II. COHERENT STATES

A. Derivation of Coherent States

The number states studied in the previous section are mathematically very simple to study, but very difficult to realize in the laboratory. Coherent states α, as we will now study, are quasi-classical states produced by lasers. The mathematics of coherent states are motivated by a simple relation, namely:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{16}$$

This equation tells us that coherent states $|\alpha\rangle$ are eigenstates of the destruction operator $\hat{a}$. A simple physical meaning is that measurement of a small portion of a coherent state does not change the coherent state as long as the field is strong (quasi-classical). For example, a small portion of a coherent state (laser) can be sampled without affecting the unsampled portion of the coherent state. This is not the case when number states are used.

The decomposition of the coherent state in terms of the number states can be found from the relation in eqn. (14). Assume,

$$|\alpha\rangle = \sum_n c_n |n\rangle, \tag{17}$$

which implies

$$\hat{a}|\alpha\rangle = \sum_n c_n \sqrt{n}|n-1\rangle. \tag{18}$$

Substituting these results into eqn. (16) yields

$$\sum_n c_n \sqrt{n}|n-1\rangle = \sum c_n \alpha |n\rangle \tag{19}$$

which brings about the recurrence relation:

$$c_{n+1}\sqrt{n+1} = c_n \alpha. \tag{20}$$

Hence,

$$c_n = \frac{\alpha}{\sqrt{n}} c_{n-1} = \frac{\alpha^2}{\sqrt{(n)(n-1)}} c_{n-1} = \frac{\alpha^n}{\sqrt{n!}} c_0. \tag{21}$$

Therefore, fixing the c_0 coefficient and using the normalization condition,

$$\sum_n |c_n|^2 = 1 \tag{22}$$

uniquely determines all of the coefficients. Using eqn. (21) and the normalization condition gives

$$\sum_n \left| \frac{\alpha^n}{\sqrt{n!}} c_0 \right|^2 = 1$$
$$\Rightarrow |c_0|^2 \sum_n \frac{|\alpha|^{2n}}{n!} = 1 \tag{23}$$

Recalling $e^x = \sum_n \frac{x^n}{n!}$, we arrive at

$$c_0 = e^{\frac{-|\alpha|^2}{2}}. \tag{24}$$

Thus,

$$|\alpha\rangle = e^{\frac{-|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{25}$$

which is the coherent state.

B. Expectation and Variance of N

The expectation value of the number operator N in the coherent basis is given by

$$\langle \alpha | N | \alpha \rangle = |\alpha|^2. \tag{26}$$

and

$$\langle \alpha | N^2 | \alpha \rangle = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle = \langle \hat{a}^\dagger (\hat{a}^\dagger \hat{a} + 1) \hat{a} \rangle = |\alpha|^4 + |\alpha|^2. \tag{27}$$

The dispersion of N is found by taking the expectation value of the variance of N, which is given by

$$(\Delta N^2) = \langle N^2 \rangle - \langle N \rangle^2 = |\alpha|^2. \tag{28}$$

Physically this means that the uncertainty in the number of photons in a coherent states goes as the square root of the number of photons in the field. The number uncertainty grows with the strength of the field. However, the relative dispersion given by

$$\frac{(\Delta N)}{N} = \frac{1}{|\alpha|} \tag{29}$$

actually gets smaller with increasing field strength. This is an important result, because lasers, or coherent laser sources thus have well defined powers at large intensities. The uncertainty of the field is considered the standard quantum limit. If the standard quantum noise is the dominant noise in the system, another trick, amplitude squeezing, can further lower this noise threshold at the expense of phase uncertainty.

C. Quadrature Uncertainties

Using the relations between the quadrature operators and creation and annihilation operators in eqns. (2) and (3), it can be seen that

$$\hat{X} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \tag{30}$$

and

$$\hat{Y} = \frac{-i}{2}(\hat{a} - \hat{a}^\dagger). \tag{31}$$

The expectation value of the quadrature operator $\hat{X}$ in a coherent state is

$$\langle \hat{X} \rangle = \langle \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \rangle = \frac{1}{2}(\alpha + \alpha^*). \tag{32}$$

