Introduction to the Transverse Spatial Correlations in Spontaneous Parametric Down-Conversion Through the Biphoton Birth Zone

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The Biphoton Wavefunction From the beginning…

$$\mathcal{H}_{EM} = \frac{1}{2}\int d^3\mathbf{r}\,(\vec{\boldsymbol{D}}\cdot\vec{\boldsymbol{E}} + \vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{H}})$$

$$\vec{\boldsymbol{D}} = \epsilon_0\vec{\boldsymbol{E}} + \vec{\boldsymbol{P}} \quad : \quad \vec{\boldsymbol{P}} = \epsilon_0\left[\chi^{(1)}\vec{\boldsymbol{E}} + \chi^{(2)}(\vec{\boldsymbol{E}})^2 + \chi^{(3)}(\vec{\boldsymbol{E}})^3 + \cdots\right]$$

With the following assumptions:

- The pump field is bright enough to be treated classically
- The pump field is not too bright compared to the Coulomb field binding electrons to atoms
  - i.e., intensities a good deal less than $\sim 10^{14}\text{W/mm}^2$
  - Note: Optical damage threshold for BBO $\sim 10^7\text{W/mm}^2$
- The pump amplitude does not significantly change over the length of the crystal
  - A.k.a. The undepleted pump approximation
- The pump field is relatively narrowband
  - The pump field's time dependence factors out (approximately)
- The pump field is paraxial
  - The longitudinal component of $\vec{k}_p$ dominates over the transverse components
  - The spatial amplitude factors in to longitudinal and transverse parts
- The nonlinear crystal in embedded in a linear material of equal refractive index
  - We can neglect internal reflections
- The nonlinear crystal is a good deal wider than the pump beam, and much wider than the pump wavelength
- And.. we examine only the nearly degenerate part of the downconverted spectrum

We can find (in 1D for a Gaussian pump beam):

$$\psi(k_+, k_-) \approx \mathcal{N} Sinc\left(\frac{L_z\lambda_p}{4\pi}k_-^2\right)e^{-2\sigma_p^2 k_+^2}$$

$$\psi(x_+, x_-) \approx \mathcal{N}\left[x_-\sqrt{2\pi}\left(\mathcal{S}\left(\frac{x_-}{\sqrt{2\pi a}}\right) - \mathcal{C}\left(\frac{x_-}{\sqrt{2\pi a}}\right)\right) + 2\sqrt{a}\left(\cos\left(\frac{x_-^2}{4a}\right) + \sin\left(\frac{x_-^2}{4a}\right)\right)\right]e^{-\frac{x_+^2}{8\sigma_p^2}}$$

$$a \equiv \frac{L_z\lambda_p}{4\pi} \quad : \quad x_+ \equiv \frac{x_1 + x_2}{\sqrt{2}} \quad x_- \equiv \frac{x_1 - x_2}{\sqrt{2}} \quad : \quad \sigma_p \equiv \sigma_{\left(\frac{x_1+x_2}{2}\right)}$$

$(x_1, x_2) \equiv$(signal,idler) photon positions

$$\mathcal{C}(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt \quad : \mathcal{S}(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt$$

