Informational steady-states in continuously monitored quantum systems

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Entropy Production in Continuously Measured Quantum Systems

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PHYSICAL REVIEW LETTERS **125,** 080601 (2020)

Editors' Suggestion

Experimental Assessment of Entropy Production in a Continuously Measured Mechanical Resonator

Massimiliano Rossi,[1,2] Luca Mancino,[3] Gabriel T. Landi,[4] Mauro Paternostro,[3] Albert Schliesser,[1,2] and Alessio Belenchia[3,*]

arXiv:2005.03429

Informational steady-states and conditional entropy production in continuously monitored systems

Gabriel T. Landi,[1,*] Mauro Paternostro,[2] and Alessio Belenchia[2]

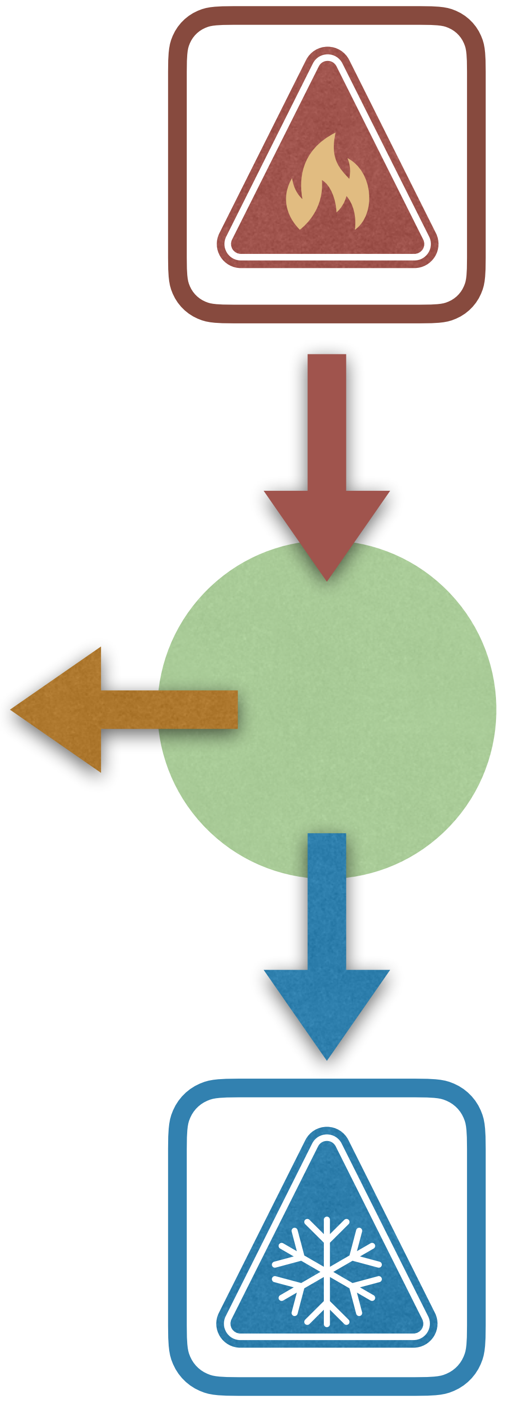
arXiv:2103.06247

THE SECOND LAW

- The 1st law puts heat and work on similar footing and says that, in principle, one can be interconverted into the other.

- For a system coupled to two baths, for instance, we have:

$$\frac{dU}{dt} = \dot{Q}_h + \dot{Q}_c + \dot{W}$$



- Not all such processes, however, are actually possible.
  - This is the purpose of the 2nd law.

GTL and M. Paternostro, "Irreversible entropy production, from quantum to classical", To appear in Review of Modern Physics. arXiv:2009.07668

- The 2nd law deals with entropy.
  - *Entropy, however, does not satisfy a continuity equation.*
- There can be a flow of entropy from the system to the environment, which is given by the famous Clausius expression $\dot{Q}/T$.
- But, in addition, there can also be some entropy which is spontaneously *produced* in the process. The entropy balance equation thus reads

$$\frac{dS}{dt} = \dot{\Sigma} + \frac{\dot{Q}_h}{T_h} + \frac{\dot{Q}_c}{T_c}$$

- The quantity $\dot{\Sigma}$ is called the **entropy production rate.**
- The second law can now be formulated mathematically by the statement

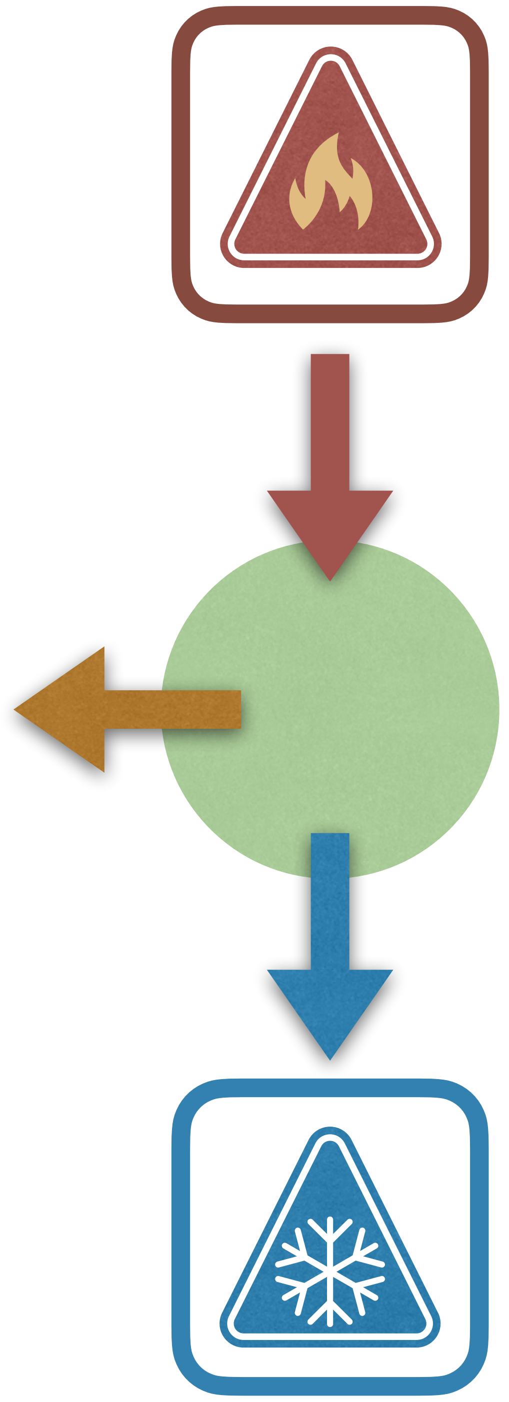
$$\dot{\Sigma} \geq 0$$

Why entropy production matters

- 1st and 2nd laws for a system coupled to two baths:

$$\frac{dU}{dt} = \dot{Q}_h + \dot{Q}_c + \dot{W} = 0$$

$$\frac{dS}{dt} = \dot{\Sigma} + \frac{\dot{Q}_h}{T_h} + \frac{\dot{Q}_c}{T_c} = 0$$



- The efficiency of the engine may then be written as

$$\eta = -\frac{\dot{W}}{\dot{Q}_h} = 1 + \frac{\dot{Q}_c}{\dot{Q}_h} = 1 - \frac{T_c}{T_h} - \frac{T_c}{\dot{Q}_h}\dot{\Sigma}$$

- *Entropy production is therefore the reason the efficiency is smaller than Carnot:*

$$\eta = \eta_C - \frac{T_c}{\dot{Q}_h}\dot{\Sigma}$$

Carnot's statement of the 2nd law

"The efficiency of a quasi-static or reversible Carnot cycle depends only on the temperatures of the two heat reservoirs, and is the same, whatever the working substance. A Carnot engine operated in this way is the most efficient possible heat engine using those two temperatures."

