III. SQUEEZED STATES

Before jumping into squeezed states, it is useful to give a little background in coherence theory and refresh our memory of the quantized electric field. A brief discussion will be given of first and second order coherence followed by their definition. A more thorough treatment will be given later when discussing interferometers such as the Mach-Zehnder interferometer.

A. First and Second Order Coherence

Coherence, as described here, is a measure of how well a field of a given space-time coordinate will interfere with a field at another space-time coordinate. Further, in most cases, we will be studying single mode quantum optics, which further reduces coherence to simply temporal coherence. In other words, we are interested in the mode overlap between temporally separated quantum states. For example, suppose a gaussian wavepacket of light with longitudinal spatial width σ is propagating in a given direction and another gaussian wavepacket of light with longitudinal spatial width σ having the same spectrum of frequencies is propagating in the same direction. The peak of the two gaussians is separated by a distance τ. As long as $\tau >> \sigma$ a high degree of interference will occur. If $\sigma >> \tau$ a low degree of interference will occur. Maximum interference obviously occurs when $\tau = 0$, when the wavepackets have maximum overlap. The factors $g^{(1)}$ and $g^{(2)}$, which will be defined below, quantify the amount of coherence or interference. The quantum mechanical degree of first and second order coherence are respectively defined as

$$g^{(1)}(\tau) = \frac{\langle \hat{E}^{-}(t)\hat{E}^{+}(t+\tau)\rangle}{\langle \hat{E}^{-}(t)\hat{E}^{+}(t)\rangle} \tag{45}$$

and

$$g^{(2)}(\tau) = \frac{\langle \hat{E}^{-}(t)\hat{E}^{-}(t+\tau)\hat{E}^{+}(t+\tau)\hat{E}^{+}(t)\rangle}{\langle \hat{E}^{-}(t)\hat{E}^{+}(t)\rangle^{2}} \tag{46}$$

B. Electric Field Operator $\hat{E}$

The electric field operator $\hat{E}$ is an observable of great interest and plays an important role in understanding the many different types of optical fields. For example, the coherence functions are defined in terms of the quantum mechanical electric field operators $\hat{E}^{-}$ and

$\hat{E}^{+}$, which are the positive and negative frequency components of the electromagnetic field operator $\hat{E}$. Following the notation of [1], the single-mode quantum mechanical electric field is defined as

$$\hat{E} = \hat{E}^{+} + \hat{E}^{-} = \frac{1}{2}\hat{a}e^{-i\chi} + \frac{1}{2}\hat{a}^{\dagger}e^{i\chi} \tag{47}$$

where the position and time information is nested in the phase factor χ. It can be seen that for $\chi = 0$, $\hat{E}(\chi = 0) = \hat{X}$ and for $\chi = \pi/2$, $\hat{E}(\chi = \pi/2) = \hat{Y}$. Thus,

$$\hat{E} = \hat{X}\cos(\chi) + \hat{Y}\sin(\chi) \tag{48}$$

The electric field variance is a measure of the noise of the field. As an example, for a number state $|n\rangle$ the variance c

$$\langle\hat{E}\rangle = \langle\frac{1}{2}\hat{a}e^{-i\chi} + \frac{1}{2}\hat{a}^{\dagger}e^{i\chi}\rangle = 0 \tag{49}$$

and

$$\langle\hat{E}^{2}\rangle = \frac{1}{4}\langle\hat{a}^{2}e^{-i2\chi} + \hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a} + \hat{a}^{\dagger 2}e^{i2\chi}\rangle = \frac{1}{4}\langle(\hat{a}^{\dagger}\hat{a}+1) + \hat{a}^{\dagger}\hat{a}\rangle = \frac{1}{4}\left(2n+1\right). \tag{50}$$

The expectation value of both quadrature operators is zero and the electric field variance goes as $\frac{1}{4}(2n+1)$. The pictorial representation is an infinitessimally thin circle drawn in the x-y quadrature plane having radius $\sqrt{\frac{1}{4}(2n+1)}$ from the origin. Thus, the expectation value of the electric field operator $\langle\hat{E}(\chi)\rangle$ is the magnitude of the vector in the x-y quadrature plane, $Noise = \langle\hat{E}(\chi)\rangle$ gives the uncertainty in the electric field vector and χ is a relative phase angle (relative to the phase of a local oscillator in homodyne detection).

With these intuitive definitions in mind, we define a value $S = \langle\hat{E}(\chi)\rangle$, which gives the length of the vector in the quadrature plane. This S value is referred to as the signal. The signal to noise ratio $\frac{S^2}{Noise}$ is a measure of the information carrying capacity of the field in homodyne detection.

The first order coherence can be determined if the functional form of the field is known. In the case that $\tau = 0$, the first order coherence is

$$g^{(1)} = \frac{\langle\hat{a}^{\dagger}\hat{a}\rangle}{\langle\hat{a}^{\dagger}\hat{a}\rangle} = 1 \tag{51}$$

which means that for a single spatial mode having two wavepackets with perfect overlap yields perfect interference as expected (it should be noted that $0 \leq g^{(1)} \leq 1$ with 1 being the maximum coherence). Under the same assumptions the second-order coherence is given by

$$g^{(2)} = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} = \frac{\langle \hat{a}^\dagger (\hat{a}\hat{a}^\dagger - 1)\hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} = \frac{\langle \hat{a}^\dagger \hat{a}\hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} = \frac{\langle \hat{a}^\dagger \hat{a}\hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} \tag{52}$$

In terms of number states, we obtain

$$g^{(2)} = \frac{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle}{\langle \hat{N} \rangle^2} \tag{53}$$

Using the relation $(\Delta N^2) = \langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2$ gives

$$g^{(2)} = \frac{(\Delta N^2) + \langle \hat{N} \rangle^2 - \langle \hat{N} \rangle}{\langle \hat{N} \rangle^2} \tag{54}$$

For number states $(\Delta N^2) = 0$ since there is no uncertainty in the number of photons in the field by definition of a number state. Hence,

$$g^{(2)} = 1 - \frac{1}{\langle \hat{N} \rangle} \tag{55}$$

Interestingly, the second order coherence vanishes for the one photon $|1\rangle$ state. The second order coherence is thus a measure of the reliability of a single photon source. Physically it means that it is impossible to measure a single photon in two places at the same time. This is often referred to as a photon antibunching experiment.

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C. Problems

1. What is the signal to noise ratio for a coherent state $|\alpha\rangle$? Draw a quadrature space diagram of this setup.
2. What are the first and second order coherence values for a coherent state?
3. Prove $\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha$, which represents an energy displacement by α.

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[1] Loudon, R. “The Quantum Theory of Light”, 3rd edn (Oxford, New York, 2000)

[2] Cohen-Tannoudji, C. “Quantum Mechanics”, Complement G_V (John Wiley and Sons, New York,)