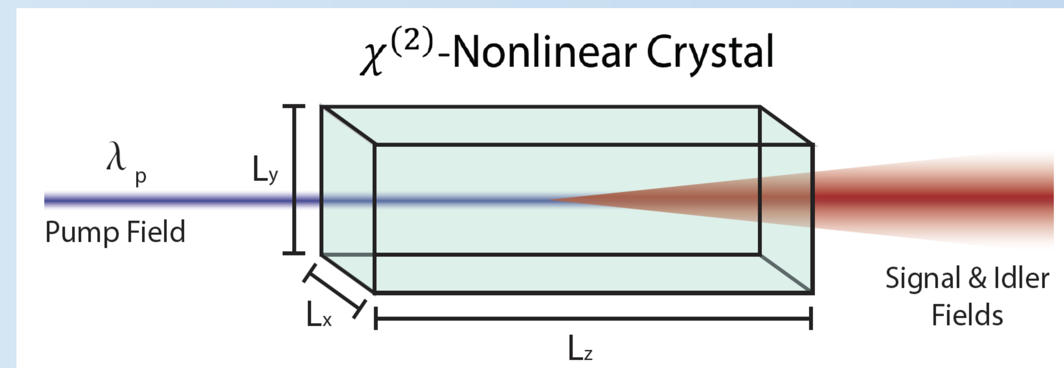


Fig. 1: Depiction of second-order nonlinear crystal axis. The direction of propagation here is the z-axis, while the transverse components are in the x and y directions. The z-axis is also called the optic axis.

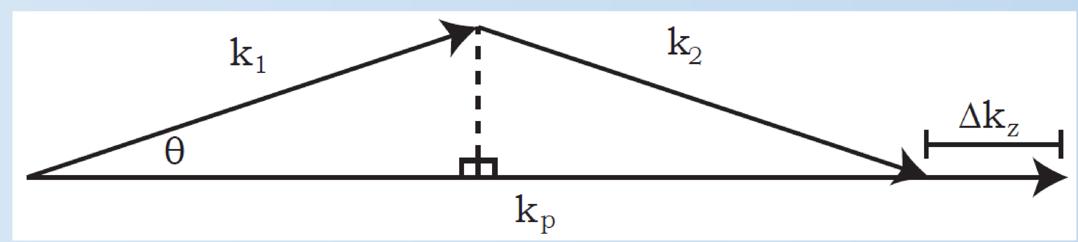


Fig. 2: Phase-Matching relationship for nearly collinear downconverted light. When θ is small, we may use the small-angle approximation to get a simple expression for our biphoton wavefunction.