Similarly, the expectation value of the squared quadrature operator $\hat{X}^2$ in a coherent state is

$$\langle \hat{X}^2 \rangle = \langle \frac{1}{4}(\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a} + \hat{a}^{\dagger 2}) \rangle = \langle \frac{1}{4}(\hat{a}^2 + (\hat{a}^\dagger \hat{a} + 1) + \hat{a}^\dagger \hat{a} + \hat{a}^{\dagger 2}) \rangle = \frac{1}{4}(\alpha^2 + 2\alpha^* \alpha + 1 + \alpha^{*2}). \tag{33}$$

The expectation value of the variance is then computed to be

$$\langle (\Delta \hat{X}^2) \rangle = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2 = \frac{1}{4}. \tag{34}$$

PROVE

$$\langle (\Delta \hat{Y}^2) \rangle = \langle \hat{Y}^2 \rangle - \langle \hat{Y} \rangle^2 = \frac{1}{4}. \tag{35}$$

The variance product of the two quadratures is then in agreement with the standard result, namely:

$$\langle (\Delta \hat{X}^2) \rangle \langle (\Delta \hat{Y}^2) \rangle = \frac{1}{16}. \tag{36}$$

D. Displacement Operator: Coherent State Generator

The operator

$$\hat{D}(\alpha) = e^{\alpha^* \hat{a}^\dagger - \alpha \hat{a}} \tag{37}$$

can be used to generate the coherent state. As can be seen this operator is unitary. Using $[\hat{a}, hata^\dagger] = 1$. If two operators $\hat{A}$ and $\hat{B}$ commute with their commutator, then

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}]}. \tag{38}$$

PROVE

$$\hat{D}(\alpha) = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} e^{\alpha^* \hat{a}}. \tag{39}$$

This form for the operator is particularly simple if one is building up the coherent state from the vacuum, since $e^{\alpha^* \hat{a}}|0\rangle = |0\rangle$. Hence,

$$D(\alpha)|0\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger}|0\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{40}$$

As can be seen this is the coherent state as derived in eqn. 25.

The $\hat{D}(\alpha)$ operator is often called the displacement operator and fulfills its meaning in the quadrature representation. The wavefunction in the position representation

$$\Psi(x) = \langle x|\alpha\rangle = \langle x|\hat{D}(\alpha)|\phi_0\rangle \tag{41}$$

where ϕ_0 is the ground state wavefunction in the quadrature representation. Using eqns. 2 and 3 which relate the quadrature operators to the annihilation and creation operators along with the normal ordering term in eqn. 38, *PROVE* we arrive at

$$D(\alpha) = e^{\frac{\alpha^{*2} - \alpha^2}{4}} e^{\hat{X}(\alpha - \alpha^*)} e^{-i\hat{Y}(\alpha + \alpha^*)}. \tag{42}$$

Substituting this result into the wavefuntion yields

$$\Psi(x) = \langle x|\hat{D}(\alpha)|\phi_0\rangle = \langle x|e^{\frac{\alpha^{*2} - \alpha^2}{4}} e^{\hat{X}(\alpha - \alpha^*)} e^{-i\hat{Y}(\alpha + \alpha^*)}|\phi_0\rangle = e^{\frac{\alpha^{*2} - \alpha^2}{4}} e^{x(\alpha - \alpha^*)} \langle x|e^{-i\hat{Y}(\alpha + \alpha^*)}|\phi_0\rangle \tag{43}$$

where the eigenvalue relation $\hat{X}|x\rangle = x|x\rangle$ was used. The operator $e^{-sP/\hbar}$ is the translation operator of s in the x direction (see Cohen-Tannoudji's treatment in complement $E_I I$ [2]). The wavefunction is then written as

$$\Psi(x) = e^{\frac{\alpha^{*2} - \alpha^2}{4}} e^{x(\alpha - \alpha^*)} \langle x - i(\alpha + \alpha^*)|\phi_0\rangle = e^{\frac{\alpha^{*2} - \alpha^2}{4}} e^{x(\alpha - \alpha^*)} \phi_0(x - i(\alpha + \alpha^*)) \tag{44}$$

which has the form of a displacement in the x-quadrature.

E. problems

1. Is there an eigenstate of the creation operator?

2. What is the variance of $\hat{N}$ for the displaced Fock state. In other words, what is the variance of $\hat{N}$ in the $\hat{D}(\alpha)|1\rangle$ state?

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[1] Loudon, R. "The Quantum Theory of Light", 3rd edn (Oxford, New York, 2000)

[2] Cohen-Tannoudji, C. "Quantum Mechanics", Complement G_V (John Wiley and Sons, New York,)