Negativity as a Measure of Entanglement

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11-21-2011
Final lecture #1

What is it?

G. Vidal, RF Werner PRA 65 032314 (2002)
"Computable measures of entanglement"

For 2 qubits

$$N(\rho) = \left(\sum_i |\lambda_i|_{PPT}\right) - 1$$

- based on the Peres separability criterion.

How to Find it?

ex/ The mixture of the $|\phi^+\rangle$ Bell state, and the best classically correlated state $\rho_{BCC} = \frac{1}{2}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{2}|\downarrow\downarrow\rangle\langle\downarrow\downarrow|$

$$\rho_{mix} = p(|\phi^+\rangle\langle\phi^+|) + (1-p)\rho_{BCC}$$

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$$

$$|\phi^+\rangle\langle\phi^+| = \frac{1}{2}\left(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\uparrow\uparrow\rangle\langle\downarrow\downarrow| + |\downarrow\downarrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right)$$

$$\rho_{\phi^+} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \rho_{BCC} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\rho_{mix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{p}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{p}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Now we need to take the partial transpose

The partial transpose is defined as $\underset{\text{over a}}{(T\otimes I)}$ or $\underset{\text{over b}}{(I\otimes T)}$

so if $\rho = \rho_A \otimes \rho_B$

$$(I\otimes T)\rho = \rho_A \otimes \rho_B^T$$

~~ex~~

$$\rho = \left(\begin{array}{cc|cc} a & b & c & d \\ e & f & g & h \\ \hline i & j & k & l \\ m & n & o & p \end{array}\right)$$

$$(T\otimes I)\rho \equiv \rho_{T_A} = \left(\begin{array}{cc|cc} a & b & i & j \\ e & f & m & n \\ \hline c & d & k & l \\ g & h & o & p \end{array}\right)$$

$$(I\otimes T)\rho \equiv \rho_{T_B} = \left(\begin{array}{cc|cc} a & e & c & g \\ b & f & d & h \\ \hline i & m & k & o \\ j & n & l & p \end{array}\right)$$

So, for ρ_{mix}

$$\rho_{T_A} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{p}{2} & 0 \\ 0 & \frac{p}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} = \rho_{T_B}$$

We then find the eigenvalues of the partial transpose

$$\begin{vmatrix} \frac{1}{2}-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & \frac{p}{2} & 0 \\ 0 & \frac{p}{2} & -\lambda & 0 \\ 0 & 0 & 0 & \frac{1}{2}-\lambda \end{vmatrix} = 0$$

$$\left(\tfrac{1}{2}-\lambda\right)^2 \begin{vmatrix} -\lambda & \frac{p}{2} \\ \frac{p}{2} & -\lambda \end{vmatrix} = 0$$

$$\left(\tfrac{1}{2}-\lambda\right)^2 \left(\lambda^2 - \left(\tfrac{p}{2}\right)^2\right) = 0$$

$$\left(\tfrac{1}{2}-\lambda\right)^2 \left(\lambda + \tfrac{p}{2}\right)\left(\lambda - \tfrac{p}{2}\right) = 0$$

$$\lambda = \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{p}{2}, -\tfrac{p}{2}$$

$$N(\rho_{mix}) = \left|\tfrac{1}{2}\right| + \left|\tfrac{1}{2}\right| + \left|\tfrac{p}{2}\right| + \left|-\tfrac{p}{2}\right| - 1$$

$$N(\rho_{mix}) = \cancel{1} + p - \cancel{1}$$

$$\boxed{N(\rho_{mix}) = p}$$

ρ_{mix} is completely unentangled only if $p = 0$

Justification & Motivation

Why are we measuring this? What does doing this tell us?
What are we trying to find by doing this?

We're trying to find how entangled the system is.
We are trying to measure the amount of entanglement exhibited by the system

This is easy for pure states

$$K = \frac{1}{Tr(\rho_A^2)} \quad \text{(participation ratio)}$$

$$S = Tr(-\rho_A \log_2(\rho_A)) \quad \text{(entropy of entanglement)}$$

For mixed states, figuring this out is decidedly more difficult

So what goes into a measure of entanglement?

- It's a continuous function of the density matrix ρ
- $N(\rho_{sep}) = 0$ it's zero for separable states
- $N(U_1 \otimes U_2 \rho U_1^\dagger \otimes U_2^\dagger) = N(\rho)$ it's invariant under local unitary transformations $(U_1 \otimes U_2)$
- $N(\rho) \geq N(\rho')$, $\sum_i p_i N(\rho_i')$ it does not increase under local operations, even with classical communication.

