Quantum Fields

I. SQUEEZED VACUUM

The squeezed states are called as such because the quadrature variance of one quadrature is smaller than $\frac{1}{4}$. This reduced or squeezed quadrature means that the variance of the other quadrature must be larger than $\frac{1}{4}$ such that the variance product of the two quadratures is still governed by the Heisenberg uncertainty relation. In pictorial form then it looks as if the phase dependent electric field variance is "squeezed". The squeeze operator is defined

$$\hat{S}(s,\theta) = e^{\left(\frac{1}{2}se^{-i\theta}\hat{a}^2 - \frac{1}{2}se^{i\theta}\hat{a}^{\dagger 2}\right)} \tag{1}$$

This operator, as will be discussed later, is created by degenerate parametric down conversion. As can be seen this operator is unitary:

$$\hat{S}(s,\theta)\hat{S}^{\dagger}(s,\theta) = \hat{S}^{\dagger}(s,\theta)\hat{S}(s,\theta) = 1 \tag{2}$$

PROVE that the operators $\hat{a}^2$ and $\hat{a}^{\dagger 2}$ do not commute with their first order commutator.

With this being the case a more general operator ordering is required to isolate the destruction operators from the creation operators. Using more general operator ordering, the squeeze operator can be written as

$$\hat{S}(s,\theta)|0\rangle = (sech(s))^{1/2}\sum_{n=0}^{\infty}\frac{[(2n)!]^{1/2}}{n!}\left[-\frac{1}{2}e^{i\theta}tanh(s)\right]^n|2n\rangle \tag{3}$$

The simplest means to solve the variance of many operators is achieved by using the following trick:

$$\langle\hat{N}\rangle = \langle\hat{a}^{\dagger}\hat{a}\rangle = \langle 0|\hat{S}^{\dagger}\hat{a}^{\dagger}\hat{S}\hat{S}^{\dagger}\hat{a}\hat{S}|0\rangle \tag{4}$$

where we drew upon the unitarity of the squeeze operator to make the symmetric form. Consider the following relation

$$\begin{aligned} e^{-\hat{B}}\hat{a}e^{\hat{B}} &= (1-\hat{B}+\frac{1}{2!}\hat{B}^2-\frac{1}{3!}\hat{B}^3+\ldots)\hat{a}(1+\hat{B}+\frac{1}{2!}\hat{B}^2+\frac{1}{3!}\hat{B}^3+\ldots) \\ &= \hat{a}+\hat{a}\hat{B}-\hat{B}\hat{a}+\frac{1}{2}\hat{a}\hat{B}^2-\hat{B}\hat{a}\hat{B}+\frac{1}{2}\hat{B}^2\hat{a}+\ldots = \hat{a}+[\hat{a},\hat{B}]+\frac{1}{2!}[[\hat{a},\hat{B}],\hat{B}]+\ldots \end{aligned} \tag{5}$$

FIND the next order term in the expansion. It can be seen that if we let $e^{\hat{B}}$ equal the squeeze operator then

$$\hat{S}^{\dagger}\hat{a}^{\dagger}\hat{S} = \hat{a}+[\hat{a},\left(\frac{1}{2}se^{-i\theta}\hat{a}^2-\frac{1}{2}se^{i\theta}\hat{a}^{\dagger 2}\right)]+\ldots \tag{6}$$

Evaluating the first order commutation relation thus yields

$$\left[\hat{a}, \left(\frac{1}{2}se^{-i\theta}\hat{a}^2 - \frac{1}{2}se^{i\theta}\hat{a}^{\dagger 2}\right)\right] = \frac{-1}{2}se^{i\theta}[\hat{a}, \hat{a}^{\dagger 2}] = \frac{-1}{2}se^{i\theta}(\hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] + [\hat{a}, \hat{a}^\dagger]\hat{a}^\dagger) = se^{i\theta}\hat{a}^\dagger \tag{7}$$

Find the next two terms in the expansion and show

$$\hat{S}^\dagger\hat{a}^\dagger\hat{S} = \hat{a}^\dagger cosh(s) - \hat{a}e^{-i\theta}sinh(s) \tag{8}$$

The Hermitian conjugate is thus

$$\hat{S}^\dagger\hat{a}\hat{S} = \hat{a}cosh(s) - \hat{a}^\dagger e^{i\theta}sinh(s) \tag{9}$$