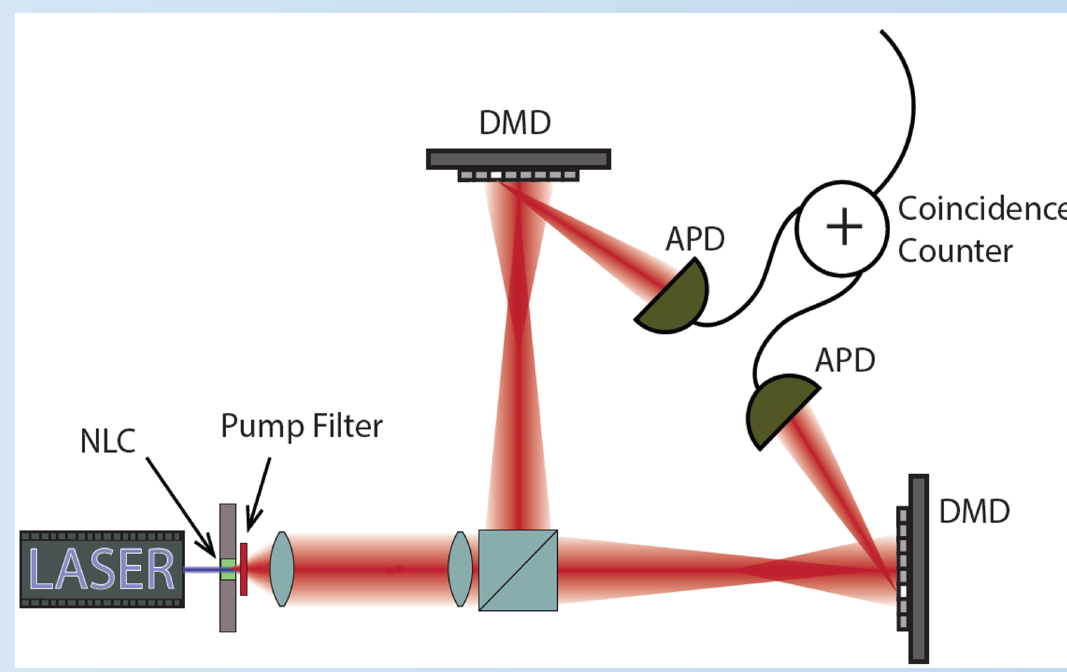
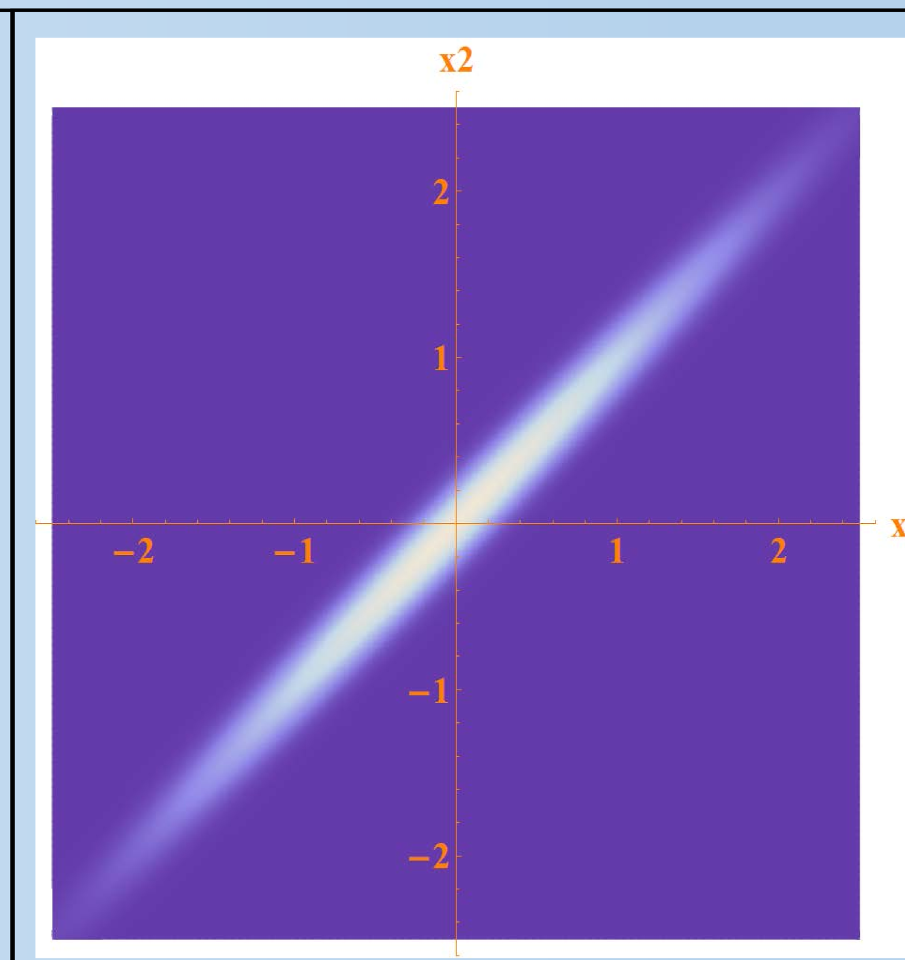
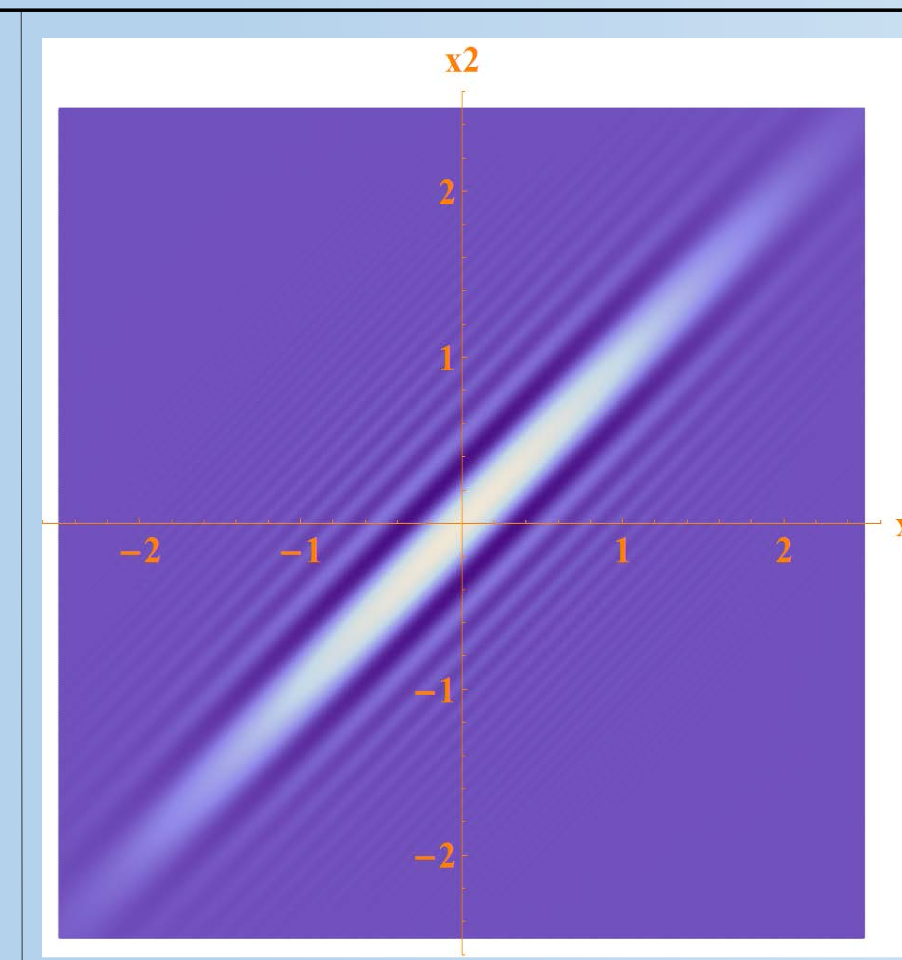