There are other attributes that are occasionally introduced, but they are not always agreed upon as fundamental

What's a separable state?

{separable states} $\Longleftrightarrow$ {unentangled states}

Separable States

A state ρ is separable if we can express it as

$$\rho = \sum_i p_i (\rho_{Ai} \otimes \rho_{Bi})$$

RF Werner PRA 40 6 1989

"Quantum States with EPR correlations admitting a hidden variable model"

Fact

We can make any unentangled state by local operations and classical communications (LOCC) on the completely uncorrelated product state $|\psi\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$

- By local Unitary transforms

$$(U_1 \otimes U_2)|\psi\rangle = |\Omega_1\rangle \otimes |\Omega_2\rangle$$

we can make any unentangled pure state by $(U_1 \otimes U_2)$ alone.

(Schmidt decomposition
$|\psi\rangle = \sqrt{\alpha}\,|\Omega_1\rangle \otimes |\Omega_2\rangle + \sqrt{1-\alpha}\,\cancel{|-\Omega_1\rangle \otimes |-\Omega_2\rangle}$)

- We can also make transformations conditioned on the outcome X_i of some event with probability $p_i = p(x_i)$.

if outcome #1, we perform $(U_{1(1)} \otimes U_{2(1)})$ on $|\psi\rangle$

if outcome #2, we perform $(U_{1(2)} \otimes U_{2(2)})$ on $|\psi\rangle$

and so on to all outcomes

- It's this way that we can create a completely unentangled mixed state

$$\rho = \sum_i p_i (U_{1(i)} \otimes U_{2(i)}) |\uparrow\uparrow\rangle\langle\uparrow\uparrow| (U_{1(i)}^\dagger \otimes U_{2(i)}^\dagger)$$

$$= \sum_i p_i (U_{1(i)} |\uparrow\rangle\langle\uparrow| U_{1(i)}^\dagger) \otimes (U_{2(i)} |\uparrow\rangle\langle\uparrow| U_{2(i)}^\dagger)$$

$$\rho = \sum_i p_i \, \rho_{Ai} \otimes \rho_{Bi}$$

IN SHORT, ANY MIXTURE OF UNENTANGLED STATES ALSO HAS NO ENTANGLEMENT

Why should a measure of entanglement be invariant under $(U_1 \otimes U_2)$?

- For pure states

$$|\psi\rangle = \sqrt{\alpha}\,|\Omega_1, \Omega_2\rangle + \sqrt{1-\alpha}\,|{-\Omega_1}, {-\Omega_2}\rangle$$

the Schmidt weights are invariant under $(U_1 \otimes U_2)$ which serve only to change the orientation of Ω_1 and Ω_2

- For mixed states

the particular choice of basis of Alice's and Bob's qubit should not affect the entanglement between them.

Why should entanglement not increase under Local operations and classical communication?

LOCC: any combination of LUT $(U_1 \otimes U_2)$ and local measurements $(M_1 \otimes I)$, $(I \otimes M_2)$ and classical communication

↳ where we could perform transformations based on the outcome of measurements.

- Nonlocal correlations are destroyed by measurement

ex $|\psi\rangle = \lambda_1 |\uparrow\uparrow\rangle + \lambda_2 |\downarrow\downarrow\rangle$

if we measure qubit A

$|\psi\rangle \Rightarrow |\uparrow\uparrow\rangle$ or $|\downarrow\downarrow\rangle$

assuming no further interaction between A and B, then

$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$

and no further measurement of $|\psi_A\rangle$ will tell you anything about $|\psi_B\rangle$

- as we saw before, Nonlocal correlations are unaffected by $(U_1 \otimes U_2)$

There is another reason we care about LOCC.

ρ_1 is more entangled than ρ_2 if

and only if we can make ρ_2 from ρ_1 by LOCC,

but not the other way around.

Now that we've gone over what goes into a measure of entanglement, I'd like to look at the principle concept that the negativity is based on.

