Quantum Cloning

Quantum cloning has been a topic of considerable interest for many years. It turns out to be quantum limit for copying an input state and is closely related to linear amplification when talking about optical cloning. It is commonly stated that it is impossible to clone. In actuallity it is possible to clone perfectly, one just needs to know the input state. As Wootters and Zurek showed it is impossible to clone any arbitrary unknown state with perfect fidelity. This became affectionately known as the "no cloning theorem", although it is more correctly called the no "perfect cloning theorem". In fact, it may be safe to say that quantum information was founded based on this theorem.

WOOTTERS-ZUREK CLONER

The Wootters-Zurek theorem goes something like this. Suppose we have a two state particle which we wish to copy. The quantum states of the particle are given by $|0\rangle_a$ and $|1\rangle_a$, where the subscript a denotes a spatio-temporal mode for the particle. Suppose initially we are given a state $|0\rangle_a$ and we have a quantum cloning machine in state $|Q\rangle_x$. The cloning machine then performs the operation

$$|0\rangle_a|Q\rangle_x \rightarrow |0\rangle_a|0\rangle_b|Q_0\rangle_x \tag{1}$$

where the subscript b denotes a unique spatio-temporal mode with respect to a. Thus, the cloning machine has made an exact copy in another place while keeping the original. Now, suppose that we are given the $|1\rangle_a$ instead. We wish to have the cloner behave as

$$|1\rangle_a|Q\rangle_x \rightarrow |1\rangle_a|1\rangle_b|Q_1\rangle_x \tag{2}$$

So far so good. These transformations are perfectly allowed by quantum mechanics. Thus, as long as we know the basis of the states, it is possible to perfectly clone. In fact, these transformations are achieved by a controlled-not gate. Now suppose that we are given a state which is a superposition of the two basis states

$$|s\rangle_a = \alpha|0\rangle_a| + \beta|0\rangle_a| \tag{3}$$

where α and β satisfy $|\alpha|^2 + |\beta|^2 = 1$. The cloning operation must then obey the following transformation

$$|s\rangle_a |Q\rangle_x \rightarrow \alpha|0\rangle_a|0\rangle_b|Q_0\rangle_x + \beta|1\rangle_a|1\rangle_b|Q_1\rangle_x \tag{4}$$

The density matrix for the output system is then given by

$$\begin{aligned} \rho &= \alpha^*\alpha|0\rangle_a|0\rangle_b|Q_0\rangle_{xx}\langle Q_0|_b\langle 0|_a\langle 0| + \alpha\beta^*|0\rangle_a|0\rangle_b|Q_0\rangle_{xx}\langle Q_1|_b\langle 1|_a\langle 1| \\ &+ \alpha^*\beta|1\rangle_a|1\rangle_b|Q_1\rangle_{xx}\langle Q_0|_b\langle 0|_a\langle 1| + \beta^*\beta|1\rangle_a|1\rangle_b|Q_1\rangle_{xx}\langle Q_1|_b\langle 1|_a\langle 1| \end{aligned} \tag{5}$$

We assume that the machine states form an orthonormal set, namely

$${}_x\langle Q_0|Q_0\rangle_x = {}_x\langle Q_1|Q_1\rangle_x = {}_x\langle Q|Q\rangle_x = 1 \tag{6}$$

and

$${}_x\langle Q_0|Q_1\rangle_x = {}_x\langle Q_0|Q\rangle_x = {}_x\langle Q_1|Q\rangle_x = 0 \tag{7}$$

We are only interested at looking at the original and the copy after the cloning. However, it can be seen that the machine states are entangled to the states of the copy and clone. To observe the copy and clone by themselves, we must trace over the machine states. The orthonormality of the machine states makes the transformation simple. The reduced density matrix of the two particle system is then given by

$$\rho_{ab}^{out} = Tr_x[\rho_{abx}^{out}] = \alpha^*\alpha|0\rangle_a|0\rangle_{bb}\langle 0_a\langle 0| + \beta^*\beta|1\rangle_a|1\rangle_{bb}\langle 1_a\langle 1| \tag{8}$$

To observe the single particle reduced density matrices, we trace over the states of the other particle. Hence,

$$\rho_a^{out} = Tr_b[\rho_{ab}^{out}] = \alpha^*\alpha|0\rangle_{aa}\langle 0| + \beta^*\beta|1\rangle_{aa}\langle 1| \tag{9}$$

and

$$\rho_b^{out} = Tr_a[\rho_{ab}^{out}] = \alpha^*\alpha|0\rangle_{bb}\langle 0| + \beta^*\beta|1\rangle_{bb}\langle 1| \tag{10}$$

It can be seen that the original and the copy have identical outputs. Unfortunately, the original has been modified relative to the input state. We will discuss two ways to measure the modification. The first method is called the "fidelity" and the second is called the Hilbert-Schmidt norm.

Fidelity

The fidelity is found by

$$F = \langle s|\rho_a^{out}|s\rangle \tag{11}$$

where $|s\rangle$ is the state of the particle before entering the cloning device. It can then be seen that the fidelity is given by

PROVE

$$F = |\alpha|^4 + |\beta|^4 \tag{12}$$

which has a maximum value of 1 for either $\alpha = 1$ or $\beta = 1$ and a minimum value of $\frac{1}{2}$ for $|\alpha| = |\beta| = \frac{1}{\sqrt{2}}$. A fidelity of 1 means that a perfect copy has been made, because the original and the copy both have the same state as the input. A fidelity of $\frac{1}{2}$ is not a copy at all, but a maximally mixed state. In fact, if we look closer at the output state in eqn. 4, it can be seen that the original, copy and machine states are maximally entangled. This implies that if one maximally entangles a state to another two state particle, one has made a perfectly bad copy. Also, we learn that a perfect copy can be made if we know the basis that a particle is in. The most important result, is that we cannot inject an arbitrary state into a cloning machine and get a perfect copy.

