Jt \sim 27$]. The reason why this happens is simply due to the way WTDs are normalized, as will be discussed further in Fig. 4.

Conversely, $P(t, 1_+|1_+)$, shown in Fig. 2(b), is not associated with transport. Instead, it describes the waiting times between consecutive firings of the same channel. It is thus zero when $t = 0$, but then rapidly increases. The peak, which occurs at $t \sim 2$, represents the most likely waiting time between two consecutive jumps. When L is small, the distribution presents a series of oscillations, associated with the confinement of the ejected excitation in a finite-size chain. But as L gets large, the distribution—and hence the spacing between firing times—quickly becomes independent of L . In fact, for large sizes $P(t, 1_+|1_+)$ essentially follows an exponential distribution, with a characteristic time $1/\gamma$, dictated precisely by the Lindblad coupling strength.

Still concerning $P(t, 1_+|1_+)$, Fig. 2(b), it is possible to draw an analogy with queuing theory—i.e., the description of customers arriving in a queue. At any given time, the environment is sending multiple excitations to the system. Precisely

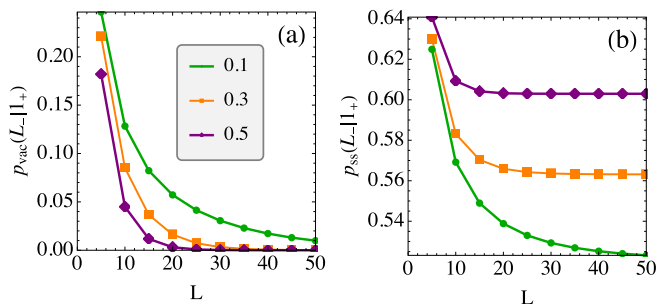


FIG. 4. Detection probabilities for the right bath $p(L_-|1_+)$, as a function of the system size L . (a) Vacuum, (b) steady state. Each curve is for a different value of γ . Other parameters are the same as in Fig. 2.

how it does that is not information that is present in the master equation, only in the microscopic model of the system-environment interactions. As far as the master equation is concerned, however, all that matters is how many of those excitations actually enter the system. In queuing theory, this would be associated with the phenomenon of *balking*, which is when a customer arrives at the line, but decides not to enter it [62]. The excitations that enter the system are those that did not balk. Except for finite-size effects, one expects that balking should be associated mostly with the environment, as well as the system-environment *boundary* (i.e., site 1). A related but different concept is *reneging*, which is when a customer enters a line but decides to leave after some time. This would be associated with the WTD $P(t, L_-|1_+)$, which will in general depend on the whole chain. In this example, however, this effect is zero since we set $f_1 = 1$.

In Fig. 3 we show similar results, but now for the system starting in the steady state. Interestingly, in this case $P(t, 1_+|1_+)$ is practically unaltered. This again corroborates the idea that $P(t, 1_+|1_+)$ is ultimately a property of site 1 and the environment. Conversely, the behavior of $P(t, L_-|1_+)$ in Fig. 3(a) is entirely different. First, it is maximal at $t = 0$. This occurs because, unlike the vacuum case of Fig. 2, the system now already has plenty of other excitations, so that a click on the left bath is not a requirement for observing a click on the right one. In fact, one can see in Fig. 3(a) the same peaks of Fig. 2(a), except that they are enveloped by a monotonically decaying distribution. When the size of the chain increases, the latter are rapidly suppressed, and $P(t, L_-|1_+)$ tends to a simple exponential decay (using larger values of γ also have the tendency to suppress the oscillations). In fact, the inset in Fig. 3(b) shows a log-scale plot of both distributions for $L = 50$. This makes it evident that, except for small deviations at early times, the distributions are essentially given by a single exponential $P \sim e^{-t/\tau}$, with $\tau = 1/\gamma$.