The Peres Separability Criterion

The Peres Separability (PPT) Criterion

James Schneeloch
11-28-2011
Final lecture #2

A Peres PRL #77, 8 (1996)
"separability criterion for density matrices"

① Since transposition of Hermitian matrices doesn't affect eigenvalues*

if ρ is a density matrix, so is ρ^T

$$* \det(A^T - \lambda I) = \det(A^T - \lambda I^T) = \det((A-\lambda I)^T) = \det(A - \lambda I)$$

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let $\mathcal{D} \equiv$ the set of separable states
$: \rho = \sum_i p_i (\rho_{Ai} \otimes \rho_{Bi})$

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② Given a separable density matrix

$$\rho = \sum_i p_i (\rho_{Ai} \otimes \rho_{Bi})$$

where the partial transpose of system B is defined as

$$\rho_{TB} \equiv \sum_i p_i (\rho_{Ai} \otimes \rho_{Bi}^T)$$

and

$$\rho_{TA} \equiv \sum_i p_i (\rho_{Ai}^T \otimes \rho_{Bi})$$

since ρ_A^T is just another density matrix ρ_A'
and $\rho_B^T = \rho_B'$

ρ_{TA} and ρ_{TB} are other separable states if ρ is separable

③ ∴ if any of the eigenvalues of ρ_{TA} or ρ_{TB} are less than zero, then ρ is not separable ...

In short

$$\text{if } \rho \in \mathcal{D} \text{ then } \rho_{TA}, \rho_{TB} \in \mathcal{D}$$

so when ρ_{TA} or $\rho_{TB} \notin \mathcal{D}$ $\quad \rho \notin \mathcal{D}$

if ρ_{TA} or ρ_{TB} are not even density matrices (negative eigenvalues) then ρ is not separable

For the $2\otimes2$ and $2\otimes3$ cases, a special situation arises

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M, P, & R Horodecki PRA 223 (1996)

"separability of mixed states; necessary and sufficient conditions"

For $2\otimes2$ and $2\otimes3$ cases, the PPT criterion is both necessary and sufficient

~~Let's look at how~~ we ~~distinguish separable from entangled states~~ with entanglement witnesses.

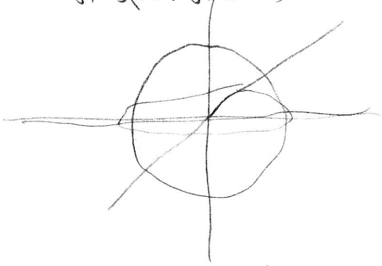
Entanglement Witness

- A hermitian operator A: For some entangled state ρ, $Tr(\rho A) < 0$ when $Tr(\rho A) \geq 0$ for all separable states
  - Nice to have
  - Hard to find in general
  - There always exists a witness A for every entangled state

WHAT DO WE MEAN BY WITNESSES?

ex/

Poincare Sphere of spin states



σ_z separates the sphere into two parts

σ_z is a "spin witness" which looks at the positivity of the z-component of spin

$Tr(\rho\sigma_z) \leq 0$ for all states with nonpositive average spin in the z-direction

so...

a state σ has positive spin in the z-direction if $Tr(\rho\sigma_z) > 0$ when $Tr(\rho\sigma_z) < 0$ for every state with nonpositive spin in the z-direction.

(a witness for spin is redundant, but helpful)

Let's look at a particular entanglement witness, ex) $W = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$

background)

let $\{\vec{V}_i\}$ be an orthogonal basis of vectors spanning vector space $\mathbb{W}$ such that

$$\vec{V}_i \cdot \vec{V}_j \propto \delta_{ij}$$

then any vector $\vec{r} \in \mathbb{W}$ can be expressed as

$$\vec{r} = \left(\frac{\vec{r}\cdot\vec{V}_1}{\vec{V}_1\cdot\vec{V}_1}\right)\vec{V}_1 + \ldots + \left(\frac{\vec{r}\cdot\vec{V}_n}{\vec{V}_n\cdot\vec{V}_n}\right)\vec{V}_n$$

ex in $\mathbb{R}^3$ with orthonomal basis $\{\hat{x}, \hat{y}, \hat{z}\}$

$$\vec{r} = (\vec{r}\cdot\hat{x})\,\hat{x} + (\vec{r}\cdot\hat{y})\,\hat{y} + (\vec{r}\cdot\hat{z})\,\hat{z}$$