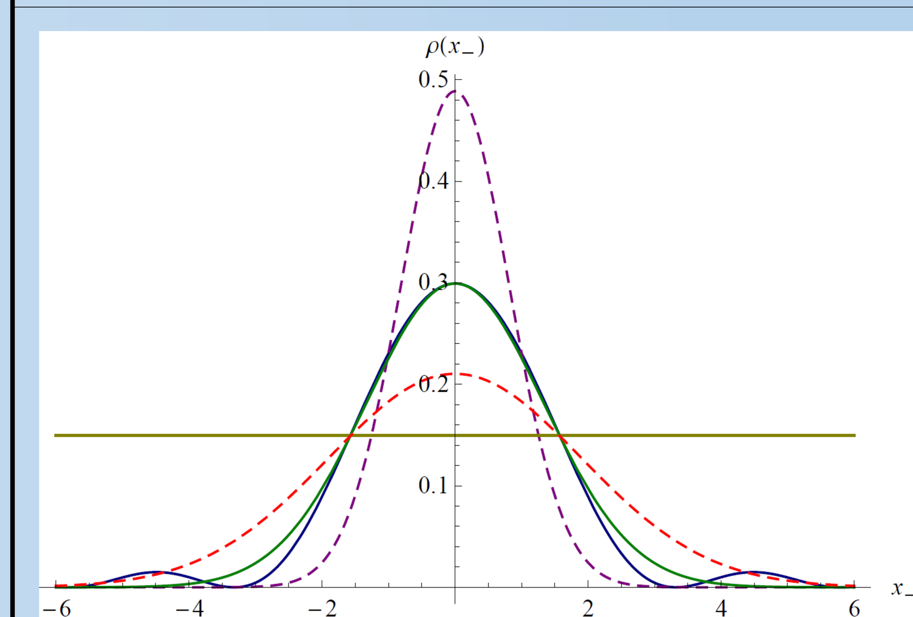
Fig. 3: Diagram of an experiment used to measure the transverse position correlations between signal and idler photons in type-I SPDC. A pair of lenses is used to image the face of the nonlinear crystal onto DMD (Digital Micro-mirror Device) arrays. By selecting pixels of these DMD arrays to reflect toward photo detectors, one obtains the joint position probabilities from the histogram of coincidence counts.