Flow of heat

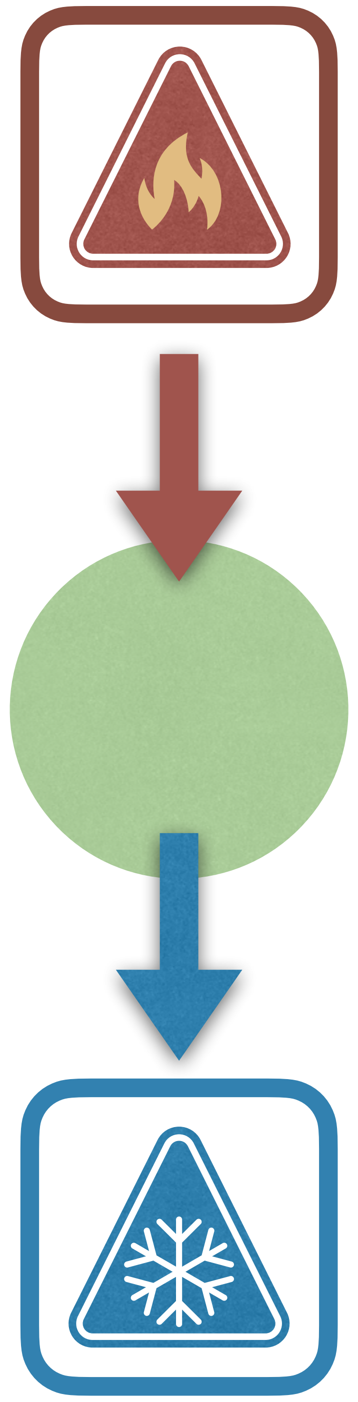
- The 2nd law reads

$$\dot{\Sigma} = -\frac{\dot{Q}_h}{T_h} - \frac{\dot{Q}_c}{T_c} \geq 0$$

- But if there is no work involved, $\dot{Q}_c = -\dot{Q}_h$

$$\therefore \qquad \dot{\Sigma} = \left(\frac{1}{T_c} - \frac{1}{T_h}\right)\dot{Q}_h \geq 0$$

- *Heat flows from hot to cold.*



Clausius’ statement of the 2nd law

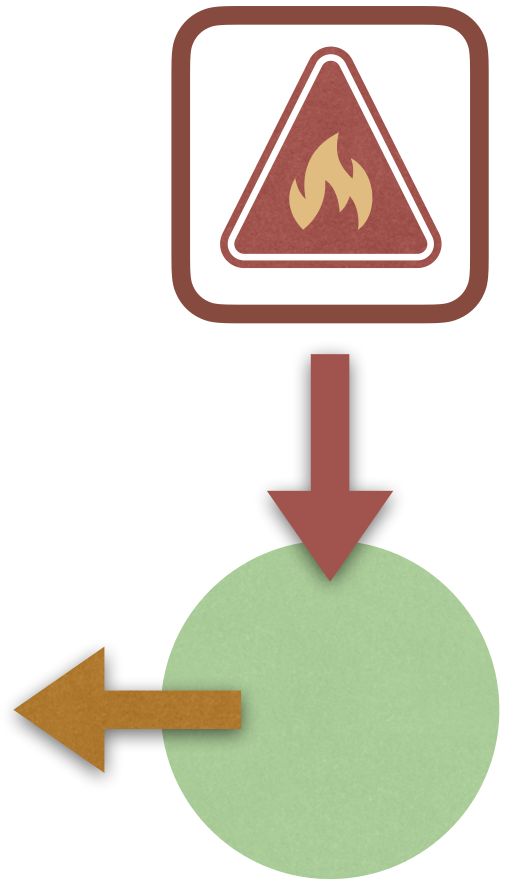
“Heat can never pass from a colder to a warmer body without some other change, connected therewith, occurring at the same time.”

Work from a single bath

- Finally, suppose there is only one bath present:

$$\dot{W} = -\dot{Q}_h$$

$$\dot{\Sigma} = -\frac{\dot{Q}_h}{T_h} = \frac{\dot{W}}{T_h} \geq 0$$



- Positive work (in my definition) means an external agent is *doing* work on the system.

Kelvin-Planck statement of the 2nd law

"It is impossible to devise a cyclically operating device, the sole effect of which is to absorb energy in the form of heat from a single thermal reservoir and to deliver an equivalent amount of work."

2nd law at the quantum level

$$\rho'_{XY} = U(\rho_X \otimes \rho_Y)U^\dagger$$

- The degree of irreversibility of this process is quantified by the entropy production:

$$\Sigma = I'(X:Y) + S(\rho'_Y || \rho_Y)$$

$$= S(X') - S(X) + \Phi$$

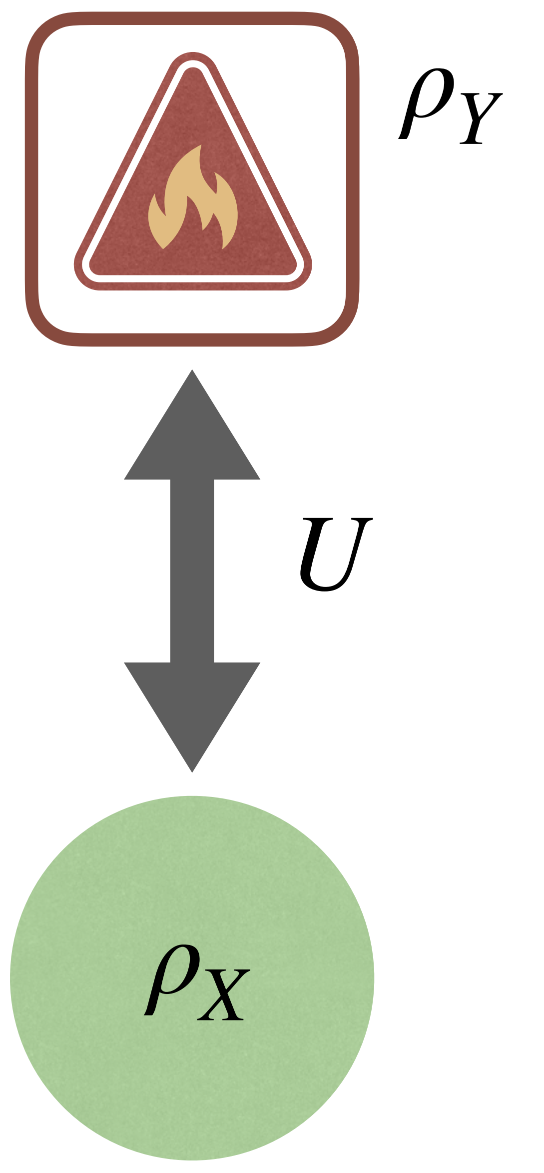
$$I'(X:Y) = S(\rho'_X) + S(\rho'_Y) - S(\rho'_{XY})$$

$$S(\rho'_Y || \rho_Y) = \mathrm{tr}\big(\rho'_Y \ln \rho'_Y - \rho'_Y \ln \rho_Y\big)$$

where

$$\Phi = \mathrm{tr}_Y\Big\{(\rho_Y - \rho'_Y)\ln \rho_Y\Big\}$$

is called the **entropy flux**.



