Lecture #15

$$\sum_i n(E_i) = \int dE\, g(E)\, n(E)$$



Let us make a quantitative estimate of the classical limit for a system of free particles in a volume V (3D)

$$\langle N_\pm \rangle = \int dE\; g(E)\, n_\pm(E)$$

occupation number; $\pm$ — Fermi, — Bose

density of single-particle states

$$g(E) = g\int \frac{d^3x\, d^3p}{h^3}\, \delta\left(E - \frac{p^2}{2m}\right) = \frac{gV}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} (E)^{1/2}$$

spin degeneracy

$$\therefore \langle N_\pm \rangle = \int dE\; g(E)\, \frac{1}{e^{\beta(E-\mu)} + a}, \quad a = \pm 1, \; \begin{matrix} + & FD \\ - & BE \end{matrix}$$

$$\langle N_\pm \rangle = \frac{gV}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty E^{1/2} dE\; e^{\beta\mu - \beta E}\left(1 - a\, e^{\beta\mu - \beta E} + \dots\right)$$

Assume $e^{\beta\mu} \ll 1$

$$= \frac{gV}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \left\{ e^{\beta\mu} \frac{\pi^{1/2}}{2\beta^{3/2}} \left[ 1 - \frac{a\, e^{\beta\mu}}{2\sqrt{2}} + \dots \right] \right\}$$

$$\therefore \langle N_\pm \rangle \simeq gV \left[\frac{mkT}{2\pi\hbar^2}\right]^{3/2} e^{\beta\mu} \equiv \frac{gV}{\lambda_T^3}\, e^{\beta\mu}$$

$$\lambda_T = \left(\frac{2\pi\hbar^2}{mkT}\right)^{1/2}$$ is the "thermal wavelength"

$$\langle N_a \rangle = \frac{gV}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \left\{ e^{\beta\mu} \frac{\sqrt{\pi}}{2\beta^{3/2}} \left[ 1 - a \frac{e^{\beta\mu}}{2\sqrt{2}} + \cdots \right] \right\}$$

Recall $\lambda_T = \left(\dfrac{2\pi\hbar^2}{mkT}\right)^{\frac{1}{2}}$ is thermal wavelength

$$\langle N_a \rangle = \frac{gV}{(\lambda_T)^3} e^{\beta\mu} \left[ 1 - a \frac{e^{\beta\mu}}{2\sqrt{2}} + \cdots \right] \quad ; e^{\beta\mu} \ll 1$$

$$\Rightarrow e^{\beta\mu} = \frac{\lambda_T^3}{V/\langle N_a \rangle} \left( 1 + \frac{a}{2\sqrt{2}} e^{\beta\mu} + \cdots \right)$$

$$e^{\beta\mu} = \frac{\lambda_T^3 \langle N \rangle}{gV}$$

$$\bigstar \quad e^{\beta\mu} = \frac{\lambda_T^3}{\left(\frac{V}{\langle N_a \rangle}\right)} \left( 1 + \frac{a}{2\sqrt{2}} \frac{\lambda_T^3}{V/\langle N_a \rangle} \right)$$

$\lambda_T$ is the wavelength of a particle with energy $= \frac{\hbar^2 k^2}{2m} = \pi\, kT$ ~ thermal energy

Thus, $e^{\beta\mu} = \frac{N\lambda_T^3}{gV} = \left[\frac{\lambda_T}{\left(\frac{gV}{N}\right)^{\frac{1}{3}}}\right]^3$

is small if the thermal wavelength is smaller compared to the inter-particle spacing $\left(\frac{V}{N}\right)^{\frac{1}{3}}$ $\Rightarrow$ no quantum effects if there is no overlap between the particle wave functions.

$$\therefore\ e^{\beta\mu} = \frac{\lambda_T^3}{gV/N}\left\{1 + \frac{a}{2\sqrt{2}}\,\frac{\lambda_T^3}{gV/N} + \cdots\right\}$$

We can now compare the ideal gas law

$$PV = \frac{kT}{a}\sum_i \ln\left[1 + a\, e^{-\beta(\epsilon_i - \mu)}\right]$$

$$\simeq \frac{kT}{a}\int dE\, g(E)\, \ln\left(1 + a\, e^{-\beta(E-\mu)}\right)$$

$$PV = \frac{kT}{a}\frac{V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} g\int_0^\infty E^{\frac{1}{2}}dE\,\ln\left(1 + a\,e^{-\beta E + \mu\beta}\right)$$

Integrate by parts, expand for $e^{\beta\mu}$ small

$$\cancel{\ln\left(1 + a e^{-\beta E} e^{\beta\mu}\right) \simeq 1 + a e^{-\beta E} e^{\beta\mu} + \dots}$$

$$= \frac{kT}{a}\frac{V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} g\left[\left(\ \right)\Big|_0^\infty - \int_0^\infty \frac{\left(-\frac{2}{3}\right)E^{\frac{3}{2}}\beta a e^{-\beta(E-\mu)}}{1 + a e^{-\beta(E-\mu)}}dE\right]$$

$$= \frac{V}{6\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} g\int_0^\infty \frac{\varepsilon^{3/2} e^{\beta\mu}\,d\varepsilon}{e^{\beta E} + a e^{\beta\mu}}$$ Expand for $e^{\beta\mu} \ll 1$

: math

$$PV = \frac{gV}{\lambda_T^3} kT e^{\beta\mu}\left[1 - \frac{a e^{\beta\mu}}{4\sqrt{2}} + \dots\right]$$

sub in our expression for $e^{\beta\mu}$ in terms of $\frac{\lambda_T^3}{gV/N}$

$$\therefore PV = NkT\left\{1 + \frac{a}{4\sqrt{2}}\frac{\lambda_T^3}{gV/N} + \dots\right\}$$

$a = \begin{cases} + \text{ FD} \rightarrow P \text{ is larger} \\ - \text{ BE} \rightarrow P \text{ is smaller} \end{cases}$ first quantum correction

$\Rightarrow$ Effect of statistics is effectively repulsive for Fermions and attractive for Bosons.