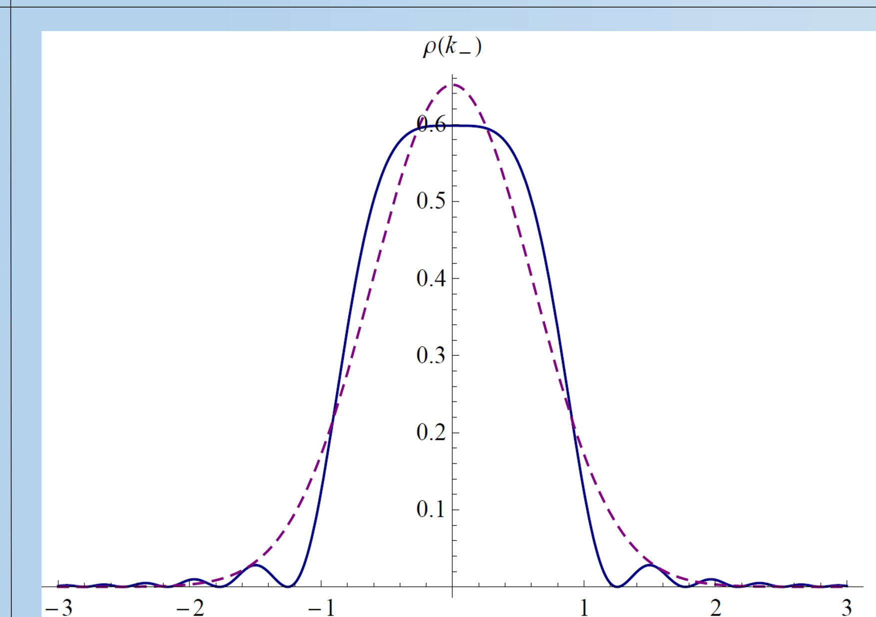
Plot of Double-Gaussian wavefunction $\psi^{DG}(x_1, x_2)$ for $\sigma_+ = 1$, and $\sigma_-^{PM} = \frac{3}{40}$ (matching statistics of biphoton wavefunction)



Plot of biphoton wavefunction $\psi(x_1, x_2)$ with $\sigma_p = \frac{1}{\sqrt{2}}$ and $\sigma_{(x_1-x_2)} = \frac{3\sqrt{2}}{40}$.



Plot comparing different estimates of $\rho(x_-)$. The solid blue wavy curve is our most accurate estimate from the transformed Sinc-based distribution. The tall dashed (magenta) curve is the Gaussian distribution obtained from matching $\langle k_-\rangle$ and $\langle k_-^2\rangle$, while the shallow dashed (red) curve is the Gaussian distribution obtained by matching $\langle x_-\rangle$ and $\langle x_-^2\rangle$. The solid (green) curve gives us a peak-matching Gaussian approximation. We see that the widths-of-half-maximum (as indicated by the gold line) are nearly identical (or by less than 0.3%) for the Sinc-based and peak-matching Gaussian distributions Again, we set a = 2 for convenience.



Plot comparing estimates of the momentum difference probability density $\rho(k_-)$. The solid (blue) curve with wavy side-bands gives our Sinc-based probability density estimate, where we set a = 2 for convenience. The dashed (magenta) Gaussian curve gives our Gaussian-based probability density estimate with matching means and variances of k_-.