$\{I, \sigma_x, \sigma_y, \sigma_z\}$ is an orthogonal basis of hermitian operators in H_1 or H_2 with inner product $\langle A, B\rangle = Tr(AB^\dagger)$
$= Tr(AB)$

$$\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$$

$$Tr(\sigma_i) = 0$$

$$\langle I, \sigma_i\rangle = Tr(\sigma_i) = 0$$

$$\langle I, I\rangle = Tr(I) = 2$$

$$\langle \sigma_i, \sigma_j\rangle = 2\delta_{ij}$$

so... we can express any spin-$\frac{1}{2}$ system in terms of this basis of operators

$$\rho = \frac{\langle\rho, I\rangle}{\langle I, I\rangle} I + \frac{\langle\rho, \sigma_x\rangle}{\langle\sigma_x, \sigma_x\rangle}\sigma_x + \ldots + \frac{\langle\rho, \sigma_z\rangle}{\langle\sigma_z, \sigma_z\rangle}\sigma_z$$

let

$$\langle\sigma_x\rangle = Tr(\rho\,\sigma_x) \equiv P_x$$

$$\langle\sigma_y\rangle \equiv P_y$$

$$\langle\sigma_z\rangle \equiv P_z$$

then

$$\rho = \frac{1}{2}I + \frac{1}{2}P_x\sigma_x + \frac{1}{2}P_y\sigma_y + \frac{1}{2}P_z\sigma_z$$

as we talked about in the section on spin squeezing, P_x, P_y, P_z form a polarization vector $\vec{P}$ whose magnitude is between 0 and 1 (0 for maximally mixed states, 1 for pure states).

so in short all spin-$\frac{1}{2}$ systems can be expressed as

$$\rho = \frac{1}{2}(I + \vec{P}\cdot\vec{\sigma})$$

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as a reference for the math ...

Ugo Fano, Reviews of Modern Physics Vol 55 #4 Oct 1983
pp 855-874
"Pairs of two-level systems

Back to the witness $W = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$

For any uncorrelated state $\rho_A \otimes \rho_B$

$$Tr((\rho_A \otimes \rho_B)W) = Tr((\rho_A \otimes \rho_B)(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z))$$

$$= Tr(\rho_A\sigma_x)Tr(\rho_B\sigma_x) + Tr(\rho_A\sigma_y)Tr(\rho_B\sigma_y) + Tr(\rho_A\sigma_z)Tr(\rho_B\sigma_z)$$

where $P_{x} = \langle\sigma_x\rangle = Tr(\rho\sigma_x)$

$$Tr((\rho_A \otimes \rho_B)W) = P_{Ax}P_{Bx} + P_{Ay}P_{By} + P_{Az}P_{Bz}$$

since $|\vec{P}_A|$ and $|\vec{P}_B| \in [0,1]$

$$Tr((\rho_A \otimes \rho_B)W) \in [-1,1]$$

also, where $\rho = \sum_i p_i(\rho_{Ai} \otimes \rho_{Bi})$; $\sum_i p_i = 1$

$$Tr(\rho W) = \sum_i p_i\, Tr((\rho_{Ai} \otimes \rho_{Bi})W) \in [-1,1]$$

so where $W = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$

if ρ is separable

$$Tr(\rho W) \in [-1, 1]$$

consider the singlet state

$$\rho_{\psi^-} = |\psi^-\rangle\langle\psi^-| = \frac{1}{4}\left(I \otimes I - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z\right)$$

$$Tr(\rho_{\psi^-} W) = \frac{1}{4}(0 - 4 - 4 - 4) = -3$$

we already knew that $|\psi^-\rangle$ was an entangled state, but we see here that W witnesses the entanglement of $|\psi^-\rangle$ since $Tr(\rho W)$ is not within the range of values for separable states.

Note: W does not witness all entangled states

e.g. $$\rho_{\psi^+} = |\psi^+\rangle\langle\psi^+| = \frac{1}{4}\left(I \otimes I + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z\right)$$

$$tr(\rho_{\psi^+} W) = \frac{1}{4}(0 + 4 + 4 - 4) = 1$$

$|\psi^+\rangle$ is entangled, but this witness doesn't show it.

Now that we've looked into the fundamentals of entanglement witnesses, it is instructive to show exactly why they exist for every entangled state.

