Separability Criterion for Continuous Variables

Following the outline of the Duan *et al* [1] criterion for inseparability, we show that momentum-position correlated photons of parametric down conversion are entangled or more correctly inseparable. A two-particle state is considered separable if and only if the density matrix of the total system can be written in the following decomposition:

$$\rho = \sum_i p_i(\rho_{i1} \otimes \rho_{i2}) \tag{1}$$

where ρ_{i1} and ρ_{i2} are states of particles one and two respectively and $\sum_i p_i = 1$. We define EPR operators

$$x_{12} = x_1 - x_2, \tag{2}$$

$$p_{12} = p_1 + p_2 \tag{3}$$

where x_1, x_2 and p_1, p_2 are position and momentum variables of particles 1 and 2 respectively. The variance of x_{12} can be found by using the decomposition in eqn. 1, namely

$$\langle(\Delta x_{12})^2\rangle_\rho = \langle(x_{12})^2\rangle_\rho + \langle(x_{12})\rangle_\rho^2 \tag{4}$$

where we recall

$$\langle(x_{12})^2\rangle_\rho = Tr[\sum_i p_i(\rho_{i1} \otimes \rho_{i2})x_{12}^2] \tag{5}$$

$$= \sum_i p_i Tr[(\rho_{i1} \otimes \rho_{i2})x_{12}^2] \tag{6}$$

$$= \sum_i p_i\langle(x_{12})^2\rangle_i \tag{7}$$

This means that

$$\langle(\Delta x_{12})^2\rangle_\rho = \sum_i p_i \left(\langle(x_{12})^2\rangle_i\right) - \langle(x_{12})\rangle_\rho^2 \tag{8}$$

Moving on, we find

$$\langle(\Delta x_{12})^2\rangle_\rho = \sum_i p_i \left(\langle(x_{12})^2\rangle_i\right) - \langle(x_{12})\rangle_\rho^2 \tag{9}$$

$$= \sum_i p_i \left(\langle(x_1)^2 - x_1x_2 - x_2x_1 + (x_2)^2\rangle_i\right) - \langle(x_{12})\rangle_\rho^2 \tag{10}$$

$$= \sum_i p_i \left(\langle(x_1)^2\rangle_i + \langle(x_2)^2\rangle_i - 2\langle x_1\rangle_i\langle x_2\rangle_i\right) - \langle(x_{12})\rangle_\rho^2 \tag{11}$$

We wish to write the variance of the two particle states in terms of the single particle variances so that it is straightforward for measurement. Then,

$$\begin{aligned}\langle(\Delta x_{12})^2\rangle_\rho &= \sum_i p_i \left(\langle(x_1)^2\rangle_i - \langle x_1\rangle_i^2 + \langle x_1\rangle_i^2 + \langle(x_2)^2\rangle_i - \langle x_2\rangle_i^2 + \langle x_2\rangle_i^2 - 2\langle x_1\rangle_i\langle x_2\rangle_i\right) - \langle(x_{12})\rangle_\rho^2 \\ &= \sum_i p_i \left(\langle(\Delta x_1)^2\rangle_i + \langle(\Delta x_1)\rangle_i^2 + \langle x_1\rangle_i^2 + \langle x_2\rangle_i^2 - 2\langle x_1\rangle_i\langle x_2\rangle_i\right) - \langle(x_{12})\rangle_\rho^2 \quad (12) \\ &= \sum_i p_i \left(\langle(\Delta x_1)^2\rangle_i + \langle(\Delta x_1)\rangle_i^2\right) + \sum_i p_i \left(x_{12}\rangle_i^2\right) - \left(\sum_i p_i (x_{12}\rangle_i)\right)^2 \quad (13)\end{aligned}$$

Using the Cauchy Schwartz inequality $\sum_i p_i \sum_i p_i \left(x_{12}\rangle_i^2\right) - \left(\sum_i p_i(x_{12}\rangle_i)\right)^2 \geq 0$, then,

$$\langle(\Delta x_{12})^2\rangle_\rho \geq \sum_i p_i \left(\langle(\Delta x_1)^2\rangle_i + \langle(\Delta x_2)\rangle_i^2\right) \quad (14)$$

A similar result is achieved for momentum entangled states,

$$\langle(\Delta p_{12})^2\rangle_\rho \geq \sum_i p_i \left(\langle(\Delta p_1)^2\rangle_i + \langle(\Delta p_2)\rangle_i^2\right) \quad (15)$$

Duan *et al* at this point summed the variances of the momentum-like and position-like observables. This is a reasonable approach to take for squeezed states, because the quadrature observables have the same dimensions. However, this is not satisfactory for a true momentum-position analysis, because of the obvious difficulty with dimensions. Therefore, we take the product of the variances to determine the inequality, which ameliorates this problem and has a very satisfying result. The product of the momentum and position variances for separable states is then given by

$$\langle(\Delta x_{12})^2\rangle_\rho\langle(\Delta p_{12})^2\rangle_\rho \geq \left(\sum_i p_i(\langle(\Delta x_1)^2\rangle_i + \langle(\Delta x_2)^2\rangle_i)\right)\left(\sum_j p_j(\langle(\Delta p_1)^2\rangle_j + \langle(\Delta p_2)^2\rangle_j)\right) \quad (16)$$

Down conversion is symmetric such that we can assume $\langle(\Delta x_1)^2\rangle_i = \langle(\Delta x_2)^2\rangle_i$ and $\langle(\Delta p_1)^2\rangle_i = \langle(\Delta p_2)^2\rangle_i$. This assumption, while it simplifies the calculation, is not necessary to achieve the *same* final result. In fact, the symmetric assumption, while valid under a simplified form of the downconversion of the Hamiltonian, is actually the minimum uncertainty state, which means that any other possibility only strengthens the inequality. Hence,

$$\langle(\Delta x_{12})^2\rangle_\rho\langle(\Delta p_{12})^2\rangle_\rho \geq 4\left(\sum_i p_i(\langle(\Delta x_1)^2\rangle_i)\right)\left(\sum_j p_j(\langle(\Delta p_1)^2\rangle_j)\right). \quad (17)$$

Using the Cauchy-Schwartz inequality

$$\left(\sum_i p_i \langle(\Delta x_1)^2\rangle_i\right)\left(\sum_j p_j \langle(\Delta p_1)^2\rangle_j\right) \geq \left(\sum_i p_i \sqrt{\langle(\Delta x_1)^2\rangle_i \langle(\Delta p_1)^2\rangle_i}\right)^2 \qquad (18)$$

and the uncertainty relation $\langle(\Delta x_1)^2\rangle_i \langle(\Delta p_1)^2\rangle_i \geq \hbar^2/4$, which is Heisenberg's uncertainty relation due to noncommuting observables $[x_1, p_1] = i\hbar$ and is responsible for the diffraction limit of optical systems, leads to the separability bound

$$\langle(\Delta x_{12})^2\rangle_\rho \langle(\Delta p_{12})^2\rangle_\rho \geq \hbar^2 \qquad (19)$$

This bound represents the smallest product of variances which can be achieved by separable states. This is a sufficient condition for inseparability, so that any two-particle system which violates this bound is entangled under the assumption that quantum mechanics is complete.

[1] L.M. Duan, Phys. Rev. Lett. **84**, 2722 (2000)

[2] C. Silberhorn, Phys. Rev. Lett. **86**, 4267 (2001)