The expectation value of the number operator is then

$$\begin{aligned}\langle\hat{N}\rangle &= \langle 0|\hat{S}^\dagger\hat{a}^\dagger\hat{S}\hat{S}^\dagger\hat{a}\hat{S}|0\rangle = \langle(\hat{a}^\dagger cosh(s) - \hat{a}e^{-i\theta}sinh(s))(\hat{a}cosh(s) - \hat{a}^\dagger e^{i\theta}sinh(s))\rangle \\ &= \langle\hat{a}^\dagger\hat{a}cosh^2(s) - \hat{a}^\dagger\hat{a}^\dagger e^{i\theta}cosh(s)sinh(s) - \hat{a}\hat{a}e^{-i\theta}sinh(s)cosh(s) + \hat{a}\hat{a}^\dagger sinh^2(s)\rangle \\ &= \langle\hat{a}^\dagger\hat{a}cosh^2(s) - \hat{a}^\dagger\hat{a}^\dagger e^{i\theta}cosh(s)sinh(s) - \hat{a}\hat{a}e^{-i\theta}sinh(s)cosh(s) + (\hat{a}^\dagger\hat{a} + 1)sinh^2(s)\rangle\end{aligned} \tag{10}$$

Only the last term on the right survives when operating on the vacuum state and we are left with

$$\langle\hat{N}\rangle = sinh^2(s) \tag{11}$$

Following a similar procedure, *PROVE* that

$$\langle\hat{N}^2\rangle = 3sinh^4(s) + 2sinh^2(s). \tag{12}$$

The degree of second order coherence is

$$g^{(2)}(\tau) = 1 + \frac{(\Delta\hat{N})^2 - \langle\hat{N}\rangle}{\langle\hat{N}\rangle^2} = 3 + \frac{1}{\langle\hat{N}\rangle}. \tag{13}$$

The expectation value of the electric field is

$$\begin{aligned}\langle\hat{E}\rangle &= \frac{1}{2}\langle 0|\hat{S}^\dagger\hat{a}\hat{S}e^{-i\chi} + \hat{S}^\dagger\hat{a}^\dagger\hat{S}e^{i\chi}|0\rangle \\ &= \langle 0|(\hat{a}^\dagger cosh(s) - \hat{a}e^{-i\theta}sinh(s))e^{-i\chi} + (\hat{a}cosh(s) - \hat{a}^\dagger e^{i\theta}sinh(s))e^{i\chi}|0\rangle \\ &= 0\end{aligned} \tag{14}$$

Thus, the mean electric field is zero.

The electric field variance is

$$\langle\hat{E}^2\rangle = \frac{1}{4}\langle\hat{a}^2e^{-2i\chi} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^{\dagger 2}e^{2i\chi}\rangle \tag{15}$$

We have already found the expectation value of

$$\langle \hat{a}^{\dagger}\hat{a}\rangle = sinh^2(s) \tag{16}$$

Accordingly

$$\langle \hat{a}\hat{a}^{\dagger}\rangle = cosh^2(s) \tag{17}$$

Prove

$$\langle \hat{a}\hat{a}\rangle = -e^{i\theta} sinh(s) cosh(s) \tag{18}$$

and

$$\langle \hat{a}^{\dagger}\hat{a}^{\dagger}\rangle = -e^{-i\theta} sinh(s) cosh(s) \tag{19}$$

Hence,

$$\langle \hat{E}^2\rangle = \frac{1}{4}(-e^{i\theta}e^{-2i\chi}\left(\frac{e^{2s}-e^{-2s}}{4}\right) + \frac{e^{2s}+e^{-2s}}{2} - e^{-i\theta}e^{2i\chi}\left(\frac{e^{2s}-e^{-2s}}{4}\right)) \tag{20}$$

where we used

$$sinh(s) = \frac{e^{s}-e^{-s}}{2} \tag{21}$$

and

$$cosh(s) = \frac{e^{s}+e^{-s}}{2} \tag{22}$$

The expectation value of the electric field squared is then

$$\langle \hat{E}^2\rangle = \frac{1}{4}(e^{2s}\sin^2(\chi - \frac{1}{2}\theta) + e^{-2s}\cos^2(\chi - \frac{1}{2}\theta)) \tag{23}$$

which is the equation of an ellipse with a circle drawn out for $s = 0$. This last term thus represents the noise or the time-averaged uncertainty in the electric field vector. The semiminor axis and semimajor axis have lengths $\frac{e^{-s}}{2}$ and $\frac{e^{s}}{2}$. Perfect squeezing occurs in the limit that $s \to \infty$. It is difficult to achieve even 5 dB of squeezing. To my knowledge, the best squeezing has been around 7 dB.

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[1] Loudon, R. "The Quantum Theory of Light", 3rd edn (Oxford, New York, 2000)

[2] Cohen-Tannoudji, C. "Quantum Mechanics", Complement G_V (John Wiley and Sons, New York,)