- Φ depends only on Y. Measures change in the "thermodynamic potential" $\ln \rho_Y$
    - If $\rho_Y = e^{-\beta H_Y}/Z_Y$ we get $\Phi = -\beta Q$.

Describes an enormous variety of processes! (maybe a complicated U)

M. Esposito, K. Lindenberg, C. Van den Broeck, "Entropy production as correlation between system and reservoir". New Journal of Physics, **12**, 013013 (2010).

Conditional entropy production

- Part of the irreversibility stems from our ignorance about the environment.
- Suppose we measure Y after it interacted with X.

$$\rho'_{XY} \to \rho'_{XY|z} = (1 \otimes M_z)\rho'_{XY}(1 \otimes M_z^\dagger)$$

$$p_z = \mathrm{tr}_Y\big(M_z^\dagger M_z \rho'_Y\big)$$

- $\{M_z\}$ = generalized measurement operators acting on Y:

This is a **conditional state**: It is the state of XY, conditioned on the measurement outcome being z.

- What is the entropy production and flux, conditioned on these outcomes?

$$\Sigma_c = S(X'|z) - S(X) + \Phi_c \qquad \text{where} \qquad S(X'|z) = \sum_z p_z S(\rho'_{X|z})$$

is the quantum-classical conditional entropy

- How to define Σ_c and Φ_c?
- Natural generalization of the flux:

$$\Phi_c = \sum_z p_z \mathrm{tr}\Big\{(\rho_Y - \rho'_{Y|z})\ln\rho_Y\Big\}$$

$$= \mathrm{tr}\Big\{(\rho_Y - \tilde{\rho}_Y)\ln\rho_Y\Big\}$$

where $\tilde{\rho}_Y = \sum_z p_z \rho'_{Y|z}$.

- But very often $\mathrm{tr}(\tilde{\rho}_Y \ln\rho_Y) = \mathrm{tr}(\rho'_Y \ln\rho_Y)$, so

$$\Phi_c = \Phi$$

Flux is physical; no subjective component associated to information acquired.

- The unconditional and conditional $\Sigma's$ are thus

$$\Sigma_u = S(X') - S(X) + \Phi$$

$$\Sigma_c = S(X'|z) - S(X) + \Phi$$

- Whence,

$$\Sigma_c = \Sigma_u - I$$

where

$$I = S(X') - S(X'|z) = \sum_z p_z S(\rho'_{X|z} || \rho'_X)$$

is the Holevo χ quantity ✅.

- One may show that

$$0 \leqslant \Sigma_c \leqslant \Sigma_u$$

- Thus, the conditional entropy production still satisfies a 2nd law ($\Sigma_c \geqslant 0$).

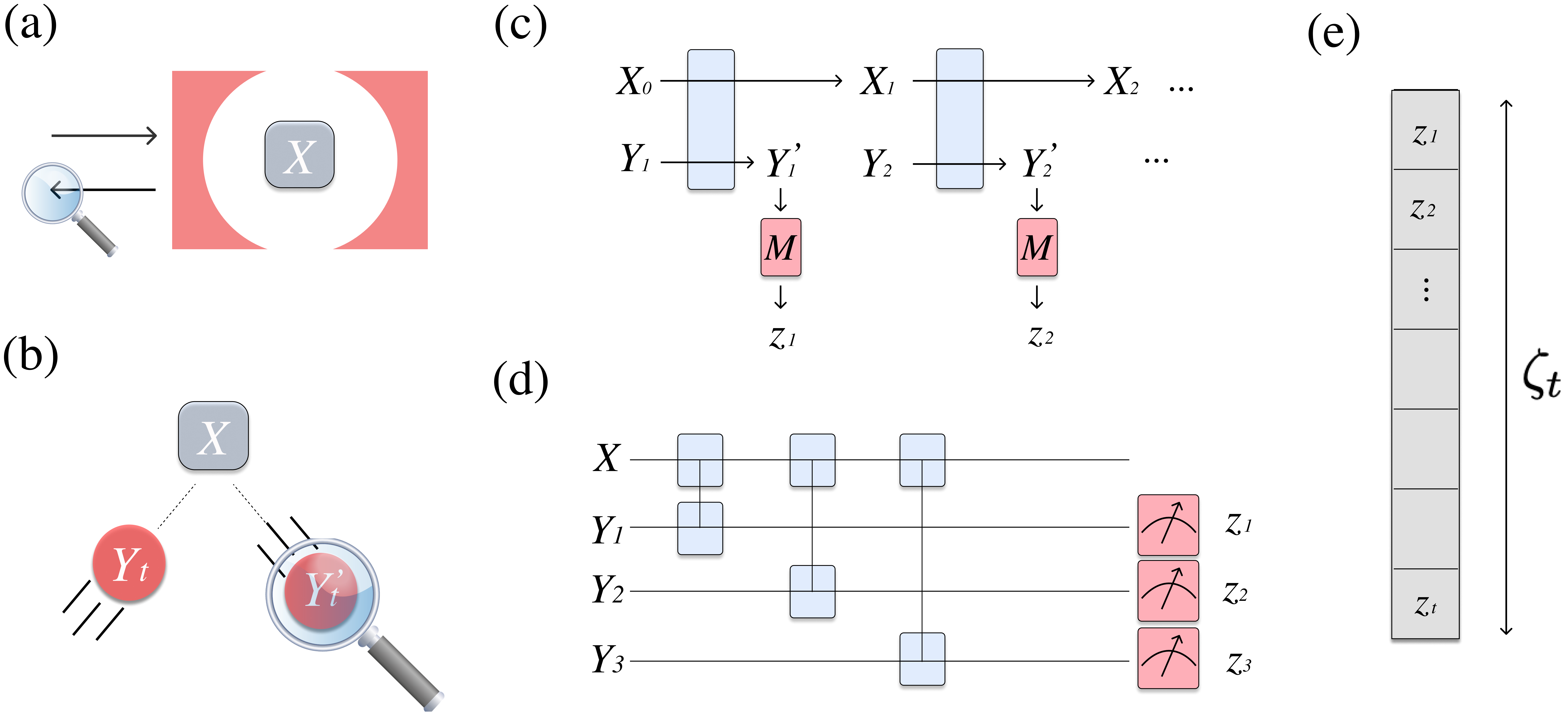
- But it is also smaller than the unconditional one:
  - Conditioning makes the process more reversible.