Hilbert-Schmidt norm

In the original Buzek-Hillery cloning paper, they used the Hilbert-Schmidt norm to measure a scalar "distance" between two density matrices. The Hilbert-Schmidt norm is defined as

$$D = Tr[(\rho_s - \rho_a^{out})^2] \tag{13}$$

where ρ_s is the density matrix of the input state

$$\begin{aligned} \rho_s &= \alpha^*\alpha|0\rangle_{aa}\langle 0| + \alpha\beta^*|0\rangle_{aa}\langle 1| \\ &+ \alpha^*\beta|1\rangle_{aa}\langle 0| + \beta^*\beta|1\rangle_{aa}\langle 1| \end{aligned} \tag{14}$$

Hence,

PROVE

$$D = 2|\alpha|^2|\beta|^2 \tag{15}$$

which has a minimum value when either $\alpha = 1, \beta = 0$ or $\beta = 1, \alpha = 0$. In this case the distance between the density matrix vanishes, which means that the original is unchanged by the cloning process. It also means that the copy is perfect, because the copy is the same as the original. The distance is maximum distance of $\frac{1}{2}$ occurs when $|\alpha| = |\beta| = \frac{1}{\sqrt{2}}$.

UNIVERSAL CLONING MACHINE

The mathematics of the universal cloning machine is considerably more involved than the Wootter-Zurek cloner. Buzek and Hillery proposed the first universal cloning machine. The idea is that a cloning machine can accept an arbitrary input state and copy it with the same fidelity for all possible states. The first experimentally realizable cloning machine was proposed by Simon *et al.* However, the proposal was different than the Buzek-Hillery cloner, because the original and copy were both in the same spatio-temporal mode. Hence, the mathmetics of the Buzek-Hillery cloner had to be neglected. Essentially, all previous proposals assumed distinguishable particles, whereas the Simon cloner assumed indistinguishable particles. After Simon's proposal, several experiments were realized.

The idea is that a seed photon, which is to be copied is injected in an optical parametric amplifier during pair production of entangled photons. Assume that the photon is initially horizontally polarized in the state

$$|s\rangle = a^{\dagger}_{hc}|0\rangle \tag{16}$$

where the hc subscript denotes a the to-be-cloned horizontally polarized photon created in mode a. The pair production state is given by

$$|\psi^{-}\rangle = \frac{1}{\sqrt{2}}(a^{\dagger}_{hd}b^{\dagger}_{v} + a^{\dagger}_{vd}b^{\dagger}_{h})|0\rangle \tag{17}$$

where the hp subscript, for example, represents a horizontally polarized photon created by It should be noticed that the original photon to be cloned and one of the photons are both created in the same mode a. Also, it is interesting to note that if one traces over the states of the photon in mode b, then the photon in mode a is a maximally mixed state, which is represented by a Bloch vector of zero length in the Bloch sphere. One can consider a Bloch vector of zero length to be a perfectly blank state (much like a white piece of paper to make a copy on). Also, the photon in mode b is the analog of the machine states discussed earlier. Without the machine state, the photon in mode a could not be completely blank. Although, this was not state, it has been shown that the machine states realize the universality of the cloning machine. The machine states, or states of the other photon or ancilla realize the universality of the cloning machine.

The state of the original and down converted pair are given by

$$|C\rangle = |s\rangle \otimes |\Psi^{-}\rangle = \frac{1}{\sqrt{2}}(a^{\dagger}_{hc}a^{\dagger}_{hd}b^{\dagger}_{v} + a^{\dagger}_{hc}a^{\dagger}_{vd}b^{\dagger}_{h})|0\rangle \tag{18}$$

If the input photon is made spatially, spectrally and temporally indistinguishable from the down converted photon in mode a, then one obtains, without worrying about normalization

$$|C\rangle = |s\rangle \otimes |\Psi^{-}\rangle = ((a^{\dagger}_{h})^{2}b^{\dagger}_{v} + a^{\dagger}_{h}a^{\dagger}_{v}b^{\dagger}_{h})|0\rangle \tag{19}$$

$$= \sqrt{2}|2,0;0,1\rangle + |1,1;1,0\rangle \tag{20}$$

The clone thus experiences a boson mode occupation enhancement for the term which has the same polarization and no enhancement for the orthogonal polarization. The fidelity is determined by using photon counting. One simply counts the number of photons weighted by their probability of occuring. The fidelity is then the number of photons with the polarization of the original divided by total of all possible events

$$F = \frac{2 \times 2 + 1 \times 1}{2 \times 2 + 2 \times 1} = \frac{5}{6} \tag{21}$$

This requires a basis-dependent verification.

PROBLEM

1. Phase covariant cloner. Show that if one interferes a circularly polarized photon with a linear polarized photon and postselects only two photon events in one arm of the beam splitter, that all possible linear polarization states will be cloned with 5/6 fidelity.