Continuously Entangled EPR States

UNCERTAINTY RELATIONS

Consider a particle with a position wavefunction

$$|\Psi\rangle = |x\rangle \tag{1}$$

The eigenvalue equation is

$$\hat{x}|x\rangle = x|x\rangle \tag{2}$$

which implies

$$\hat{x}^2|x\rangle = x^2|x\rangle \tag{3}$$

The variance of the position is then found to be

$$(\Delta x)^2 = \langle x|\hat{x}^2|x\rangle - \langle x|\hat{x}|x\rangle^2 = x^2 - x^2 = 0 \tag{4}$$

where it was assumed that $\langle x|x\rangle = 1$. The uncertainty in the position vanishes and thus the position of the particle is perfectly known. To compute the variance of the momentum we apply the continuous version of the closure relation $\int_{-\infty}^{\infty} |k\rangle\langle k| dk = 1$ to obtain

$$|\Psi\rangle = \int_{-\infty}^{\infty} |k\rangle\langle k|x\rangle dk \tag{5}$$

Using the fact that $\langle k|x\rangle = \frac{1}{\sqrt{2\pi}}e^{ikx}$ then

$$|\Psi\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |k\rangle e^{ikx} dk \tag{6}$$

The eigenvalue equation for momentum is also given by

$$\hat{k}|k\rangle = k|k\rangle \tag{7}$$

which implies

$$\hat{k}^2|k\rangle = k^2|k\rangle \tag{8}$$

Hence,

$$k|\Psi\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k|k\rangle e^{ikx} dk \tag{9}$$

which means that

$$\langle\Psi|\hat{k}|\Psi\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle k'|\hat{k}|k\rangle e^{ikx}e^{-ik'x}dkdk' \tag{10}$$

Recall that $\langle k'|k\rangle = \delta(k'-k)$ which yields

$$\langle\Psi|\hat{k}|\Psi\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k'dk' = 0 \tag{11}$$

since it is an odd function integrated from negative to positive infinities. On the other hand,

$$\langle\Psi|\hat{k}^2|\Psi\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k'^2dk' = \infty \tag{12}$$

since it is an even function integrated between negative to positive infinity. Hence, the variance of the momentum is infinite. This satisfies our intuition for the uncertainty principle.

ENTANGLED STATES

Now consider the state of the form

$$|\Psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(x_1,x_2)|x_1\rangle|x_2\rangle dx_1dx_2 \tag{13}$$

where $A(x_1,x_2)$ cannot be separated into a function of x_1 and a function of x_2. The simplest and most interesting example of a two-particle correlation function is the Dirac delta function. Suppose,

$$|\Psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta x_1 - x_2|x_1\rangle|x_2\rangle dx_1dx_2 \tag{14}$$

$$= \int_{-\infty}^{\infty} |x_1\rangle|x_1\rangle dx_1 \tag{15}$$

It should be noted that each ket represents a particle in a unique quantum state. So, even though they have the same values in their kets, they have different quantum states.

Now consider a measurement of the first particle. Suppose, that we are able to make a measurement of the first particle so that we know the position of the first particle to within an infinitessimally small position. The measurement is given by the position operator $|M\rangle = |x'\rangle$. Hence, the measurement yields

$$|\Psi_p\rangle = \langle M|\Psi\rangle \tag{16}$$

$$= \int_{-\infty}^{\infty}\langle x'|x_1\rangle|x_1\rangle dx_1 \tag{17}$$

$$= \int_{-\infty}^{\infty}\delta(x'-x_1)|x_1\rangle dx_1 \tag{18}$$

$$= |x'\rangle \tag{19}$$

Hence, a measurement of the position of the first particle within an infinitessimally small diameter denoted by x' projects the other particle into an infinitessimally small diameter a x'. Interestingly, the momentum variance of both particles is now infinite as was derived in the first section.

Entanglement Symmetry

Instead of choosing to perform a projective measurement in position, consider the effect of applying the momentum closure relations to both particles.

$$|\Psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |k_1\rangle\langle k_1|x_1\rangle|k_2\rangle\langle k_2|x_1\rangle dx_1 dk_1 dk_2 \tag{20}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |k_1\rangle|k_2\rangle e^{ik_1x_1}e^{ik_2x_1} dx_1 dk_1 dk_2 \tag{21}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(k_1 - k_2)|k_1\rangle|k_2\rangle dk_1 dk_2 \tag{22}$$

$$= \int_{-\infty}^{\infty} |k_1\rangle| - k_1\rangle dk_1 \tag{23}$$

Hence, the momentum correlation function is also a Delta function. However, this time the momenta of the invidividual particles is anticorrelated. If a measurement of the momentum of the first particle is taken then so that it can be determined to within a very small momentum distribution dk, then

$$\langle k'|\Psi_k\rangle = \int_{-\infty}^{\infty} |\langle k'|k_1\rangle| - k_1\rangle dk_1 \tag{24}$$

$$= | - k'\rangle \tag{25}$$

Using the momentum eigenvalue equations the variance is then found to be

$$(\Delta k)^2 = \langle \hat{k}^2\rangle - \langle \hat{k}\rangle^2 = 0 \tag{26}$$

This represents the Einstein Podolsky Rosen paradox. EPR assumed that if two distant particle did not interact then a measurement of one would have no effect on the measurement of the other. If this assumption of local realism were true, then the projective measurement would mean that a measurement would not actually project the other particle into a well defined state. The EPR paradox is mathematically written as

$$(\Delta x_2|_{x_1})^2(\Delta k_2|_{k_1})^2 < \frac{1}{4} \tag{27}$$

which means that the variances of particle 2 are computed after performing conditional measurements of the position or momentum of the first particle. If local realism were true, EPR would be correct and the variance product would be bounded by the original uncertainty principle.