BLACK HOLES & HOW TO PREVENT THEM

PROBLEM SET #4 ON WEBWORK – DUE NEXT WEDNESDAY, 10/9, AT MIDNIGHT

BLACK HOLES & HOW TO PREVENT THEM

Singularities in physics: why no one worried about Schwarzschild's black holes for awhile

The Schwarzschild singularity and the sizes of black holes

Degeneracy pressure: a quantum-mechanical effect that might stop matter from collapsing to form a black hole when gas pressure or material strength are not enough.

The central star in this planetary nebula, NGC 6543, is well on its way to becoming a white dwarf (HST and Chandra/NASA, STScI, CfA)

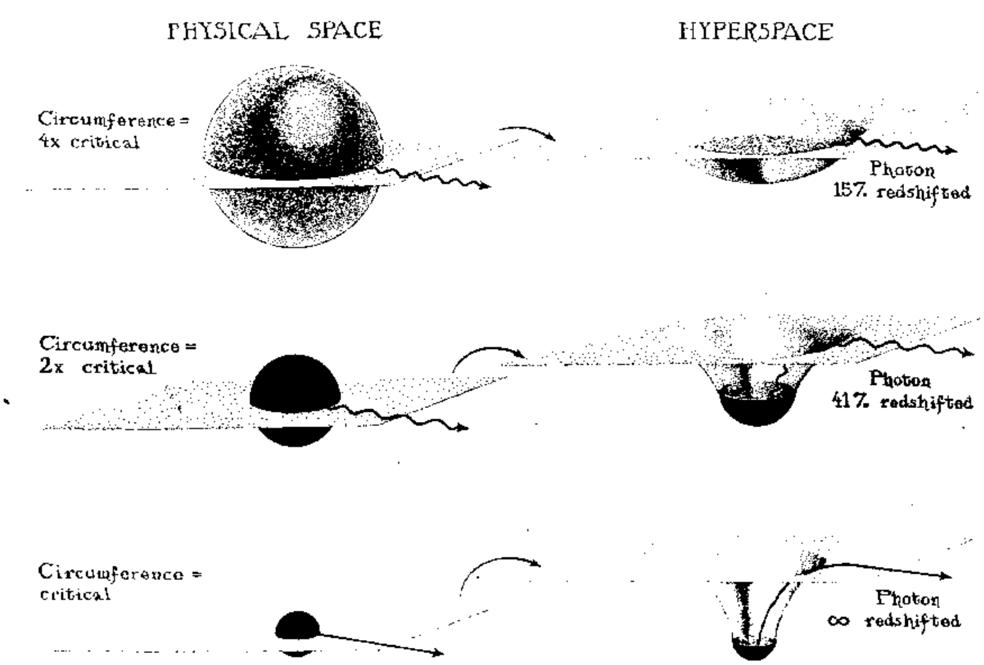
THE SCHWARZSCHILD SINGULARITY

According to Schwarzschild's solution to Einstein's field equation for spherical objects, the gravitational redshift becomes infinite (i.e. time appears to a distant observer to stop) if an object having mass M is confined within a sphere of circumference C_S, given by

$$C_S = \frac{4\pi GM}{c^2}$$ **Schwarzschild circumference**

Here, $G = 6.674\times10^{-8}$ cm^3/(g s^2) is Newton's gravitational constant, and $c = 2.9979\times10^{10}$ cm/s is the speed of light (and $\pi = 3.14159$…)

You need to understand this formula.



WHITE STARS

Schwarzschild found that, as the size of a white star was reduced and began to approach the size of the singularity, it would begin to appear to an observer as

A. Spherical and red
B. Funnel-shaped and red
C. Spherical and white
D. Funnel-shaped and white

IMPLICATIONS OF "SCHWARZSCHILD'S SINGULARITY"

- If a star is made too small in circumference for a given mass, nothing can escape from it, **not even light**.
  - This would be a black hole, and the critical size is the size of the black hole's horizon.
- This is similar to an 18th century idea for "dark stars" (John Michell and Pierre LaPlace, independently): if light were subject to gravitational force, there could be stars from which light could not escape.
  - The critical size of a star with Schwarzschild's singularity turns out to be the same as Michell and LaPlace determined (from classical physics!) for their "dark star."

