Spin 1/2 systems

CANNONICAL COMMUTATION RELATIONS

In this section we will derive the spin observables for two-photon polarization entangled states. Instead of using the spin-1/2 basis, we will use the polarization basis, which has essentially the same physics and the possible measurement outcomes are, for example a horizontally polarized photon. In a two-state system, there are three mutually unbiased bases, which we will call the 0,90 (Horizontally and Vertically polarized) basis, the 45,135 (Horizontal prime and Vertical prime polarized) basis and the R,L (Right and left circularly polarized) basis. These are analogs of the spin bases. Accordingly, we will use three operators P_0, P_{45} and P_c. The eigenstates of the P_0 operator are the $|H\rangle$ and $|V\rangle$ states with eigenvalues

$$P_0|H\rangle = |H\rangle \tag{1}$$

$$P_0|V\rangle = -|V\rangle \tag{2}$$

The eigenstates of the P_{45} operator are the $|H'\rangle$ and $|V'\rangle$ states with eigenvalues

$$P_{45}|H'\rangle = |H'\rangle \tag{3}$$

$$P_{45}|V'\rangle = -|V'\rangle \tag{4}$$

and the eigenstates of the P_c operator are the $|R\rangle$ and $|L\rangle$ states with eigenvalues

$$P_c|R\rangle = |R\rangle \tag{5}$$

$$P_c|L\rangle = -|L\rangle. \tag{6}$$

The polarization states of a photon are then a particularly simple context in which study basis transformation. For example, if one wishes to transform a state from one basis to another, it is as simple as placing the appropriate waveplate at the correct angle in the path of the photon. For example, a half waveplate oriented at 22.5° with respect to the horizontal vertical basis, will reorient the photon to the horizontal-vertical primed basis. The relationships between the bases are very straightforward. Writing all of the bases in terms of the horizontal-vertical basis we have

$$|H'\rangle = \frac{1}{\sqrt{2}}|H\rangle + \frac{1}{\sqrt{2}}|V\rangle \tag{7}$$

$$|V'\rangle = -\frac{1}{\sqrt{2}}|H\rangle + \frac{1}{\sqrt{2}}|V\rangle \tag{8}$$

$$|R\rangle = \frac{1}{\sqrt{2}}|H\rangle + \frac{i}{\sqrt{2}}|V\rangle \tag{9}$$

$$|L\rangle = \frac{1}{\sqrt{2}}|H\rangle - \frac{i}{\sqrt{2}}|V\rangle \tag{10}$$

We wish to have a matrix representation for each of the polarization operators in a single basis. By convention, the horizontal vertical basis, which is analogous to the z basis in the spin notation is the chosen basis of representation. Thus, we must expand the horizontal-vertical basis in terms of the primed basis

$$|H\rangle = \frac{1}{\sqrt{2}}|H'\rangle - \frac{1}{\sqrt{2}}|V'\rangle \tag{11}$$

$$|V\rangle = \frac{1}{\sqrt{2}}|H'\rangle + \frac{1}{\sqrt{2}}|V'\rangle \tag{12}$$

and the circularly polarized basis

$$|H\rangle = \frac{1}{\sqrt{2}}|R\rangle + \frac{1}{\sqrt{2}}|L\rangle \tag{13}$$

$$|V\rangle = -\frac{i}{\sqrt{2}}|R\rangle + \frac{i}{\sqrt{2}}|L\rangle \tag{14}$$

We can notice some interesting properties about the operators in the new bases. Consider

$$P_c|H\rangle = \frac{1}{\sqrt{2}}|R\rangle - \frac{1}{\sqrt{2}}|L\rangle = -i|V\rangle \tag{15}$$

$$P_c|V\rangle = -\frac{i}{\sqrt{2}}|R\rangle - \frac{i}{\sqrt{2}}|L\rangle = -i|H\rangle \tag{16}$$

and

$$P_{45}|H\rangle = \frac{1}{\sqrt{2}}|H'\rangle + \frac{1}{\sqrt{2}}|V'\rangle = |V\rangle \tag{17}$$

$$P_{45}|V\rangle = \frac{1}{\sqrt{2}}|H'\rangle - \frac{1}{\sqrt{2}}|V'\rangle = |H\rangle \tag{18}$$

Thus, the operators expanded in the H-V basis are given by

$$P_0 = \begin{pmatrix} \langle H|P_0|H\rangle & \langle H|P_0|V\rangle \\ \langle V|P_0|H\rangle & \langle V|P_0|V\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{19}$$

$$P_{45} = \begin{pmatrix} \langle H|P_{45}|H\rangle & \langle H|P_{45}|V\rangle \\ \langle V|P_{45}|H\rangle & \langle V|P_{45}|V\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{20}$$

and

$$P_c = \begin{pmatrix} \langle H|P_c|H\rangle & \langle H|P_c|V\rangle \\ \langle V|P_c|H\rangle & \langle V|P_c|V\rangle \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{21}$$

As can be seen these have the same matrix representation as Pauli spin matrices. Indeed any two state system can be represented in the same way.

Find the commutation relation for $[P_0, P_{45}]$. *Problem* expand the operators in the primed basis.