The relative frequency with which the jump $1_+ \rightarrow L_-$ occurs is given by $p(L_-|1_+)$, Eq. (7). This is presented in Fig. 4, as a function of L , for both steady state and vacuum. When the system starts in the vacuum [Fig. 4(a)] $p(L_-|1_+)$ is exponentially suppressed with increasing L , for all values of γ . This happens because, when the chain is large, it takes a long time for an excitation to be transported to the other side. In contrast, $1_+ \rightarrow 1_+$ refers to two events at the same site, and is thus independent of L . This explains why the curves in Fig. 2(a) are suppressed with increasing L . Conversely, if the system starts in the steady state [Fig. 4(b)], the probabilities tend to a finite value when $L \rightarrow \infty$. This means that the frequencies with which $L_-|1_+$ and $1_+|1_+$ occur remain comparable in magnitude, even in the thermodynamic limit.

Finally, we turn to the net activity time distribution (NATD) $P(t)$ in Eq. (11), which describes the waiting time between any two events. The results are shown in Fig. 5. Due to the symmetry $P(L_-|L_-) = P(1_+|1_+)$ and $P(1_+|L_-) = P(L_-|1_+)$ of the present choice of parameters, it reduces in this case to $P(t) = P(t, L_-|1_+) + P(t, 1_+|L_-)$. Thus, $P(t)$ behaves as a mixture of the two distributions in Fig. 3, serving as a good summary of the typical activities happening in the system. In the inset we show the mean and standard deviation as a function of L . Quite remarkably, even though the distributions themselves depend sensibly on L (main plot), the mean $E(T)$

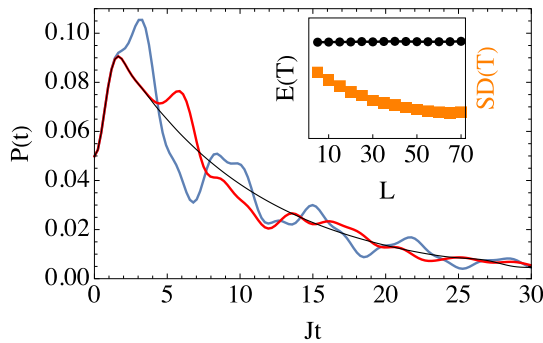


FIG. 5. Activity time distribution, Eq. (11), as a function of time for the same values of L (and other parameters) as in Fig. 2. The inset shows the mean $E(T)$ and standard deviation $SD(T) = \sqrt{\text{var}(T)}$ as a function of the system size L .

is absolutely flat. The standard deviation, on other hand, depends weakly on L and is also very close to the mean.

To shed further light on the NATD, we look at the case $L = 2$, where it can actually be computed analytically. The result is

$$P(t) = \frac{\gamma}{2(\gamma^2 - 4J^2)} e^{-\gamma t} [\gamma^2 - 8J^2 + 4J^2 \cos(t\sqrt{4J^2 - \gamma^2})]. \quad (45)$$

The average time between clicks is thus

$$E(T) = \frac{1}{\gamma} + \frac{\gamma}{4J^2}. \quad (46)$$

The first contribution is associated solely with the stochastic nature of the baths, which generates a typical exponential distribution with rate γ ; it is therefore consistent with the approximate exponential behavior shown in the inset of Fig. 3. The second term, on the other hand, is associated with the coherent hopping J . Hence, it yields a correction to the average waiting time due to the presence of the system. Since $E(T)$ is found to be independent of L , we therefore conclude that the contributions from the hopping persist even in the thermodynamic limit.

VI. SIGNIFICANCE AND APPLICATIONS

The goal of this paper was to provide closed expressions for the waiting time distribution in free fermion chains, written solely in terms of the $L \times L$ matrices characterizing the problem. We believe this is of value for three reasons. First, WTDs represent a somewhat unexplored aspect of full counting statistics, with rich physics. For example, in the simple tight-binding model studied in this paper we have shown how WTDs can clearly capture dynamical aspects of transport through boundary-driven chains. This includes, in particular, insights on how the chain size L influences the time between absorption/emission events, the relative probabilities, and the overall dynamical activities within the chain. Second, WTDs are usually difficult to compute, especially for many-body systems. Being able to study them for arbitrary chain sizes is thus extremely valuable. For instance, they can be used to benchmark simulations for interacting systems using, e.g., tensor networks [63–68]. The third reason why these results should be of value is that, although free fermions are

sometimes regarded as not so interesting (e.g., when compared to interacting models), there has recently been a surge of interest in exotic fermionic chains, such as those exhibiting quasiperiodic behavior. In fact, as illustrated in Refs. [69–75], quasiperiodic noninteracting chains can exhibit any kind of transport, not only ballistic [70,72,76,77]. A study of WTDs for these models will be the subject of future work.

