Quantum transport of non-commuting charges
Overview

• The laws of thermodynamics in the quantum regime.
• Thermodynamics beyond thermal.
• Transport of non-commuting charges.
• Thermosqueezing effects.

References:


The laws of thermodynamics in the quantum regime

Scenario

- For simplicity: unitary interaction between two systems, AB.
  \[ \rho'_{AB} = U(\rho_A \otimes \rho_B)U^\dagger \]

- \( U \) = generic unitary, produced by some means.
  - Could be \( U = \exp\{-it(H_A + H_B + H_I)\} \)
  - Or from time-dependent \( H(t) \) (“work protocol”).
  - All that matters is that some protocol generates the said \( U \).
    - More general.

1st law

- There is some arbitrariness in how to define heat and work (big debate!).
- My view: there is no unique way of doing so (has to be operational).
- Here, I will adopt the following convention:
  - Work = mismatch between changes in local energies of A and B.
  - (assumes initial and final Hamiltonians are the same; not necessary).

Strict energy conservation (SEC) condition:

$$ W = \Delta H_A + \Delta H_B = \text{tr} \left\{ (H_A + H_B) (\rho_A' \otimes \rho_B - \rho_A \otimes \rho_B) \right\} $$

$$ [U, H_A + H_B] = 0 \rightarrow W \equiv 0 $$

Example: two interacting qubits

$$ U = e^{-iHt} \text{ with } $$

$$ H = \frac{\omega_A}{2} \sigma_z^A + \frac{\omega_B}{2} \sigma_z^B + g (\sigma_+^A \sigma_-^B + \sigma_-^A \sigma_+^B) $$

Works when $\omega_A = \omega_B$: resonant energy exchange.

Note: $[U, H_{A(B)}] \neq 0$

(only commutes with the sum)

Example: light-matter interaction

$$ H = \frac{\omega}{2} \sigma_z + \sum_k \left\{ \Omega_k b_k^\dagger b_k + g_k (\sigma_+ b_k + \sigma_- b_k^\dagger) \right\} $$

Generally $[U, H_A + H_B] \neq 0$.

Problem is in the side-bands, $\Omega_k \neq \omega$.

Effect very small at weak coupling

$$ [U, H_A + H_B] \simeq 0 $$

“SEC = weak coupling a priori”

Example (extreme):

B uncoupled from A.

$$ U = U_A \otimes I_B, $$

so we only require

$$ [U, H_A] = 0 $$

Work in this case is simply

$$ W = \int_0^t dt' \langle \frac{\partial H_A}{\partial t} \rangle $$
2nd law - quantifies irreversibility

- Entropy production $\Sigma \geq 0$. Shape of sigma depends on process:
  - $\Sigma = \Delta S_S - \beta \Delta Q$
  - or $\Sigma = \beta (W - \Delta F)$

- Quantum: If we have full control, then the process is reversible.

- (same is true for a gas with $10^{23}$ particles)

  Irreversibility is an emergent property.

- Operational approach (à la Jensen): irreversible is that which we do not have access to.

\[ \Sigma = I(A:B) + D(\rho_A' || \rho_A) + D(\rho_B' || \rho_B) \geq 0 \]

- No access to correlations
- No access to internal states

The thermal scenario

Take $B = \text{bath}$.

\[
\Sigma = I(A:B) + D(\rho_B' || \rho_B) = \Delta S_S + \text{tr}\{(\rho_B - \rho_B')\ln \rho_B\}
\]

If $\rho_B = e^{-\beta_B H_B / Z_B}$ we recover the Clausius inequality

\[ \Sigma = \Delta S_S - \beta_B \Delta H_B \geq 0 \]

[Esposito, Lindenberg, Van den Broek, NJP 12, 013013 (2010).]
Collisional models

- Stroboscopic dynamics from combining multiple interactions (collisions).

\[ \rho_{S}^{n+1} = \text{tr}_A \{ U (\rho_{S}^n \otimes \rho_A) U^\dagger \} \]

- All of the previous results continue to hold, but now referring to each collision.

- Matches well with Boltzmann’s Stosszahlansatz.

- Full control over the “bath”: crucial for thermodynamics.