K. Funo, Y. Watanabe and M. Ueda, "Integral quantum fluctuation theorems under measurement and feedback control". PRE, **88**, 052121 (2013).

GTL and M. Paternostro, "Irreversible entropy production, from quantum to classical", To appear in Review of Modern Physics. arXiv:2009.07668

M. Naghiloo, J. J. Alonso, A. Romito, E. Lutz, K. Murch, "Information Gain and Loss for a Quantum Maxwell's Demon". PRL **121**, 030604 (2018).

CM²: Continuously measured collisional models



Information-theoretic quantities

- The unconditional dynamics is governed by the stroboscopic map

$$\rho_{X_t} = \mathscr{E}(\rho_{X_{t-1}}) = \mathrm{tr}_{Y_t}\left\{U_t\left(\rho_{X_{t-1}} \otimes \rho_{Y_t}\right)U_t^\dagger\right\}$$

- And its information content is thus summarized by the von Neumann entropy

$$S(X_t) = -\,\mathrm{tr}\left\{\rho_{X_t} \ln \rho_{X_t}\right\}$$

- The conditional dynamics, on the other hand, is governed by (up to a normalization)

$$\rho_{X_t|\zeta_t} = \mathscr{E}_{z_t}(\rho_{X_{t-1}|\zeta_{t-1}}) = \mathrm{tr}_{Y_t}\left\{M_{z_t}U_t\left(\rho_{X_{t-1}} \otimes \rho_{Y_t}\right)U_t^\dagger M_{z_t}^\dagger\right\}$$

- And its information content is thus summarized by the quantum-classical conditional entropy

$$S(X_t|\zeta_t) = \sum_{\zeta_t} P(\zeta_t)\; S(\rho_{X_t|\zeta_t})$$

Their difference is the Holevo information:

$$I(X_t : \zeta_t) = S(X_t) - S(X_t|\zeta_t) = \sum_{\zeta_t} P(\zeta_t) D\left(\rho_{X_t|\zeta_t} || \rho_{X_t}\right) \geqslant 0$$

Gain rate/Loss rate - ISS

- The change in Holevo information can have any sign:

$$\Delta I_t = I(X_t : \zeta_t) - I(X_{t-1} : \zeta_{t-1})$$

- But we can split it into a Gain rate and a Loss rate

$$\Delta I_t = G_t - L_t$$

$$G_t = I(X_t : z_t | \zeta_{t-1}) = I(X_t : \zeta_t) - I(X_t : \zeta_{t-1}) \geqslant 0$$

$$L_t = I(X_{t-1} : \zeta_{t-1}) - I(X_t : \zeta_{t-1}) \geqslant 0$$

Informational steady-state:

$$\Delta I_{ISS} = 0$$

but

$$G_{SS} = L_{SS} \neq 0.$$

Thermodynamics

- The entropy flux/production is now the same as before:
  - Unconditional:

$$\Delta\Sigma_t^u = S(X_t) - S(X_{t-1}) + \Delta\Phi_t$$

  - Conditional:

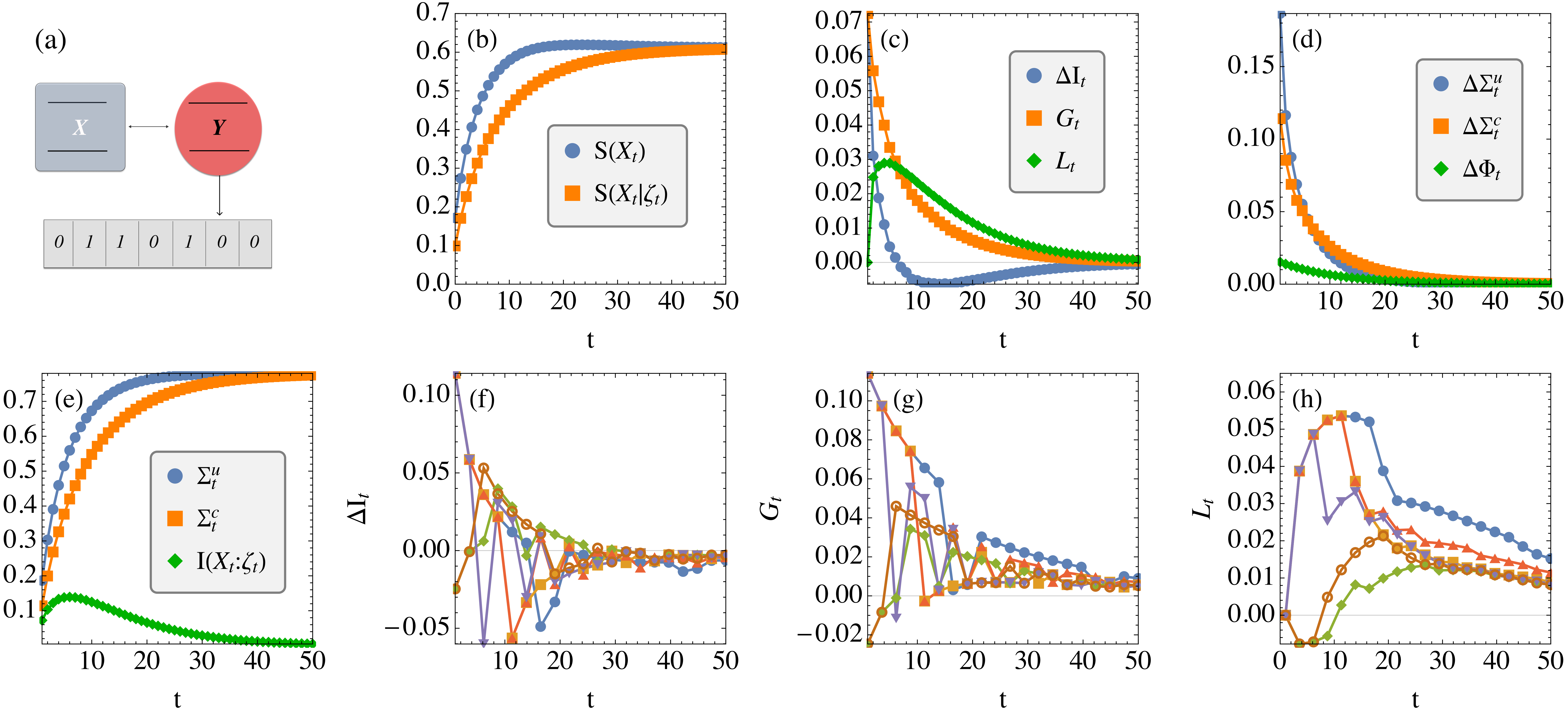
$$\Delta\Sigma_t^c = S(X_t \,|\, \zeta_t) - S(X_{t-1} \,|\, \zeta_{t-1}) + \Delta\Phi_t$$

$$= \Delta\Sigma_t^u - \Delta I_t$$

  - Flux is again the same in both.
- In an ISS $\Delta I_{ISS} = 0$ so $\Delta\Sigma_{ISS}^c = \Delta\Sigma_{ISS}^u$.