As for other future extensions, it would be interesting to extend this to Gaussian bosonic states, as they may have applications in, e.g., optomechanical systems [78,79]. Another extension is to include WTDs in which not all channels are monitored. In this case it is not possible to decompose the free evolution as $\mathcal{L}_0 = -i(H_e \rho - \rho H_e^\dagger)$. Notwithstanding, the resulting Liouvillian is still quadratic, so it should be possible to derive the WTD using, e.g., third quantization [80], or a similar vectorization method [32].

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APPENDIX: PROOF OF EQUATIONS (18) AND (21)

1. Proof of Equation (18)

We will prove Eqs. (18) and (21) using Eq. (17). For simplicity, it will be assumed that $i' = i$ and $j' = j$, but the proof when they are different is quite similar. Due to the fermionic algebra, it holds that for any constant α [81],

$$e^{\alpha c_i^\dagger c_i} = 1 + (e^\alpha - 1)c_i^\dagger c_i, \quad c_i^\dagger c_i = \frac{e^{\alpha c_i^\dagger c_i} - 1}{e^\alpha - 1}. \quad (A1)$$

With this, we can write

$$\text{tr}\{c_i^\dagger c_i e^X\} = \frac{1}{e^\alpha - 1} [\text{tr}(e^{\alpha c_i^\dagger c_i} e^X) - \text{tr}(e^X e^\alpha)]. \quad (A2)$$

Equation (17) is now applicable to each term individually. Here I have assumed only a single exponential e^X . But since Eq. (17) holds for an arbitrary number of quadratic forms, the results can be readily extended. Of course, the result must be independent of α , so this constant must eventually factor out. In the first term of (A2), the quantity $e^{\alpha c_i^\dagger c_i}$ is a quadratic form, with a matrix $R_{ii} = |i\rangle\langle i|$, i.e., with all elements being zero except the entry (i, i) . Here we also introduced for convenience the notation $|i\rangle$ to represent single-particle kets (from a basis of L elements). Thus we can write

$$\text{tr}\{c_i^\dagger c_i e^X\} = \frac{1}{e^\alpha - 1} [\det(1 + e^{\alpha R_{ii}} e^X) - \det(1 + e^X)]. \quad (A3)$$

However, we also have that $e^{\alpha R_{ii}} = 1 + (e^\alpha - 1)|i\rangle\langle i|$, so the first term is written as

$$\det(1 + e^{\alpha R_{ii}} e^X) = \det[1 + e^X + (e^\alpha - 1)|i\rangle\langle i|e^X]. \quad (A4)$$

Next we use the Sylvester determinant identity, which states that

$$\det(A + |\psi\rangle\langle\phi|) = \det(A)(1 + \langle\phi|A^{-1}|\psi\rangle). \quad (A5)$$

This yields

$$\det(1 + e^{\alpha Z} e^X) = \det(1 + e^X)[1 + (e^\alpha - 1)\langle i|e^X(1 + e^X)^{-1}|i\rangle]. \quad (\text{A6})$$

Plugging this in Eq. (A2) finally leads to a cancellation of the factor $e^\alpha - 1$, as expected. The only thing left is

$$\text{tr}\{c_i^\dagger c_i e^X\} = \det(1 + e^X)\langle i|e^X(1 + e^X)^{-1}|i\rangle. \quad (\text{A7})$$

This is almost Eq. (18). To finish, we extend it to multiple matrices, X, Y, Z , leading to

$$\text{tr}\{c_i^\dagger c_i e^X e^Y e^Z\} = \det(1 + e^X e^Y e^Z)\langle i|e^X e^Y e^Z(1 + e^X e^Y e^Z)^{-1}|i\rangle. \quad (\text{A8})$$

The form shown in Eq. (18) is finally obtained by writing, e.g., $e^X(1 + e^X)^{-1} = (e^{-X} + 1)^{-1}$.

