Bell's Inequalities

In 1935 Einstein, Podolsky and Rosen put forward a seminal paper questioning the completeness of the wavefunction description of quantum mechanics. They analyzed the situation of two spatially separated particles that are entangled in position and momentum, which, we will cover later in the course. In 1951 Bohm framed the EPR problem in terms of a Gedanken experiment based on the dissociation of a spin-zero two-atom molecule where each atom had a spin of 1/2. After dissociation the wave function of the system has the well known form $\frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right)$ Concerning measurements on the two-atom system Boym wrote: "Suppose now that one measures the spin angular momentum of any one of the particles... Because of the existence of correlations, one can immediately conclude that the angular-momentum vector of the other particle is equal and opposite".

The debate about the completeness of quantum mechanics was considered to be merely philosophical until 1964 when John Bell showed that quantum mechanics and hidden-variable theories were mathematically imcompatible. He derived an inequality based on Bohm-type quantum systems which showed that any local realistic theory and quantum mechanics predicted two different probabilistic outcomes. His work was further elaborated on by Clauser, Horne, Shimony and Holt.

The derivation of Bell's inequalities start with the following consideration. If a local realistic theory can account for the correlations that Alice and Bob measure, then it must be true that Alice's measurement outcome A which can be either 1/2 or -1/2 must be independent of Bob's analyzer orientation b and measurement outcome B, which can be either +1/2 or -1/2. Alice's and Bob's joint probability function must then decouple as

$$P(A, B|a, b, \lambda) = P(A|a, \lambda)P(B|b, \lambda) \tag{1}$$

where λ accounts for all possible local hidden variables. Assuming local hidden variables one can define a measurement outcome given by

$$E^{HV}(a, b) \equiv \int d\lambda f(\lambda)\bar{A}(a, \lambda)\bar{B}(b, \lambda) \tag{2}$$

where

$$\bar{A}(a,\lambda) \equiv P(+|a,\lambda) - P(-|a,\lambda) \tag{3}$$

$$\bar{B}(b,\lambda) \equiv P(+|b,\lambda) - P(-|b,\lambda) \tag{4}$$

where $P(+|a,\lambda)$ is the probability of measuring the $|+\rangle$ eigenstate for analyzer setting a. Because of the signs of the probabilities in $\bar{A}(a,\lambda)$ and $\bar{B}(b,\lambda)$, it must be true that $\bar{A}(a,\lambda) \leq 1$ and $\bar{B}(b,\lambda) \leq 1$. Now consider,

$$E^{HV}(a,b) - E^{HV}(a,b') = \int d\lambda f(\lambda)\bar{A}(a,\lambda)(\bar{B}(b,\lambda) - \bar{B}(b',\lambda)) \tag{5}$$

since $|\bar{A}(a,\lambda)| \leq 1$, then

$$|E^{HV}(a,b) - E^{HV}(a,b')| \leq \int d\lambda f(\lambda)|(\bar{B}(b,\lambda) - \bar{B}(b',\lambda))|. \tag{6}$$

Similarly

$$|E^{HV}(a',b) + E^{HV}(a',b')| \leq \int d\lambda f(\lambda)|(\bar{B}(b,\lambda) + \bar{B}(b',\lambda))|. \tag{7}$$

Since $|(\bar{B}(b,\lambda) \leq 1$ then

$$|\bar{B}(b,\lambda) - \bar{B}(b',\lambda) + \bar{B}(b,\lambda) + \bar{B}(b',\lambda)| \leq 2. \tag{8}$$

and knowing that $\int d\lambda f(\lambda) = 1$ then

$$S \equiv |E^{HV}(a,b) - E^{HV}(a,b') + E^{HV}(a',b) + E^{HV}(a',b')| \leq 2. \tag{9}$$

Hence, the maximum possible value of S that can be achieved assuming locally explicable outcomes is 2. On the other hand, quantum mechanics predicts nonseparable joint probability functions of the form

$$E^{QM} = P(+,+|a,b) - P(+,-|a,b) - P(-,+|a,b) + P(-,-|a,b). \tag{10}$$

For maximally entangled photons entangled in polarization, (i.e., replacing $|+\rangle$ with the analogous $|H\rangle$) and using angles of $a = 0$, $a' = 45°$, $b = 22.5°$ and $b' = 67.5$ a maximum value of 2.82 can be achieved. This is a clear violation of the bound set by the inequality.

PROBLEMS

Derive a spin-1 Bell inequality.