Rotations

We must also consider the effects of rotations of the states. For example a general rotation in the 2-dimensional complex space is given by

$$P_c = \begin{pmatrix} \cos\phi & \sin\phi e^{i\theta} \\ -\sin\phi & \cos\phi e^{i\theta} \end{pmatrix} \tag{22}$$

For example, the rotation by ϕ with $\theta = 0$ can be physically realized by rotating a half wave plate by an angle $\phi/2$ with respect to the incoming linear polarization or semimajor axis.

Density Matrix

We now wish to describe state of the system. Because the density matrix formalism can describe both pure and mixed states as well as is a simple means of computing expectation values, the formalism will be used throughout. Consider initially that the polarization state of the field is unpolarized. The unpolarized density matrix is given by the maximally mixed state

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{23}$$

The expectation value of all the polarization operators vanishes for the unpolarized state, as expected. The variance of the P_0 operator is then

$$(\Delta P_0)^2 = Tr[\rho P_0^2] = 1 \tag{24}$$

Find the variance of the P_c operator.

Now consider the effect of placing a linear polarizer in the path of the photon. A linear polarizer "filters" the state into a well-defined linear polarization state. For example, a polarizer which is set to pass horizontal polarization then performs the operation

$$\rho_f = |H\rangle\langle H|\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{25}$$

where the subscript f denotes the density matrix after the filtering process. Also one should note the density matrix has been renormalized. The expectation value of the P_0 operator after the filtering is then

$$\langle P_0 \rangle = Tr[\rho_f P_0] = 1 \tag{26}$$

and the expectation value of P_0^2 is

$$\langle P_0^2 \rangle = Tr[\rho_f P_0^2] = 1 \tag{27}$$

which means that the variance is zero (sharp dispersion). This seems physically obvious, if we know that the polarization state after the polarizer is horizontal to perfect accuracy, the uncertainty must be zero. However, the variance of the P_{45} operator is nonvanishing. The expectation value of the P_{45} operator after the filtering is then

$$\langle P_{45} \rangle = Tr[\rho_f P_{45}] = 0 \tag{28}$$

and the expectation value of P_{45}^2 is

$$\langle P_0^2 \rangle = Tr[\rho_f P_{45}^2] = 1 \tag{29}$$

which means that the variance is 1.

Find the variance of P_c.

TWO-PHOTON HILBERT SPACE

The previous section is essentially a review of spin-1/2 physics. However, it is important to lay the foundation before moving on to the more complicated entangled states. With two photons each having two polarization states, the basis vectors (using the eigenstates of the P_0 operator) are $|H_1, H_2\rangle$, $|H_1, V_2\rangle$, $|V_1, H_2\rangle$ and $|V_1, V_2\rangle$, where the indices denote the particle. Now one can write the matrices for the operators in the new 4 dimensional basis. This can be done by determining each of the matrix elements using the standard procedure or one can use the properties of tensors to simplify the task. For example, if one wishes to determine the elements of the P_0 matrix when operating on particle 1 in the two-photon basis, one can use

$$P_{01} = P_0 \otimes \mathbf{1} = \begin{pmatrix} 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & -1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{30}$$

If on the other hand the P_0 operator were acting on particle 2, then the

$$P_{02} = \mathbf{1} \otimes P_0 = \begin{pmatrix} 1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{31}$$

Find the matrix for the P_c operator acting on particle 2.

ROTATIONAL INVARIANCE

One of the crowning features of entangled states is rotational invariance. Rotational invariance leads to a conditional vanishing variance (dispersion) in any basis, which is not possible using nonentangled photons. Consider the wavefunction

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|H_1, V_2\rangle - |V_1, H_2\rangle \right) \tag{32}$$

This wavefunction leads to the 4×4 density matrix

$$\rho = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{33}$$

Now consider a rotation of the type in eqn. 22 acting on particle 1,

$$R_1 = R \otimes \mathbf{1} = \begin{pmatrix} \cos\phi & 0 & \sin\phi e^{i\theta} & 0 \\ 0 & \cos\phi & 0 & \sin\phi e^{i\theta} \\ -\sin\phi & 0 & \cos\phi e^{i\theta} & 0 \\ 0 & -\sin\phi & 0 & \cos\phi e^{i\theta} \end{pmatrix} \tag{34}$$

and

$$R_2 = \mathbf{1} \otimes R = \begin{pmatrix} \cos\phi & \sin\phi e^{i\theta} & 0 & 0 \\ -\sin\phi & \cos\phi e^{i\theta} & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi e^{i\theta} \\ 0 & 0 & -\sin\phi & \cos\phi e^{i\theta} \end{pmatrix} \tag{35}$$

After polarization rotations of particles 1 and 2, the new matrix is given by

$$\rho' = R_2[\phi_2, \theta_2] R_1[\phi_1, \theta_1] \rho R_1^\dagger[\phi_1, \theta_1] R_2^\dagger[\phi_2, \theta_2] \tag{36}$$

When $\phi_1 = \phi_2$ and $\theta_1 = \theta_2$, then $\rho' = \rho$ and the matrix is left unchanged (this is tedious but straightforward to prove, it is easiest to use the wavefunction form or even better, use computer simulation) by the rotation and thus the title "rotational invariance". Physically this means that the wavefunction can be written in the same way using the two eigenstates of any of the three polarization operators. When one thinks about this, this is a truly remarkable phenomenon. However, rotational invariance, as interesting as it is, does not rule out local hidden variables.

Prove that this matrix is rotationally invariant.