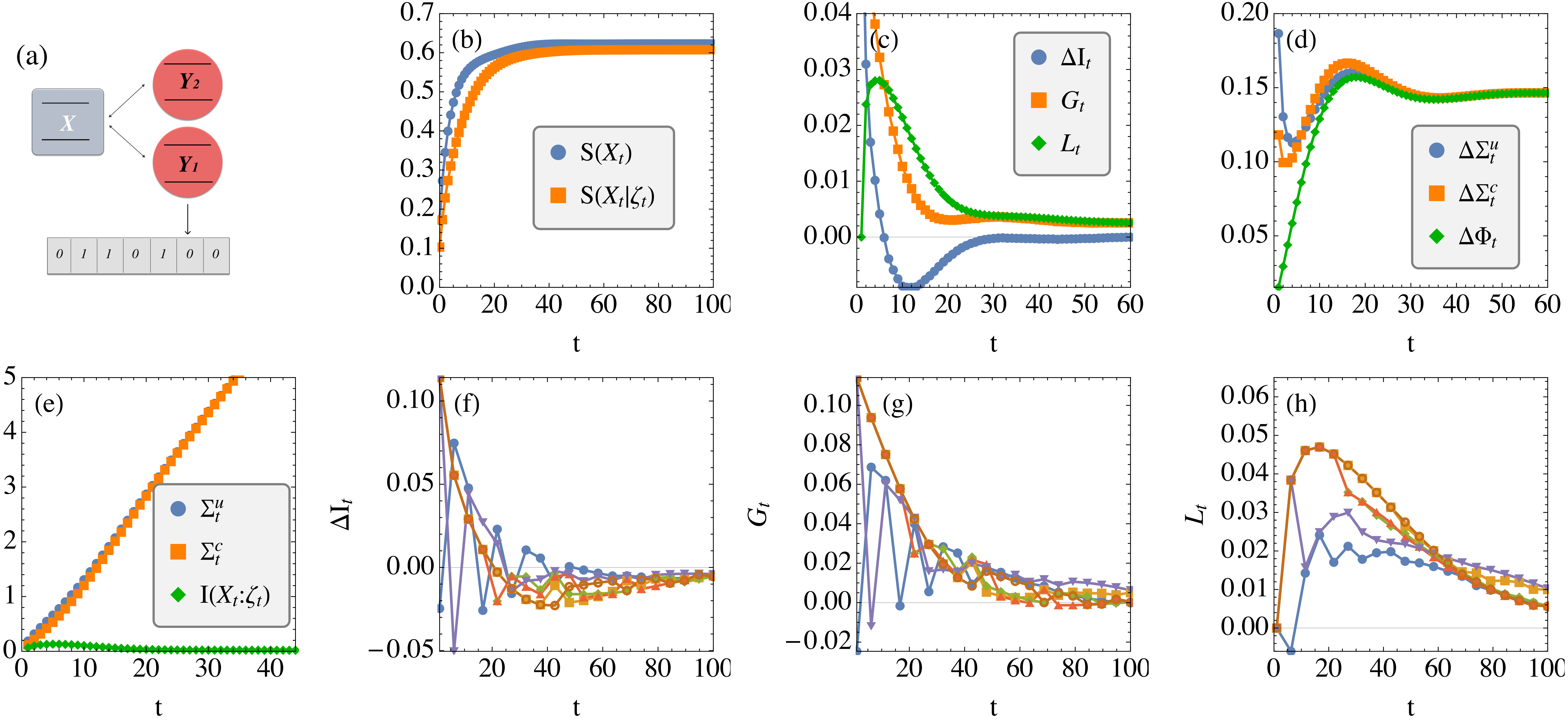
Minimal qubit models - Single-qubit ancilla

Thermal ancilla qubit + partial SWAP.

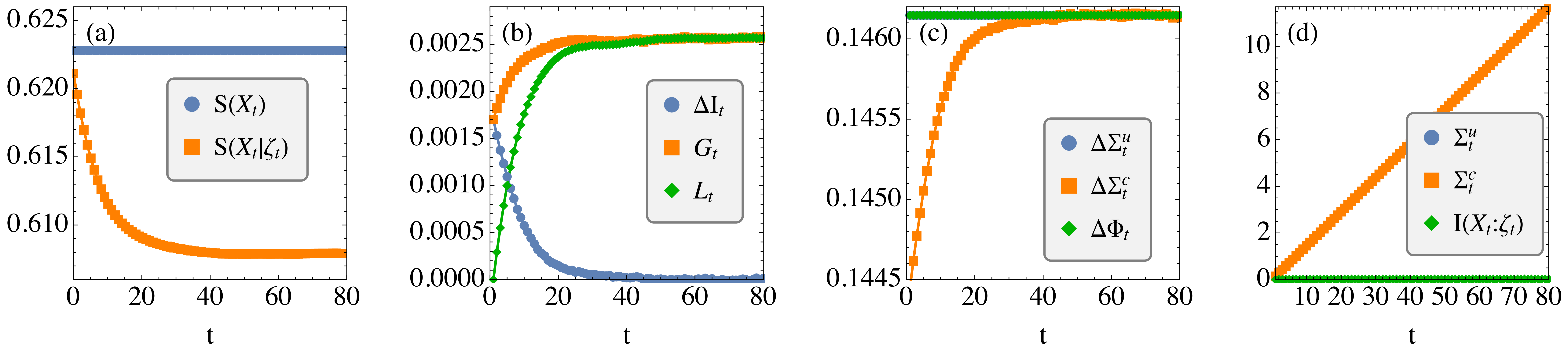


Minimal qubit models - Two-qubit ancilla

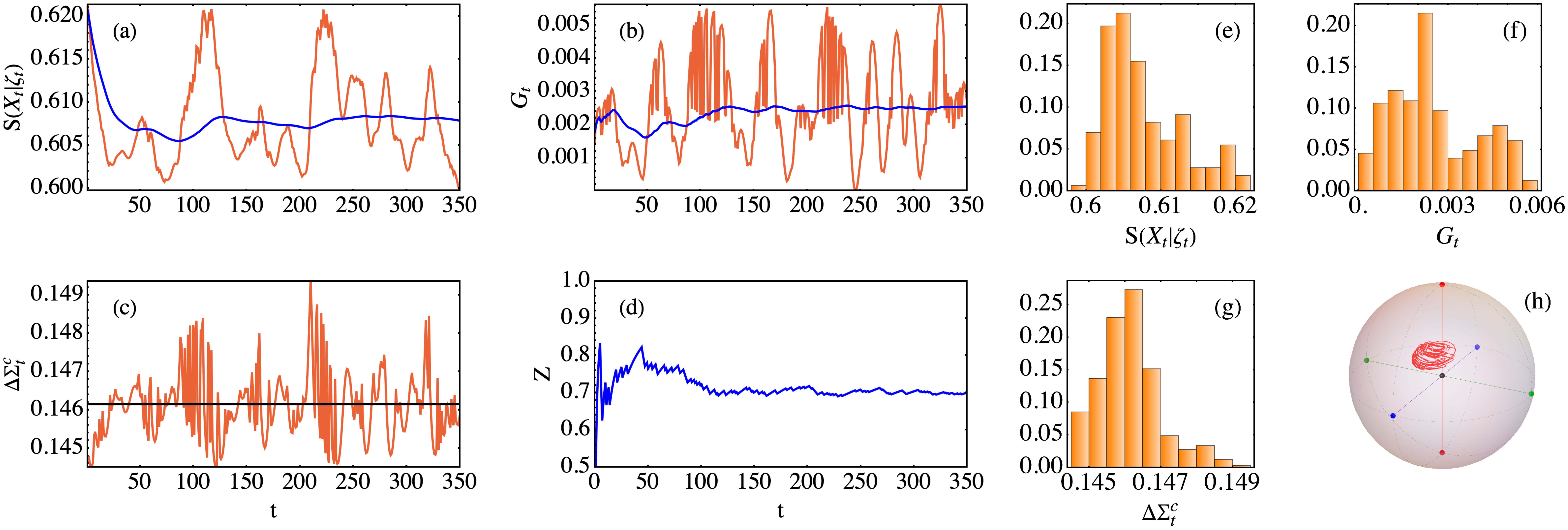
One ancilla thermal. The other prepared in $|+\rangle$
Sequential partial SWAPs