The Double-Gaussian Approximation

$$\psi^{DG}(x_1, x_2) = \frac{1}{\sqrt{2\pi\sigma_+\sigma_-}}e^{-\frac{(x_1+x_2)^2}{8\sigma_+^2}}e^{-\frac{(x_1-x_2)^2}{8\sigma_-^2}}$$

$$\sigma_+ = \sigma_{\left(\frac{x_1+x_2}{\sqrt{2}}\right)} = \sigma_p\sqrt{2} \quad : \quad \sigma_- \approx \frac{\Delta^{PM}}{\sqrt{2}} = \sqrt{\frac{2L_z\lambda_p}{9\pi}}$$

Correlation measures

The Schmidt Number $\mathcal{K}$

$$\mathcal{K} \equiv \frac{1}{Tr[\hat{\rho}_A^2]}$$

For the Double-Gaussian State:

$$\mathcal{K} = \frac{1}{2}\left(\frac{\sigma_+}{\sigma_-} + \frac{\sigma_-}{\sigma_+}\right)$$

The Fedorov Ratio $\mathcal{R}$

$$\mathcal{R} \equiv \frac{\sigma_{(x_1)}}{\sigma_{(x_1|x_2)}}$$

For the Double-Gaussian State:

$$\mathcal{R} = \frac{1}{2}\left(\frac{\sigma_+}{\sigma_-} + \frac{\sigma_-}{\sigma_+}\right) = \mathcal{K}$$

The Birth Zone Number N

$$\text{N} \equiv \left(\frac{\sigma_{(x_1+x_2)}}{\sigma_{(x_1-x_2)}}\right)^d$$

d = number of dimensions (1 or 2, usually)

For the Double-Gaussian State:

$$\text{N} = \left(\frac{\sigma_+}{\sigma_-}\right)^d$$

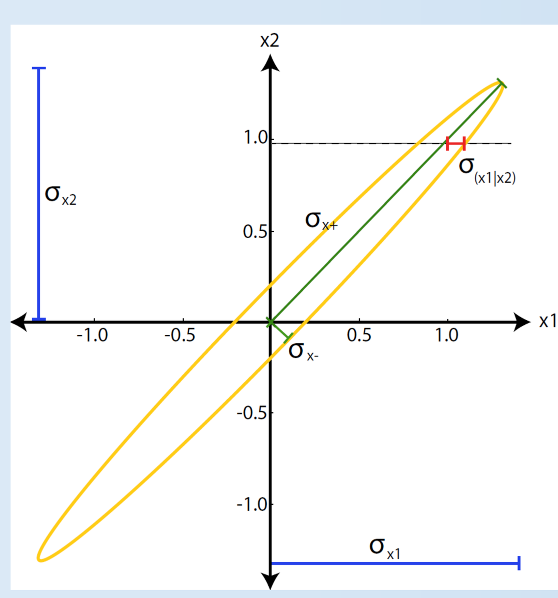


TABLE I: Statistics of the Double Gaussian

Name	Value
marginal means	$\langle x_1\rangle = \langle x_2\rangle = 0$
conditioned mean	$\langle x_2\rangle_{\rho(x_2\|x_1)} = x_1 - \frac{2x_1\sigma_{x_-}^2}{\sigma_{x_+}^2+\sigma_{x_-}^2} = rx_1$
marginal variance	$\sigma_{x_1}^2 = \sigma_{x_2}^2 = \frac{\sigma_{x_+}^2+\sigma_{x_-}^2}{2}$
conditioned variance	$\sigma_{(x_1\|x_2)}^2 = \sigma_{(x_2\|x_1)}^2 = \frac{2\sigma_{x_+}^2\sigma_{x_-}^2}{\sigma_{x_+}^2+\sigma_{x_-}^2}$
co-variance	$\langle x_1x_2\rangle - \langle x_1\rangle\langle x_2\rangle = \frac{\sigma_{x_+}^2-\sigma_{x_-}^2}{2}$
Pearson r value	$r = \frac{\langle x_1x_2\rangle-\langle x_1\rangle\langle x_2\rangle}{\sigma_{x_1}\sigma_{x_2}} = \frac{\sigma_{x_+}^2-\sigma_{x_-}^2}{\sigma_{x_+}^2+\sigma_{x_-}^2}$
joint entropy	$h(x_1, x_2) = \log(2\pi e\sigma_{x_+}\sigma_{x_-})$
marginal entropy	$h(x_1) = \frac{1}{2}\log(\pi e(\sigma_{x_+}^2 + \sigma_{x_-}^2))$
mutual information	$h(x_1 : x_2) = \log\left(\frac{\sigma_{x_+}^2+\sigma_{x_-}^2}{2\sigma_{x_+}\sigma_{x_-}}\right) = \log\left(\frac{\sigma_{x_1}}{\sigma_{(x_1\|x_2)}}\right)$

