

Peres Separability Criterion

While Bell's inequalities was the first quantitative means of determining the separability of a state, it is by no means the optimal method for determining or detecting entanglement of nonseparable states. A stronger method for detecting entanglement, at least for spin-1/2 states, is the Peres' separability criterion.

Before studying Peres' criterion, let us review two very important properties of linear algebra for two-dimensional states. The first property is that the eigenvalues of the matrix are nonnegative. To prove this consider a density matrix of the form

$$\rho = \begin{pmatrix} \alpha^* \alpha & \alpha^* \beta \\ \beta^* \alpha & \beta^* \beta \end{pmatrix} \quad (1)$$

To determine the Eigenvalues we take the determinant of the characteristic equation $|\rho - \lambda \mathbf{1}| = 0$, which yields

$$(|\alpha|^2 - \lambda)(|\beta|^2 - \lambda) - |\alpha|^2 |\beta|^2 = 0. \quad (2)$$

After simplifying, the eigenvalues are then found to be $\lambda = 0$ and $\lambda = |\alpha|^2 + |\beta|^2 = 1$ for all amplitudes α and β . Hence the eigenvalues are rotationally invariant. In other words, applying any unitary operation on the density matrix yields the identical eigenvalues. For a mixed state in which the off diagonal elements are zero, the eigenvalues are simply $|\alpha|^2$ and $|\beta|^2$, which are positive definite.

The second property which Peres exploits is the fact that the transpose of a density matrix still has the same nonnegative eigenvalues. The transposed density matrix given by

$$\rho^T = \begin{pmatrix} \alpha^* \alpha & \beta^* \alpha \\ \alpha^* \beta & \beta^* \beta \end{pmatrix} \quad (3)$$

once again has eigenvalues $\lambda = 0$ and $\lambda = |\alpha|^2 + |\beta|^2 = 1$ for all amplitudes α and β .

So far, all of the linear algebra has described a single quantum particle with spin-1/2 behavior. Now, consider two-particle states. Two separable particles may be written as the tensor product of two single particles

$$\rho_{12} = \sum_n P_n \rho_{1n} \otimes \rho_{2n} \quad (4)$$

Since the individual particles have nonnegative eigenvalues, then the two-particle density matrix must also have all nonnegative eigenvalues.

Peres then defines a new matrix σ , which is constructed by taking the partial transpose of only one particle in the two-particle density matrix. For separable states given by eqn. 5 then σ is given by

$$\sigma_{12} = \sum_n P_n (\rho_{1n})^T \otimes \rho_{2n} \quad (5)$$

Drawing upon the two properties for single particle density matrices, σ_{12} must then also have all nonnegative eigenvalues. It is sufficient then to say that a state is nonseparable if it has at least one negative eigenvalue.

As an example, we will consider the Werner state which is the singlet state with noise of the form

$$\rho_{12} = \frac{p}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{(1-p)}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6)$$

Taking the partial transpose yields

$$\sigma_{12} = \frac{p}{2} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \frac{(1-p)}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1-p}{4} & 0 & 0 & \frac{-p}{2} \\ 0 & \frac{1+p}{4} & 0 & 0 \\ 0 & 0 & \frac{1+p}{4} & 0 \\ \frac{-p}{2} & 0 & 0 & \frac{1-p}{4} \end{pmatrix} \quad (7)$$

The deterministic equation is then found to be

$$\left(\frac{1-p}{4} - \lambda\right)^2 \left(\frac{1+p}{4} - \lambda\right)^2 - \left(\frac{p}{4}\right)^2 \left(\frac{1+p}{4} - \lambda\right)^2 = 0 \quad (8)$$

which after some algebra yields 3 eigenvalues of $\frac{1+p}{4}$ and one eigenvalue of $\frac{1-3p}{4}$. Then as long as $p > \frac{1}{3}$ then the entanglement can be “detected”. This represents both a necessary and sufficient condition for nonseparability, given a spin-1/2 Werner state.

The Peres criterion for separability has been shown to be valid for 2×2 and 2×3 composite systems. However, for higher dimensional systems, states exist which have all positive eigenvalues, but which are nonseparable. These states are called positive partial transpose (PPT) states. The entanglement of these states cannot be extracted for quantum information however. This type of entanglement is called “Bound entanglement”.

Now consider the topology of the entangled states. From previous work we have seen that no linear combination of separable states can result in an entangled state. On the other

hand a linear combination of entangled states can yield a separable state. Lastly, PPT states incorporate all separable states in addition to bound entangled states.

PROBLEMS

1. The use of spontaneous parametric downconversion, the process in which one photon splits into two photons, to produce polarization entangled pairs does not yield Werner states. When trying to produce the singlet state, the noise is of the form $|H_1, V_2\rangle\langle H_1, V_2| + |V_1, H_2\rangle\langle V_1, H_2|$. What are the eigenvalues? What does this say about the entanglement?