SINGULARITIES IN MATH, PHYSICS, & ASTRONOMY

A formula is called **singular** if, when one puts the numbers into it in a calculation, the result is either infinity or not well defined. The particular combination of numbers is called the **singularity**.

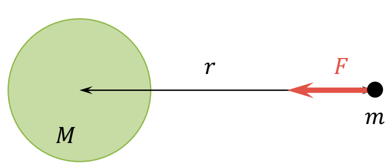
Singularities often arise in the formulae of physics and astronomy. They usually indicate either

- **Invalid approximations** – not all the necessary physical laws have been accounted for in the formula (no big deal)
- That the singularity is **not realizable** (also no big deal)
- That a mathematical **error** was made in obtaining the formula (just plain wrong)

They are hardly ever real – infinity is hard to come by!

SINGULARITIES IN MATH, PHYSICS, & ASTRONOMY

Example of a classical physics law with a singularity: Newton's law of gravitation.



$$\boxed{F = \frac{GMm}{r^2}}$$

- r is the distance between the centers of the two spherical masses. A spherical mass exerts force as if its mass is concentrated at its center.
- If r were zero, the force would be infinite!

This formula will not appear on homework or exams. It is used only because it is a good example of a singularity.

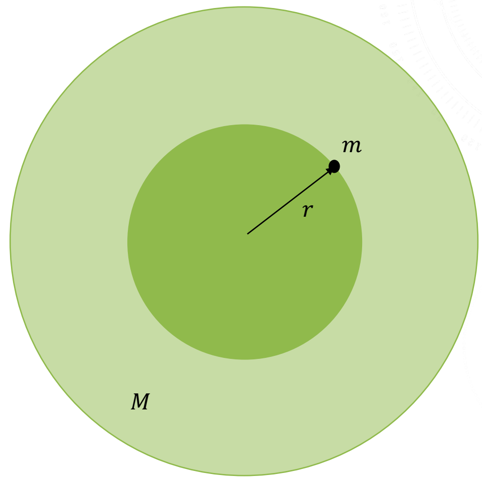
SINGULARITIES IN MATH, PHYSICS, & ASTRONOMY

This singularity is **not realized**, however, because

- **The mass M is not really concentrated at a point**
- A spherical shell of matter does not exert a net gravitational force on a mass inside it.

Consider mass m inside mass M: the outer (light green) matter's forces on m cancel out, and only the inner (solid green) exerts a force. As m gets closer to the center ($r \to 0$), the force gets smaller, not larger.

No singularity!



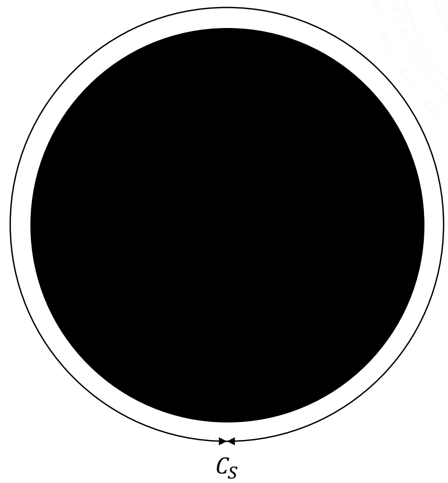
REACTION TO THE SCHWARZSCHILD SINGULARITY

- Schwarzschild's solution to the Einstein field equation was demonstrated to be correct – the singularity is not the result of a math error.
- Most physicists and astronomers assumed that the singularity would not be physically realizable (just like the singularity in Newton's law of gravitation) or that accounting for other physical effects would remove it.
- Einstein (1939) eventually tried to prove this in a general-relativistic calculation of stable (non-collapsing or exploding) stars of size equal to the Schwarzschild circumference.
- He found that this would require either **infinite gas pressure** or **particle speed greater than the speed of light,** both of which are not possible.
- Einstein's results show that a stable object with a singularity cannot exist.
- From this, he concluded **(incorrectly)** that this meant that the singularity could not exist in nature.

THE SCHWARZSCHILD SINGULARITY

Any object with mass M and circumference smaller than C_S would not be able to send light (or anything else) to an outside observer; it would be a black hole.