Fourier Transform Properties of the Double-Gaussian

The Heisenberg relations:

$$\sigma_{x_+}\sigma_{k_+} \geq \frac{1}{2}$$

$$\sigma_{x_-}\sigma_{k_-} \geq \frac{1}{2}$$

Are saturated by the Double-Gaussian state.

In addition, the conditional Heisenberg relations:

$$\sigma_{x_1}\sigma_{(k_1|k_2)} \geq \frac{1}{2}$$

$$\sigma_{(x_1|x_2)}\sigma_{k_1} \geq \frac{1}{2}$$

..are also saturated by the Double-Gaussian state.

Lower bound for birth zone size

Since $\sigma_{(x_1-x_2)} \geq \sigma_{(x_1|x_2)}$,

We know that

$$\sigma_{(x_1-x_2)} \geq \frac{1}{2\sigma_{k_1}}$$

Typical Lab Parameters:

For:
$\lambda_p = 325\text{nm}$
$\sigma_p = 0.3\text{mm}$
$L_z = 1.0\text{mm}$

We find:
$\Delta^{PM} = 6.78\mu\text{m}$;
$\Delta^{EX} = 9.65\mu\text{m}$;
$N^{PM} = 7{,}830$: d = 2
$\mathcal{R}, \mathcal{K} = 44.25$;

$h(\vec{x}_1 : \vec{x}_2) =$10.9 bits

Temporal Correlation Width

Type-I SPDC

$$\sigma_{(t_1-t_2)}^{(exact)} = \sqrt{\frac{9L_z\kappa_1}{10}}$$

$$\sigma_{(t_1-t_2)}^{(PM)} = \sqrt{\frac{4L_z\kappa_1}{9}}$$

Type-II SPDC

$$W_{(t_1-t_2)} = \frac{L_z|\Delta n_g|}{c}$$

Notes:

$\kappa_1 \equiv \frac{d^2k_1}{d\omega^2}\Big|_{\frac{\omega_p}{2}}$: (GVD constant)

$|\Delta n_g| \equiv$(group index mismatch)

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The Transverse Correlation Width

$$\Delta \equiv 2\sigma_{(x_1|x_p)} \;:\; x_p = \frac{x_1 + x_2}{2}$$

When the joint distribution factors:
$\rho(x_1, x_2) = \rho_+(x_+)\rho_-(x_-)$,
We get the simple formula:

$$\Delta = \sigma_{(x_1-x_2)}.$$

For the Biphoton state:

$$\boxed{\boldsymbol{\Delta} = \sqrt{\frac{\boldsymbol{9L_z\lambda_p}}{\boldsymbol{10\pi}}}}$$

Consequence:
Thinner crystals give better (smaller) correlation widths, at the expense of brightness.

If we want to approximate our statistics with a Double-Gaussian, the Peak-Matching width,

$$\Delta^{PM} = \sqrt{\frac{4L_z\lambda_p}{9\pi}}$$

fits the full width at 48.2% of the maximum.