---

Usefulness of the thermal de-Broglie wavelength.

$$\lambda_T \ll \left(\frac{gV}{\langle N\rangle}\right)^{\frac{1}{3}} \rightarrow \text{no quantum effects (classical)}$$

Quantum limit if $E = \frac{p^2}{2m} = \frac{\left(\frac{\hbar}{\lambda}\right)^2}{2m} = \frac{\hbar^2}{2m}\frac{1}{\lambda^2} = \pi kT$

$$\lambda_T = \left(\frac{2\pi\hbar^2}{mkT}\right)^{\frac{1}{2}} \gg \left(\frac{gV}{N}\right)^{\frac{1}{3}}$$

$$\Rightarrow kT \ll \left(\frac{N}{gV}\right)^{\frac{2}{3}} \frac{2\pi\hbar^2}{m} \equiv kT^*$$

$kT^*$ depends on both the mass and the density

Quantum effects $\longleftrightarrow$ large density or small mass.

Example: Electrons in a Metal: $\frac{N}{V} \sim \left(\frac{1}{3\text{Å}}\right)^3$
Typical densities

$g = 2$, $m = 9\times10^{-31}\,kg = .5\,MeV$

$kT^* = 3.3\,eV \Rightarrow T^* \simeq 40{,}000\,K$

$\Rightarrow$ electrons always in quantum regime

$g = 1$

Helium (4) $\frac{N}{V} = \left(\frac{1}{3.6\,\mathring{A}}\right)^3$, but $m \approx 4000$ MeV

So $kT^* = 4.7 \times 10^{-4}$ eV ; $T^* \cong 5.6$ K

→ above 5K, He behaves ~classically.

<u>Protons</u> in the interior core of the sun.

Density ~ $6 \times 10^{31}\,m^{-3}$ ; $kT^* \sim .24$ eV

$T^* \sim 2820$ K

Core temperature ~ $10 \times 10^6$ K ⇒ fairly classical sun.

(Neutron stars have $\rho \sim 10^{12}$ kg/m$^3$)

---

Focus on 3D Fermi gas (Non-interacting)

$$PV = kT \int dE\, g(E) \ln\left(1 + e^{-\beta(E-\mu)}\right)$$

$$g(E) = \frac{gV}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\sqrt{E}$$

$$\frac{P}{kT} = \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{g}{4\pi^2} \int_0^\infty dE\, E^{\frac{1}{2}} \ln\left(1 + e^{-\beta(E-\mu)}\right)$$

Integrate by Parts: $u = \ln(\,)$ ; $dv = \sqrt{E}\,dE$

$$\frac{P}{kT} = \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{g}{4\pi^2}\left[ \cancel{E^{\frac{3}{2}} \ln(\,)}\Big|_0^\infty - \int_0^\infty \frac{2}{3} E^{\frac{3}{2}} dE \left(\frac{(-\beta) z e^{-\beta E}}{1 + e^{-\beta E} \cdot z}\right)\right]$$

$$\frac{P}{kT} = \frac{2}{3kT}\left(\frac{g}{4\pi^2}\right)\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_0^\infty dE\, \frac{E^{\frac{3}{2}} e^{-\beta E} z}{1 + z e^{-\beta E}}$$

$$\lambda_T^3 = \left(\frac{2\pi\hbar^2}{mkT}\right)^{\frac{3}{2}} \longleftrightarrow \lambda_T = \frac{h}{\sqrt{2\pi m k T}}$$

define $x = \beta E$; $dx = \beta\, dE$

$$\frac{P}{kT} = \frac{2\beta}{3}\left(\frac{g}{4\pi^2}\right)\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\left(\frac{1}{\beta}\right)^{\frac{5}{2}} \int_0^\infty dx\, \frac{x^{\frac{3}{2}} e^{-x} z}{1 + z e^{-x}}$$

$$= \frac{2}{3}(kT)^{\frac{3}{2}}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{g}{4\pi^2} \int_0^\infty dx\, \frac{x^{\frac{3}{2}}}{z^{-1} e^{x} + 1}$$

$$= \frac{2}{3}\, \frac{g}{4\pi^2}\left(\frac{2mkT}{4\pi\hbar^2}\right)^{\frac{3}{2}} \cdot (4\pi)^{\frac{3}{2}} \int_0^\infty dx\, \frac{x^{\frac{3}{2}}}{z^{-1} e^{x} + 1}$$

$$\frac{P}{kT} = g\lambda_T^{-3}\, \frac{2}{3} \cdot \frac{1}{\pi}\sqrt{4\pi} \cdot \int_0^\infty dx \ldots$$

$$= \frac{g}{\lambda_T^3}\, \frac{2}{3} \cdot \frac{1}{\sqrt{\pi}} \int_0^\infty dx\, \frac{x^{\frac{3}{2}}}{z^{-1} e^{x} + 1}$$

$$\frac{P}{kT} = \frac{g}{\lambda_T^3} f_{5/2}(z)$$

sim $\frac{N}{V} = \frac{g}{\lambda_T^3} f_{3/2}(z)$

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}}{z^{-1}e^x + 1}\, dx \simeq z - \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} \ldots$$

[are Fermi-Dirac functions]

$$U = -\frac{\partial}{\partial \beta} \Xi$$