The spherical boundary with this critical circumference – the Schwarzschild singularity itself – is what we have been calling the **event horizon,** or simply the **horizon,** of a black hole.



EXAMPLE CALCULATION USING THE HORIZON (SCHWARZSCHILD) CIRCUMFERENCE

Example 1: What is the horizon circumference of a $10M_{\odot}$ black hole?

$$C_S = \frac{4\pi GM}{c^2}$$

$$= \frac{4\pi(6.67\times10^{-8}\ \mathrm{cm^3/g\ s^2})(10M_{\odot})(2\times10^{33}\ \mathrm{g}/M_{\odot})}{(3\times10^{10}\ \mathrm{cm/s})^2}$$

$$= 1.86\times10^{7}\ \mathrm{cm}$$

$$C_S = 1.86\times10^{7}\,\mathrm{cm}\times\frac{1\ \mathrm{km}}{10^5\ \mathrm{cm}} = 186\ \mathrm{km}$$

EXAMPLE CALCULATION USING THE HORIZON (SCHWARZSCHILD) CIRCUMFERENCE

Example 2: What is the horizon circumference of a black hole with the same mass as the Earth (6.0×10^{27}g)?

$$C_S = \frac{4\pi GM}{c^2}$$

$$= \frac{4\pi(6.67\times10^{-8}\ \mathrm{cm^3/g\ s^2})(6\times10^{27}\ \mathrm{g})}{(3\times10^{10}\ \mathrm{cm/s})^2}$$

$$C_S = 5.6\ \mathrm{cm}$$

EXAMPLE CALCULATION USING THE HORIZON (SCHWARZSCHILD) CIRCUMFERENCE

Example 3: What is the mass of a black hole that has a horizon circumference equal to that of the Earth (4.0×10^{9} cm)?

First, rearrange the formula:

$$C_S = \frac{4\pi GM}{c^2}$$

$$M = \frac{C_S c^2}{4\pi G} = \frac{(4\times10^9\ \mathrm{cm})(3\times10^{10}\ \mathrm{cm/s})^2}{4\pi(6.67\times10^{-8}\ \mathrm{cm^3/g\ s^2})}$$

Then, plug in the numbers

$$= 4.3\times10^{36}\ \mathrm{g}$$

$$M = 4.3\times10^{36}\ \mathrm{g}\times\frac{1M_\odot}{2.0\times10^{33}\ \mathrm{g}} = 2.15\times10^3 M_\odot$$

YOUR TURN!

What is the horizon circumference of a black hole with a mass equal to that of the bright star Vega, $M = 2.9M_{\odot}$?

A. 54.01584595 km
B. 54 km
C. 65.1429 km
D. 69.8 km
E. 186 km

REACTION TO THE SCHWARZSCHILD SINGULARITY

- Einstein's results show that a stable object with a singularity cannot exist.
- From this, he concluded (**incorrectly**) that this meant that the singularity could not exist in nature.
- Einstein's calculation was correct, but the correct inference from the result is that **gas pressure cannot support the weight of stars similar in size to the Schwarzschild circumference**.
- If nothing stronger than gas pressure holds them up, such stars will collapse to form black holes – the singularity is real.
  - Stronger than gas pressure: **degeneracy pressure**

DEGENERACY PRESSURE

This involves a concept from quantum mechanics called the **wave-particle duality**.

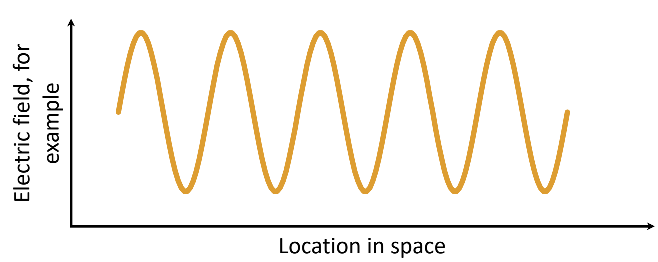
- All elementary particles from which matter and energy are made (including light, electrons, protons, neutrons,…) simultaneously have the properties of both particles and waves.
- Which property they display depends on the situation that they are in.