Background

Definition

Given a set of vectors $\{v_i\} \in V$

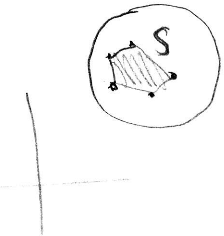
A convex combination of these vectors is a linear combination with non-negative weights which add up to 1.

$$V_{cc} = \sum_i p_i v_i : \sum_i p_i = 1, \quad p_i \in [0, 1]$$

Definition

A set of vectors $\mathcal{S}$ is <u>convex</u> if all convex combinations of vectors in $\mathcal{S}$ are also vectors in $\mathcal{S}$

eg. In $\mathbb{R}^2$

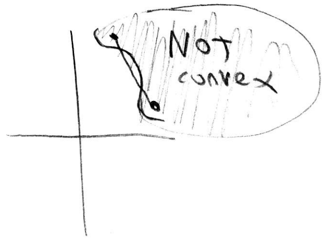


For 3 vectors (or any number), the set of all convex combinations is a set of all weighted averages of these vectors which forms a triangle.

jargon

The set of convex combinations of vectors $\{v_i\}$ is called the <u>convex hull</u> of $\{v_i\}$

$\mathcal{S}$ is convex if for any two points in $\mathcal{S}$, the line between them is also in $\mathcal{S}$.



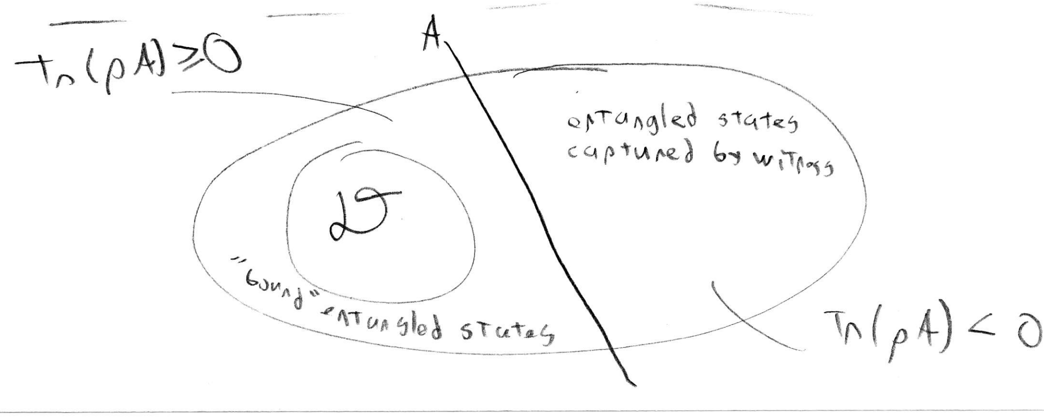
Fact

Since any mixture of separable states is also a separable state, the set of separable states $\mathcal{D}$ is a <u>convex set</u> in $H_1 \otimes H_2$

for each $\rho_j = \sum_i p_i \rho_{Aij} \otimes \rho_{Bij}$

or $\rho = \sum_j q_j \rho_j = \sum_{ij} p_i q_j \rho_{Aij} \otimes \rho_{Bij}$

$\rho = \sum_k g_k \rho_{Ak} \otimes \rho_{Bk}$ is also separable



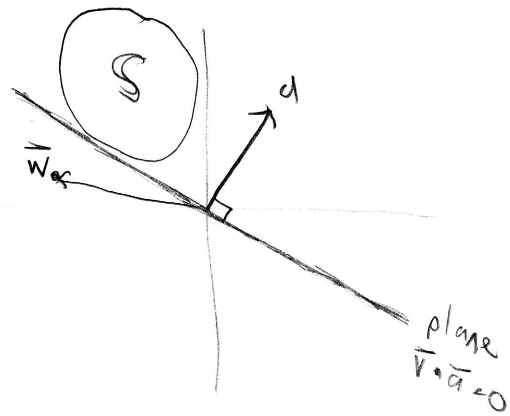
Knowing that $\mathcal{D}$ is a convex set, how does that helps us to show that entanglement witnesses always exist?

Consequence of Hahn-Banach Theorem

For R^N

- a single vector $\vec{a}$ defines a plane is $\{\vec{v} : \vec{v} \cdot \vec{a} = 0\}$
- We can always tell which side of the plane a vector $\vec{w}$ is on if $\vec{w} \cdot \vec{a} > 0$ or if $\vec{w} \cdot \vec{a} < 0$

☆ Given a convex set of vectors $\mathbb{S}$, and a vector $\vec{w}$ outside of $\mathbb{S}$, there always exists a vector $\vec{a}$ such that $\vec{w} \cdot \vec{a} < 0$ when for every $\vec{v}$ in $\mathbb{S}$. $\vec{v} \cdot \vec{a} \geq 0$



In other words...