Starting from the ISS:



Single-shot scenario



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Entropy production in continuously measured Gaussian quantum systems

Alessio Belenchia[1], Luca Mancino[1], Gabriel T. Landi[2] and Mauro Paternostro[1] ✉

Gaussian continuous weak measurements

- The theory of continuous measurements is further developed, and can go much deeper, in the case of continuous variables undergoing Gaussian-preserving dynamics.
- Let $x = (q_1, p_1, q_2, p_2, \ldots)$ denote the vector of quadrature operators. Gaussian systems are fully characterized by their 2 first moments:
  - the average $\bar{x} = \langle x \rangle$
  - and the covariance matrix (CM) $\sigma_{ij} = \frac{1}{2}\langle \{x_i, x_j\} \rangle - \langle x_i \rangle \langle x_j \rangle$.
- We must track both the conditional and unconditional dynamics.
  - Unconditional means we monitor (there is still backaction) but we don't care about the results. Described by a Lindblad MEq.
  - Conditional dynamics is stochastic because we condition on random outcomes. Described by a stochastic MEq.

A. Serafini, "Quantum Continuous Variables: A Primer of Theoretical Method".

M. G. Genoni, L. Lami, and A. Serafini, "Conditional and unconditional Gaussian quantum dynamics", Contemp. Phys. **57**, 331 (2016).

- Unconditional variables evolve as in a Lindblad master equation:

$$\frac{d\bar{x}_u}{dt} = A\bar{x}_u + b$$

where A, b depend on both unitary and dissipative dynamics.

- Similarly, the CM evolves according to the Lyapunov equation:

$$\frac{d\sigma_u}{dt} = A\sigma_u + \sigma_u A^T + D$$

where D is called the diffusion matrix.

- The continuous measurement will cause the mean $\bar{x}_c$ to evolve stochastically according to the Langevin equation:

$$\frac{d\bar{x}_c}{dt} = (A\bar{x}_c + b) + (\sigma_c C^\mathsf{T} + \Gamma^\mathsf{T})\xi(t)$$

where C, Γ are matrices and $\xi(t)$ is a vector of white noises.

- The CM, on the other hand, evolves deterministically:

$$\frac{d\sigma_c}{dt} = A\sigma_c + \sigma_c A^\mathsf{T} + D - \chi(\sigma_c)$$

where

$$\chi(\sigma) = (\sigma_c C^\mathsf{T} + \Gamma^\mathsf{T})(C\sigma + \Gamma) \geqslant 0$$

describes the information gained due to the measurement.

M. G. Genoni, L. Lami, and A. Serafini, "Conditional and unconditional Gaussian quantum dynamics", Contemp. Phys. **57**, 331 (2016).

🔁 Thermodynamics of Gaussian CMs

- In the case of continuous measurements, the relevant quantity is the entropy production **rate**.

- We formulate the thermodynamics of this model using a semi-classical representation in terms of the Wigner function $W(x)$ (standard approach does not work).

  - The Wigner function, conditioned on a given outcome for the average, is $W_c(x|\bar{x})$.

  - The variable $\bar{x}$ is classical, with probability distribution $p(\bar{x})$.

  - The conditional and unconditional Wigner functions are thus associated by a Kalman filter:

$$W_u(x) = \int W_c(x|\bar{x})p(\bar{x})d\bar{x}$$

A. Belenchia, L. Mancino,GTL and M. Paternostro, "Entropy Production in Continuously Measured Quantum Systems", arXiv:1908.09382. npj Quantum Inf 6, 97 (2020).

- As an alternative representation of entropy, we can use

$$S_u = -\int W_u(x) \ln W_u(x) dx$$

and

$$S_c = -\int p(\bar{x}) d\bar{x} \int W_c(x|\bar{x}) \ln W_c(x|\bar{x}) dx$$

- Their difference represents the net amount of information acquired by the measurement record:

$$I = S_u - S_c \geqslant 0$$

- This is the phase-space analog of the Holevo quantity. Exactly the same idea ✅.

$$\left(\chi_M(\rho'_S) = S(\rho'_S) - \sum_k p_k S(\rho'_{S|k}) \right)$$

G. Adesso, D. Girolami, A. Serafini, "Measuring gaussian quantum information and correlations using the Rényi entropy of order 2". PRL **109**, 190502 (2012).

🅿️ Unconditional production/flux

- The unconditional Wigner function evolves according to a Fokker-Planck equation:

$$\frac{\partial W}{\partial t} = \mathrm{div}\left[J + J_{\mathrm{sto}}\right]$$

where

$$J = (Ax + b)W - \frac{D}{2}\nabla W$$

is a quasi-probability current.

- The entropy production and flux rates are

$$\Pi_u = 2\int \frac{dx}{W_u}\, J^T D^{-1} J \geqslant 0$$

$$\Phi_u = -2\int J^T D^{-1} A\, dx$$

- The stochastic MEq is translated into a stochastic Fokker-Planck equation:

$$\frac{\partial W_c}{\partial t} = \mathrm{div}\left[J + J_{\mathrm{sto}}\right]$$

where

$$J_{\mathrm{sto}} = W_c(\sigma_c C^T + \Gamma^T)\xi(t)$$

- One can show that the flux does not change:

$$\Phi_c = \Phi_u$$

as intuitively expected.

J. P. Santos, GTL, M. Paternostro, "The Wigner entropy production rate", PRL **118**, 220601 (2017).

- Hence, as before, we will have

$$\Pi_u = \dot{S}_u + \Phi_u \qquad \qquad \therefore \qquad \Pi_c = \Pi_u - \dot{I}$$
$$\Pi_c = \dot{S}_c + \Phi_u$$

- In particular, the net rate of information gain can be shown to be

$$\dot{I} = \frac{1}{2}\mathrm{tr}\left[D(\sigma_c^{-1} - \sigma_u^{-1})\right] - \frac{1}{2}\mathrm{tr}\left[\chi(\sigma_c)\sigma_c^{-1}\right] := \dot{L} - \dot{G}$$

  ✓ The 1st term is the information loss rate due to the dissipation ($\propto D$).