- Can also be used to derive quantum master equations (infinitesimal collision times).

F. Ciccarello, S. Lorenzo, V. Giovannetti, G. Massimo Palma
“Quantum collision models: open system dynamics from repeated interactions”,
arXiv:2106.11974
Thermodynamics beyond thermal

\[ \beta_A - \beta_B \geq \Delta I(A:B) \]

**Motivation**

• Thermodynamics offers simple rules for doing useful things.

• Are there (restricted) scenarios, beyond the thermal paradigm, where such simple rules also exist?

\[ (\beta_A - \beta_B)Q \geq \Delta I(A:B) \]

**Nature Communications** 10, 2456 (2019).


**Weakly coherent thermal ancillas**

\[ \rho_A = \rho_A^{th} + \sqrt{\lambda} \chi_A, \quad \text{tr}(\chi_A) = 0 \]

Strict energy preservation:

\[ [U, H_S + H_A] = 0 \]

Means there is no work involved.

But, due to coherence, the heat can be split as

\[ \Delta H_A = \Delta H_A^{th} + \Delta H_A^{\chi} \]

We show that when \( \lambda \ll 1 \)

\[ \Sigma = \beta(\Delta H_A^{\chi} - \Delta F) \geq 0 \] (work like).

Moreover,

\[ \beta \Delta H_A^{\chi} \geq - \Delta C_A \]

Relative entropy of coherence:

\[ C_A = S(\rho_A^d) - S(\rho_A) \]
Transport of non-commuting charges

Onsager theory

- Gradients in temperature and chemical potential lead to currents of particles and energy.

- Fluxes vs. forces (affinities): $J_k$ vs. $\delta \lambda_k$

  $$J_N \leftrightarrow \delta \beta = \beta_L \mu_L - \beta_R \mu_R$$

  and

  $$J_E \leftrightarrow -\delta \beta = -\beta_L \mu_L + \beta_R \mu_R$$

- Entropy production (rate): $\dot{\Sigma} = \sum_k \delta \lambda_k J_k$ (fluxes $\times$ forces)

- At linear response, things become clearer:

  **Awesome result**
  - $L$ is symmetric: Peltier & Seebeck are equal.
  - And positive semi-definite:

    $$\dot{\Sigma} = \sum_{kj} L_{kj} \delta \lambda_k \delta \lambda_j \geq 0$$

  $L_{11}$: Fick's law of diffusion
  Particles flow due to gradient of concentration.

  $L_{22}$: Fourier's law
  Heat flows due to gradient of temperature.

  $L_{12}$: Seebeck effect
  Gradient of temperature generates a flow of particles/electrons.

  $L_{21}$: Peltier effect
  Gradient of concentration generates heat flow.

Thermoelectric plates in our laptops.
Transport of non-commuting charges

- At the quantum level, energy and particle number are associated to commuting observables, $H$ and $N$,

$$[H, N] = 0$$

- This means they can be simultaneously measured.

- What if we try to transport charges which do not commute?

- Our approach: Collisional model between two baths, where units (ancillas) are prepared in a Generalized Gibbs Ensemble

$$\pi_\lambda = \frac{e^{-\sum_k \lambda_k Q_k}}{Z_\lambda} \quad \text{where } Q_k = \text{set of thermodynamic charges, } \lambda_k = \text{conjugated affinity.}$$

  - e.g. grand-canonical: $\pi = \frac{e^{-\beta(H-\mu N)}}{Z}$

- When can we talk about transport?

  - Transport means the thing leaving one place enters the other.

  - Condition for strict charge conservation (SCC):

$$[U, Q^A_k + Q^B_k] = 0, \quad \forall k$$

  - In this case, we can properly define currents

$$J_k = \text{tr} \left\{ Q^A_k (\rho'_{AB} - \rho^A_{\lambda_A} \otimes \rho^B_{\lambda_B}) \right\}$$
Transport of non-commuting charges

In our work, we focused on linear response:

The currents may thus be expanded as

where $\lambda$ is the Onsager matrix.

Main result

If the charges $Q_k$ and the dynamics are time-reversal invariant, then the Onsager matrix can be written as

where

is the $y$-covariance, with $\rho$ being the equilibrium state.