Since $\mathcal{D}$ is a convex set in $H_1 \otimes H_2$,

- For any ρ not in $\mathcal{D}$ there always exists a hermitian operator A such that $(\langle \rho, A \rangle = \mathrm{Tr}(\rho A)) < 0$ when for every state $q \in \mathcal{D}$ $\langle q, A \rangle \geq 0$

thus.. Entanglement witnesses exist for every entangled state.

What do we take away from this?

- Entanglement witnesses always exist, but a single witness won't separate all entangled states from separable ones any more than a flat plane can cover the surface of a sphere.

Having shown that entanglement witnesses exist for every entangled state, it is good to see how this among other things is used to show that the PPT criterion is necessary and sufficient for $2\otimes2$ and $2\otimes3$ cases.

Before our next step, we need to talk about positive maps and how they connect to entanglement witnesses.

Definitions

A Positive map Φ is...

- linear $\quad \Phi(\alpha M + \beta N) = \alpha\Phi(M) + \beta\Phi(N)$
- self adjoint (maps hermitian operators to other Hermitian operators)
- positive (it maps positive operators onto other positive operators)

  if eigenvalues $\{\lambda_i\}$ of M are ≥ 0 then eigenvalues of $\Phi(M)$ are also ≥ 0

Given a hermitian operator M, for any $|\psi\rangle$

- $\langle\psi|M|\psi\rangle > 0$ iff M is positive definite.
  The eigenvalues $\lambda_i(M) > 0$ iff ↑
- $\langle\psi|M|\psi\rangle \geq 0$ iff M is positive semi-definite.
  The eigenvalues $\lambda_i(M) \geq 0$ iff ↑

> Note: Any map that leaves the eigenvalues unchanged is a positive map.

CP Maps "completely positive"

- A subset of positive maps where if Φ is a CP map, so is $\Phi\otimes I$

so... if Φ is a CP map

if $M > 0$ so is $\Phi(M)$

and if $M \otimes N > 0$, so is $(\Phi \otimes I)(M \otimes N)$

ex any change of basis (any unitary evolution) is a CP map.

- We want to look at only positive maps, because if $(\Phi \otimes I)$ is not necessarily positive, we can use it to witness entanglement between subsystems in $H_1 \otimes H_2$

Thus...

If we can find a positive map $\Phi : (\Phi \otimes I) < 0$ for entangled states, but not for separable states, our work will be nearly done.

Note: For a positive map Φ, $(\Phi \otimes I) > 0$ for separable states by default.

for $\rho = \sum_i p_i \, \rho_{Ai} \otimes \rho_{Bi}$

$$(\Phi \otimes I)\rho = \sum_i p_i (\Phi(\rho_{Ai})) \otimes \rho_{Bi}$$

since Φ is positive

$\Phi(\rho_{Ai})$ is positive

and $\Phi(\rho_{Ai}) \otimes \rho_{Bi}$ is positive

and so $(\Phi \otimes I)(\rho)$ is positive if ρ is separable

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Let us look at the connection between positive maps and entanglement witnesses...

The Jamiolkowski Isomorphism

There is an isomorphism between hermitian operators $A \in H_1 \otimes H_2$ and positive maps

(for 2 qubits) $A = (I \otimes \Phi)(|\phi^+\rangle\langle\phi^+|)$: $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$

hermitian ↑ positive ↑

(in general $|\phi^+\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |ii\rangle$)

proof

$$A = \sum_i a_i |a_i\rangle\langle a_i|$$

let A be a positive operator : $a_i > 0$

then there exists CP - operators M_i, M_i^+

: $(I \otimes M_i)|\phi^+\rangle = \sqrt{a_i}|a_i\rangle$

then $A = \sum_i (I \otimes M_i)|\phi^+\rangle\langle\phi^+|(I \otimes M_i^+)$

$$= \sum_{i,j} \sqrt{a_i}\sqrt{a_j}\,|a_i\rangle\langle a_i|a_j\rangle\langle a_j|$$

$$= \sum_i a_i |a_i\rangle\langle a_i|$$

and so $A = (I \otimes \Phi)(|\phi^+\rangle\langle\phi^+|)$

where: $\Phi = \sum_i M_i(\quad)M_i^+$ is a CP map

if A is to have negative eigenvalues as well, then Φ should be only positive.

theorem

A state σ is entangled if and only if there exists a positive map Φ : $(I \otimes \Phi)(\sigma) < 0$

since entanglement witnesses always exist...