  ✓ The 2nd term is the information gain rate, due to the update matrix $\chi(\sigma_c)$

- In the steady-state $\dot{I} = 0$. But this does not mean we are no longer acquiring information.
  - What it means is that $\dot{G} = \dot{L}$: the information acquired is balanced by the information dissipated.

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arXiv:2005.03429

🇩🇰 Copenhagen setup

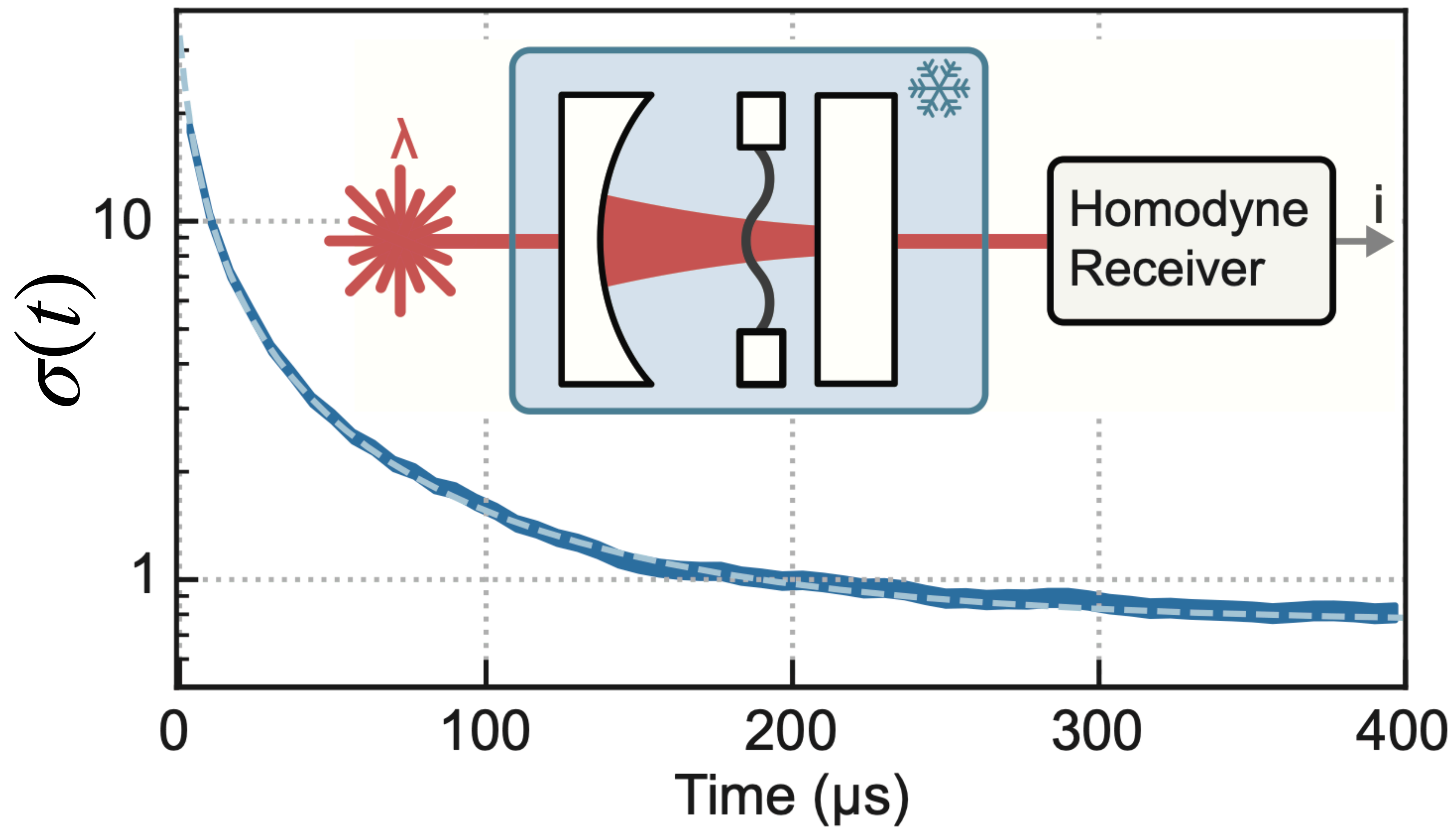
- Optomechanical system continuously monitored by an optical field.
- Competition: Thermal bath vs. Measurement.
- Quadratures of the mechanical mode: $x = (q, p)$
- Unconditional dynamics tends to $\bar{x}_u = 0$

$$\sigma_u = \bar{n} + 1/2 + \Gamma_{qba}/\Gamma_m$$

- Conditional dynamics evolves instead to

$$\frac{dx}{dt} = -\frac{\Gamma_m}{2}x + \sqrt{4\eta\Gamma_{\mathtt{qba}}}\sigma_c(t)\xi(t)$$

$$\frac{d\sigma_c}{dt} = \Gamma_m(\sigma_u - \sigma_c) - 4\eta\Gamma_{\mathtt{qba}}\sigma_c^2$$



Informational steady-state:

Conditional dynamics relaxes to a colder state, $\sigma_c < \sigma_u$, which can only be maintained by continuously monitoring S.