Consequence

The entropy production can be written as

\[
\Sigma = \frac{1}{2} \int_0^1 dy \, \text{cov}_y(D, D), \quad D = \sum_k \delta \lambda_k (\tilde{Q}_k^A - Q_k^A)
\]

This can be further split as

\[
\Sigma = \frac{1}{2} \text{var}(D) - \frac{1}{2} \int_0^1 dy \, I_y(\pi, D)
\]

where $I_y$ is the Wigner-Yanase-Dyson skew information (a quantifier of coherence)

\[
I_y(\pi, D) = -\frac{1}{2} \text{tr} \left( [\pi^y, D][\pi^{1-y}, D] \right) \geq 0
\]

Reduction in the entropy production due to quantum coherence.


\[\rho_{AB}' = U(\rho_A^\lambda \otimes \rho_B^\lambda) U^\dagger\]

Main result

If the charges $Q_k$ and the dynamics are time-reversal invariant, then the Onsager matrix can be written as

\[
L_{kj} = \frac{1}{2} \int_0^1 dy \, \text{cov}_y(\tilde{Q}_k^A - Q_k^A, \tilde{Q}_j^A - Q_j^A)
\]

where $\tilde{Q} = U^\dagger Q U$ and

\[
\text{cov}_y(A, B) = \text{tr} \left( A \pi^y B \pi^{1-y} \right) - \text{tr}(A \pi) \text{tr}(B \pi)
\]

is the $y$-covariance, with $\pi = \rho_A^\lambda \otimes \rho_B^\lambda$ being the equilibrium state.
Transport of non-commuting charges

Efficiency of heat engines coupled to nonequilibrium reservoirs
Obinna Abah and Eric Lutz
Published 2 May 2014 · Copyright © EPLA, 2014
EPL (Europhysics Letters), Volume 106, Number 2
Citation Obinna Abah and Eric Lutz 2014 EPL 106 20001

Entropy production and thermodynamic power of the squeezed thermal reservoir
Gonzalo Manzano, Fernando Galve, Roberta Zambrini, and Juan M. R. Parrondo
Phys. Rev. E 93, 052120 – Published 10 May 2016

Squeezed Thermal Reservoirs as a Resource for a Nanomechanical Engine beyond the Carnot Limit
Jan Klaers, Stefan Faelt, Atac Imamoglu, and Emre Togan
Phys. Rev. X 7, 031044 – Published 13 September 2017
Thermosqueezing effects

• Bosonic mode in a squeezed thermal state (non-Abelian GGE)

• Two charges, \( (\text{energy}) \) and \( (\text{asymmetry}) \).

• Depending on \( \rho \), we can construct a total of 3 independent charges.

• For simplicity, we focus only on \( \rho \) and \( A \).

\[
\rho = \frac{1}{Z} \exp\left\{ -\beta H - \beta \mu A \right\}
\]

\[
H = \omega^2 \left( p^2 + x^2 \right)
\]

\[
A = \omega^2 \left( p^2 - x^2 \right)
\]

\[
Q_1 = H = \omega^2 (p^2 + x^2), \quad Q_2 = A = \omega^2 (p^2 - x^2)
\]

\[
Q_3 = \omega^2 \{ x, p \}
\]

\[
\rho' = U (\rho \lambda) \otimes (\rho' \lambda) U^\dagger
\]

\[
U = \exp\{ -g\tau (a_1^\dagger a_2 - a_2^\dagger a_1) \}
\]

Actually the only one which is also Gaussian (quadratic).

\[
\bar{n} = (e^{\beta \omega} - 1)^{-1}, \quad f_\tau = \omega^2 \sin^2 (g\tau), \quad \alpha = \beta \omega \sqrt{1 - \mu^2}
\]

Transport coefficients

Thermal conductance: \( \kappa = -\beta^2 L_{22} \)

Squeezing conductance: \( G = -\beta L_{22} \)

Entropy production/dissipated heat reads

\[
\dot{Q}_{\text{diss}} = \frac{\Sigma}{\beta} = \kappa \delta T^2 / T + J_A G
\]

New Joule-like heating term due to squeezing.