σ is entangled iff there exists a hermitian operator A: $Tr(\sigma A) < 0$ when $Tr(\rho A) \geq 0$ for every separable state ρ.

Since $A = (I \otimes \Phi)(|\phi^+\rangle\langle\phi^+|)$

$$Tr(A\sigma) = Tr((I \otimes \Phi)(|\phi^+\rangle\langle\phi^+|)\,\sigma)$$

$$= \sum_i Tr((I \otimes M_i)|\phi^+\rangle\langle\phi^+|(I \otimes M_i^*)\,\sigma)$$

$$= \sum_i Tr((I \otimes M_i^+)\,\sigma(I \otimes M_i)|\phi^+\rangle\langle\phi^+|)$$

$M_i^+ = M_i$ $\longrightarrow$ $$= \sum_i \langle\phi^+|(I \otimes M_i^+)\,\sigma(I \otimes M_i)|\phi^+\rangle$$

$$Tr(A\sigma) = \langle\phi^+| I \otimes \Phi(\sigma)|\phi^+\rangle$$

thus σ is entangled iff there exists a positive map Φ: $(I \otimes \Phi) < 0$

$\hookrightarrow$ $Tr(A\sigma) < 0$ if and only if $(I \otimes \Phi)(\sigma) < 0$

as stated previously, we know by default that if ρ is separable $((I \otimes \Phi)(\rho))$ is positive for any positive map Φ. thus if there is a Φ such that $(I \otimes \Phi(\sigma))$ is negative then σ is not a separable state.

having illustrated the connection between entanglement witnesses and positive maps, let's look at the peculiar case of $2 \otimes 2$ and $2 \otimes 3$

Consequence of Størmer & Woronowicz Theorem

For $2 \otimes 2$ and $2 \otimes 3$, any positive operator can be expressed as the sum of a CP operator, and another CP operator followed by a transposition.

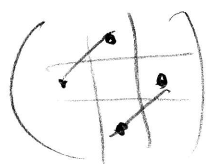
For $2 \otimes 2$ and $2 \otimes 3$

any Φ can be written as

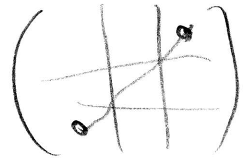
$$\Phi = (CP_1) + (CP_2)(T)$$

- this is not true for $3 \otimes 3$ and higher, where there are other quasi-transpositions which can be made

ex



or



Looking at some $2 \otimes 2$ or $2 \otimes 3$ state ρ

$$(I \otimes \Phi)(\rho) = I \otimes ((CP_1) + (CP_2)T)\rho$$
$$= (I \otimes CP_1)(\rho) + (I \otimes CP_2 T)(\rho)$$

since CP_1 and CP_2 are completely positive

$(I \otimes \Phi)(\rho) < 0$ if and only if $(I \otimes T) < 0$

And so...

$$\boxed{\begin{array}{l}\text{For } 2 \otimes 2 \text{ and } 2 \otimes 3 \\ \rho \text{ is entangled if and only if} \\ (I \otimes T)\rho < 0 \text{ or } (T \otimes I)(\rho) < 0\end{array}}$$

Knowing this, the PPT criterion tells us when a state is separable for $2 \otimes 2$ and $2 \otimes 3$ and when it is not. The negativity looks at just how nonseparable the states are by the magnitude of the negativity of the eigenvalues of the partial transpose.


The Negativity
What is it?
How do you find it?
What is it based on? Motivation & Justification
What's in an entanglement measure?
The Peres Separability (PPT) Criterion
What's a separable state?
LOCC & Local Unitary transformations
Why is it both necessary and sufficient for 2x2 and 2x3 cases?
how do we distinguish between separable & entangled states?
P and CP maps
Witnesses! What are they?
Starmer Woronowicz Theorom
the connection between maps and witnesses
Witnesses always exist for every entangled state.
∴ Necessary and sufficient for 2x2 and 2x3 cases
There is always a map for every entangled state.
Measures based on the (PPT) criterion for 2x2 and 2x3 cases are well justified.