Production and flux at the trajectory level

Information gain/loss rates characterizing the information steady-state

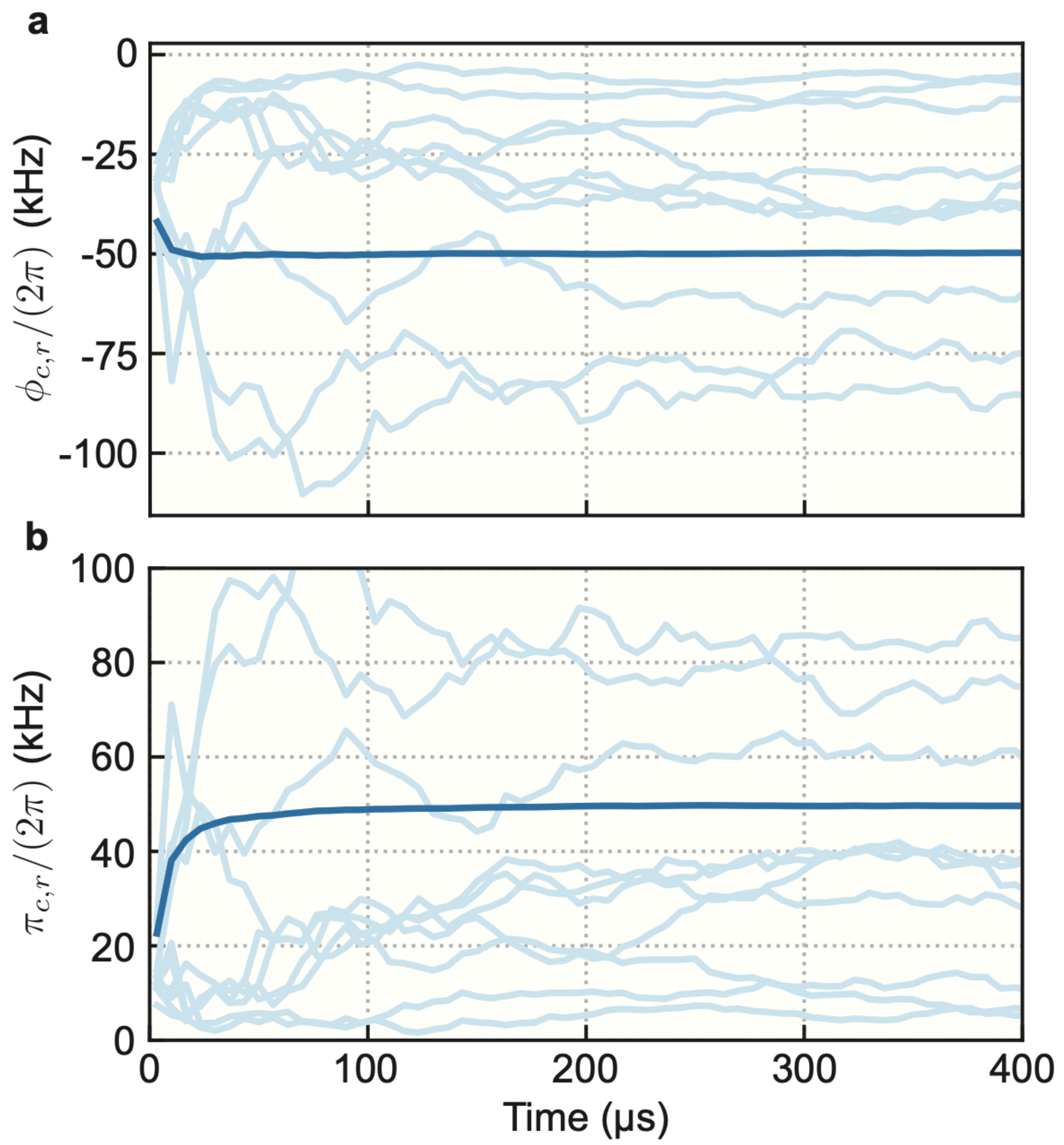


FIG. 2. **Stochastic entropy flux and production rates. a,** The stochastic entropy flux rates (light blue) for a sample of 10 trajectories. The dark blue line is the ensemble average over all the trajectories. **b,** The stochastic entropy production rates (light blue) and the ensemble average (dark blue), for the same sample of trajectories.

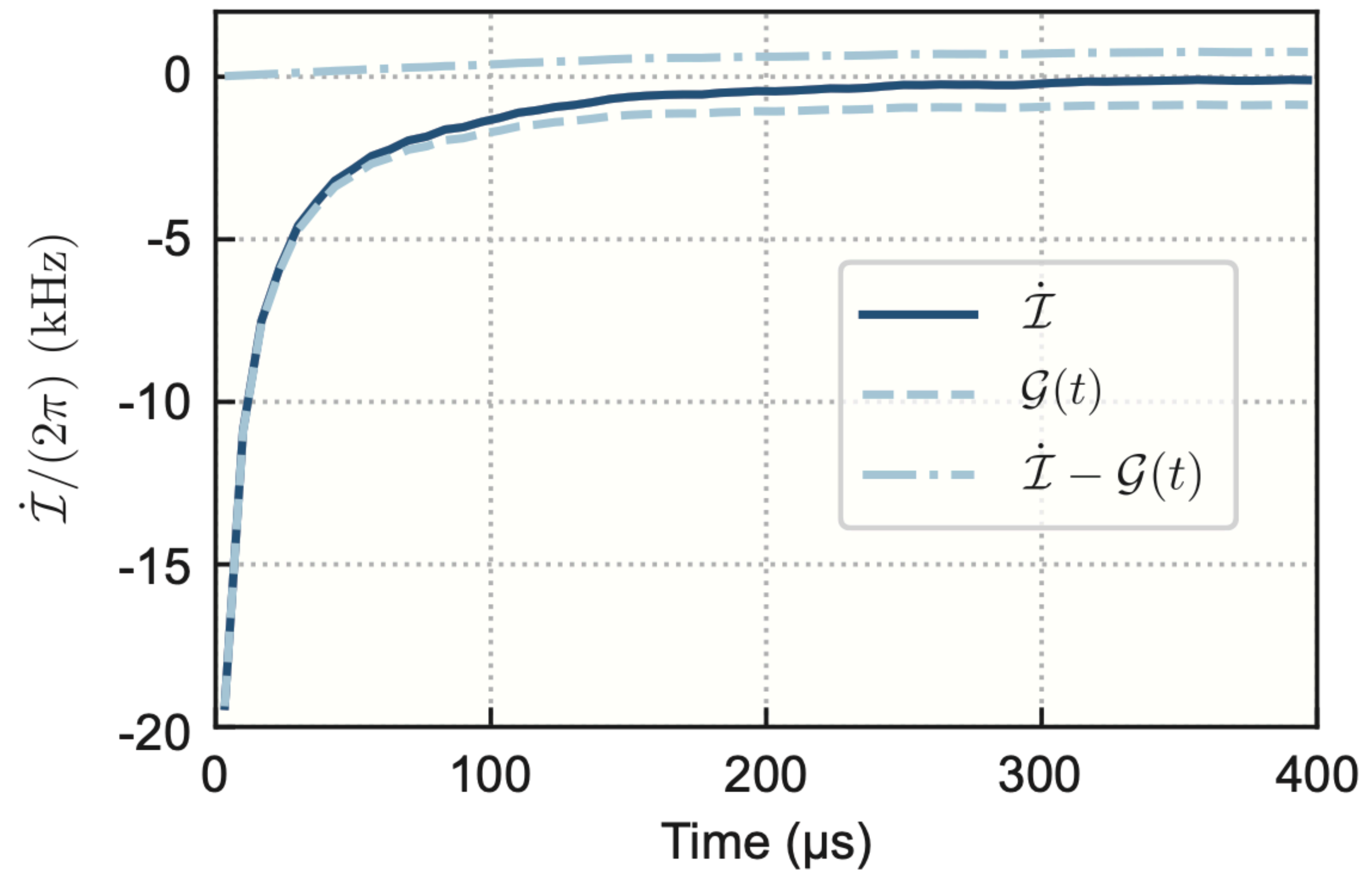


FIG. 3. **Informational contribution to the entropy production rate.** We obtain the informational contribution (dark blue) from the entropy production. The dashed (dot-dashed) line is the differential gain of information due to the measurement (loss of information due to noise input by the phonon bath).

Conclusions

- Knowing something about the bath makes the process less irreversible.
- The **conditional entropy production** quantifies this effect.
- We put forth a framework based on **continuously monitored collisional models** to address this scenario:
    - Clear conditions for identifying **informational steady-states**.
    - We also provide an **experimental assessment** of the entropy production at the level of stochastic trajectories in a quantum optomechanical system.

Thank you! 🙃

Quantum Thermodynamics QT² Quantum Transport

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