2. Proof of Equation (21)

Next, we turn to Eq. (21), which is harder. We again use the factorization in (A1) to write

$$\text{tr}\{c_i^\dagger c_i e^X c_j^\dagger c_j e^Y e^Z\} = \frac{1}{e^\alpha - 1} \{\text{tr}[c_i^\dagger c_i e^X e^{\alpha c_j^\dagger c_j} e^Y e^Z] - \text{tr}[c_i^\dagger c_i e^X e^Y e^Z]\}. \quad (\text{A9})$$

Both terms can now be computed from Eq. (18). The last is in fact *exactly* Eq. (18). For simplicity, we are going to define

$$\mathbb{D} = \det(1 + e^X e^Y e^Z), \quad \mathcal{T} = (e^{-Z} e^{-Y} e^{-X} + 1)^{-1} = e^X e^Y e^Z (1 + e^X e^Y e^Z)^{-1}. \quad (\text{A10})$$

Then the last term in (A9) becomes

$$\text{tr}[c_i^\dagger c_i e^X e^Y e^Z] = \mathbb{D} \mathcal{T}_{ii}. \quad (\text{A11})$$

Conversely, the first term reads

$$\text{tr}[c_i^\dagger c_i e^X e^{\alpha c_j^\dagger c_j} e^Y e^Z] = \det(1 + e^X e^{\alpha R_{jj}} e^Y e^Z) [e^{-Z} e^{-Y} e^{-\alpha R_{jj}} e^{-X} + 1]_{ii}^{-1}. \quad (\text{A12})$$

This formula still requires some working. We again write $e^{\alpha R_{jj}} = 1 + (e^\alpha - 1)|j\rangle\langle j|$. Using Sylvester's identity (A5), the part associated with the determinant can be written as

$$\begin{aligned} \det(1 + e^X e^{\alpha R_{jj}} e^Y e^Z) &= \det[1 + e^X e^Y e^Z + (e^\alpha - 1)e^X |j\rangle\langle j| e^Y e^Z] \\ &= \det(1 + e^X e^Y e^Z) \{1 + (e^\alpha - 1)\langle j|e^Y e^Z(1 + e^X e^Y e^Z)^{-1}e^X|j\rangle\} \\ &= \mathbb{D}[1 + (e^\alpha - 1)(e^{-X} \mathcal{T} e^X)_{jj}], \end{aligned} \quad (\text{A13})$$

where Eq. (A10) was used in the last line.

To treat the second term in Eq. (A12), we first write it as

$$[e^{-Z} e^{-Y} e^{-\alpha R_{jj}} e^{-X} + 1]_{ii}^{-1} = [1 + e^{-Z} e^{-Y} e^{-X} + (e^{-\alpha} - 1)e^{-Z} e^{-Y} |j\rangle\langle j| e^{-X}]^{-1}, \quad (\text{A14})$$

and then use the Sherman-Morrison formula, which states that

$$(A + |\psi\rangle\langle\phi|)^{-1} = A^{-1} - \frac{A^{-1}|\psi\rangle\langle\phi|A^{-1}}{1 + \langle\phi|A^{-1}|\psi\rangle}. \quad (\text{A15})$$

In our case $A = e^{-Z} e^{-Y} e^{-X} + 1$ so $A^{-1} \equiv \mathcal{T}$ Eq. (A10). As a result, we get that the i, i element of this will be

$$[e^{-Z} e^{-Y} e^{-\alpha R_{jj}} e^{-X} + 1]_{ii}^{-1} = \mathcal{T}_{ii} - (e^{-\alpha} - 1) \frac{(\mathcal{T} e^{-Z} e^{-Y})_{ij} (e^{-X} \mathcal{T})_{ji}}{1 + (e^{-\alpha} - 1)(e^{-X} \mathcal{T} e^{-Z} e^{-Y})_{jj}}. \quad (\text{A16})$$