\[
J_Q = L_{QQ} \delta_\beta - L_{QA} \beta \delta_\mu
\]

\[
J_A = L_{AA} \delta_\beta - L_{AA} \beta \delta_\mu
\]

with

\[
L_{QQ} = f_\tau (1 - \mu^2) \bar{n} (\bar{n} + 1)
\]

\[
L_{QA} = L_{AQ} = f_\tau \mu \bar{n} (\bar{n} + 1)
\]

\[
L_{AA} = f_\tau (1 - \mu^2)^{-1} \left[ \mu \bar{n} (\bar{n} + 1) + \frac{\tanh \alpha}{\alpha} (\bar{n}^2 + \bar{n}/2 + 1/2) \right]
\]

Onsager matrix

Unitary which preserves both heat (\( J_Q = J_H - \mu J_A \)) and squeezing:

\[
U = \exp\{ -g\tau (a_1^\dagger a_2 - a_2^\dagger a_1) \}
\]

\[
\bar{n} = (e^{\beta \omega} - 1)^{-1}, \quad f_\tau = \omega^2 \sin^2 (g\tau), \quad \alpha = \beta \omega \sqrt{1 - \mu^2}
\]

Charge preserving Gaussian unitary

Unitary which preserves both energy and squeezing:

\[
U = \exp\{ -g\tau (a_1^\dagger a_2 - a_2^\dagger a_1) \}
\]

SU(1,1) algebra

The charges \( Q_1, Q_2, Q_3 \) form a non-Abelian group:

\[
\left[ Q_1, Q_2 \right] = 2i Q_3
\]

\[
\left[ Q_2, Q_3 \right] = -2i Q_1
\]

\[
\left[ Q_3, Q_1 \right] = 2i Q_2
\]

 SU(1,1) algebra
Cross coefficients

Thermopower, or Squeezing-Seebeck (Squeebeck) coefficient

\[ S = \frac{1}{T} \frac{L_{AQ}}{L_{AA}} \]

(flow of squeezing due to gradient of temperature)

Squeezing-Peltier (Squeetier (?)) coefficient:

\[ \Pi = \frac{L_{AQ}}{L_{AA}} \]

(flow of heat due to gradient in squeezing)

The two are related by

\[ S = \frac{1}{T} \frac{L_{AQ}}{L_{AA}} \]

Entropy reduction

Recall that

\[ \Sigma = \frac{1}{2} \text{var}(D) - \frac{1}{2} \int_0^1 dy \, I_y(\pi, D) \]

Define the entropy reduction due to non-commutativity

\[ \mathcal{R} = \frac{1}{2\Sigma} \int_0^1 dy \, I_y(\pi, D) \]

Classical case corresponds to \( \mathcal{R} = 0 \).
Conclusions and outlook

• Quantum mechanics opens up the way for performing transport of non-commuting charges.

• We put forth a framework suitable for describing this in the linear response regime.

GGE: $\pi_{\lambda} = \frac{e^{-\sum_k \lambda_k Q_k}}{Z_{\lambda}}$

Charges $Q_k$ and affinities $\lambda_k$.

$J_k = \sum_j L_{kj} \delta \lambda_j$

$\dot{\Sigma} = \sum_{kj} L_{kj} \delta \lambda_k \delta \lambda_j \geq 0$

Strict Charge Conservation

$[U, Q_k^A + Q_k^B] = 0, \quad \forall k$

Onsager matrix:

$L_{kj} = \frac{1}{2} \int_0^1 dy \text{cov}_y (\tilde{Q}_A^k - Q_k^A, \tilde{Q}_j^A - Q_j^A)$

Entropy production

$D = \sum_k \delta \lambda_k (\tilde{Q}_k^A - Q_k^A)$

$\Sigma = \frac{1}{2} \int_0^1 dy \text{cov}_y (D, D) = \frac{1}{2} \text{var}(D) - \frac{1}{2} \int_0^1 dy I_y(\pi, D)$

References:


Perspectives

• If the charges do not commute, how can we actually measure them?
  • Current fluctuations and Thermodynamic Uncertainty Relations.
• Concrete applications of thermosqueezing devices.

THANK YOU!