Inserting Eqs. (A13) and (A16) into Eq. (A12) leads to

$$\begin{aligned} \text{tr}[c_i^\dagger c_i e^X e^{\alpha c_j^\dagger c_j} e^Y e^Z] &= \mathbb{D}[1 + (e^\alpha - 1)(e^{-X} \mathcal{T} e^X)_{jj}] \left[\mathcal{T}_{ii} - (e^{-\alpha} - 1) \frac{(\mathcal{T} e^{-Z} e^{-Y})_{ij} (e^{-X} \mathcal{T})_{ji}}{1 + (e^{-\alpha} - 1)(e^{-X} \mathcal{T} e^{-Z} e^{-Y})_{jj}} \right] \\ &= \mathbb{D} \left[\mathcal{T}_{ii} + (e^\alpha - 1)(e^{-X} \mathcal{T} e^X)_{jj} \mathcal{T}_{ii} - (e^{-\alpha} - 1) \frac{(\mathcal{T} e^{-Z} e^{-Y})_{ij} (e^{-X} \mathcal{T})_{ji}}{1 + (e^{-\alpha} - 1)(e^{-X} \mathcal{T} e^{-Z} e^{-Y})_{jj}} \right. \\ &\quad \left. - (e^\alpha - 1)(e^{-\alpha} - 1) \frac{(e^{-X} \mathcal{T} e^X)_{jj} (\mathcal{T} e^{-Z} e^{-Y})_{ij} (e^{-X} \mathcal{T})_{ji}}{1 + (e^{-\alpha} - 1)[e^{-X} \mathcal{T} e^{-Z} e^{-Y}]_{jj}} \right]. \end{aligned}$$

Finally, we insert this in Eq. (A9). In light of Eq. (A11), this simply means we cancel out the term $\mathbb{D} \mathcal{T}_{ii}$. Hence, we are only left with

$$\text{tr}\{c_i^\dagger c_i e^X c_j^\dagger c_j e^Y e^Z\} = \mathbb{D} g \left\{ (e^{-X} \mathcal{T} e^X)_{jj} \mathcal{T}_{ii} + (\mathcal{T} e^{-Z} e^{-Y})_{ij} (e^{-X} \mathcal{T})_{ji} \frac{(e^\alpha - 1)(e^{-X} \mathcal{T} e^X)_{jj} + 1}{e^\alpha [1 - (e^{-X} \mathcal{T} e^{-Z} e^{-Y})_{jj}] + (e^{-X} \mathcal{T} e^{-Z} e^{-Y})_{jj}} \right\}.$$

Using the structure of \mathcal{T} in Eq. (A10), one may verify that the matrix appearing in the denominator is actually related to the matrix $e^{-X}\mathcal{T}e^X$ according to

$$e^{-X}\mathcal{T}e^{-Z}e^{-Y} = 1 - e^{-X}\mathcal{T}e^X. \quad (\text{A17})$$

This allows for the expression to be simplified, finally leading to a cancellation of the factor of e^α (as it must, since α is arbitrary). As a result, we are left only with

$$\text{tr}\{c_i^\dagger c_i e^X c_j^\dagger c_j e^Y e^Z\} = \mathbb{D}\{(e^{-X}\mathcal{T}e^X)_{jj}\mathcal{T}_{ii} + (\mathcal{T}e^{-Z}e^{-Y})_{ij}(e^{-X}\mathcal{T})_{ji}\}. \quad (\text{A18})$$

The formula in the case when $i' \neq i$ and $j' \neq j$ is similar, and reads

$$\text{tr}\{c_i^\dagger c_{i'} e^X c_j^\dagger c_{j'} e^Y e^Z\} = \mathbb{D}\{(e^{-X}\mathcal{T}e^X)_{j'j}\mathcal{T}_{i'i} + (\mathcal{T}e^{-Z}e^{-Y})_{i'j}(e^{-X}\mathcal{T})_{j'i}\}. \quad (\text{A19})$$

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