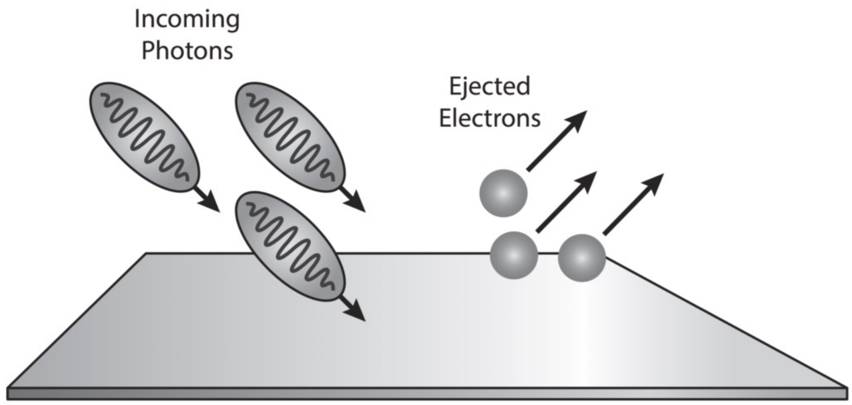
Degeneracy pressure consists of a powerful resistance to compression that is exhibited by the elementary constituents of matter when these particles are confined to spaces small enough to reveal their wave properties.

PARTICLES & WAVES

Particles exist only at one point in space

Waves extend over a region of space



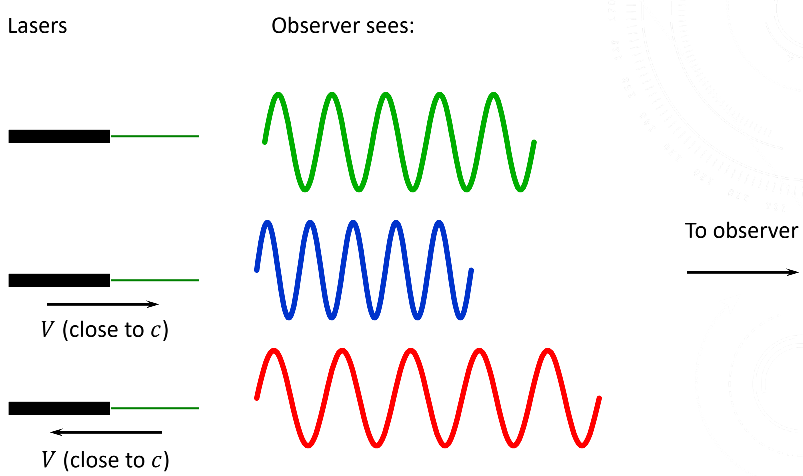


LIGHT CAN BE EITHER A PARTICLE…

Particle example: the **photoelectric effect** – the 1905 explanation of which, in these terms, won Einstein the 1921 Nobel Prize in physics.

…OR A WAVE

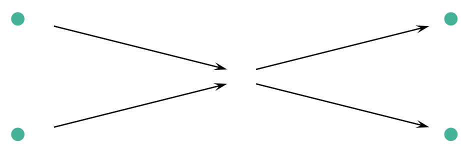
Wave example: the **Doppler effect**



ELECTRONS CAN BE PARTICLES OR WAVES

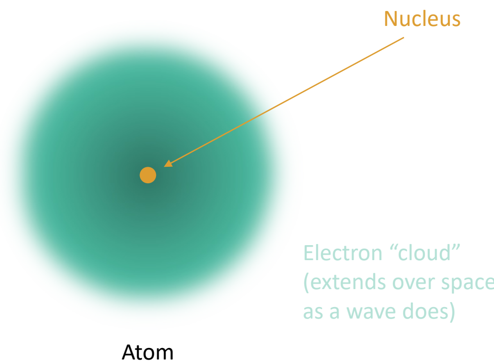
Particle example

Collisions between free electrons are "elastic" (they behave like billiard balls).



Wave example

Electrons confined to atoms behave like waves.



HOW TO EVOKE THE WAVE PROPERTIES OF MATTER

All the elementary constituents of matter have both wave and particle properties.

If a subatomic particle (like an electron, proton, or neutron) is **confined to a very small space**, it acts like a **wave** rather than a particle.

How small a space?

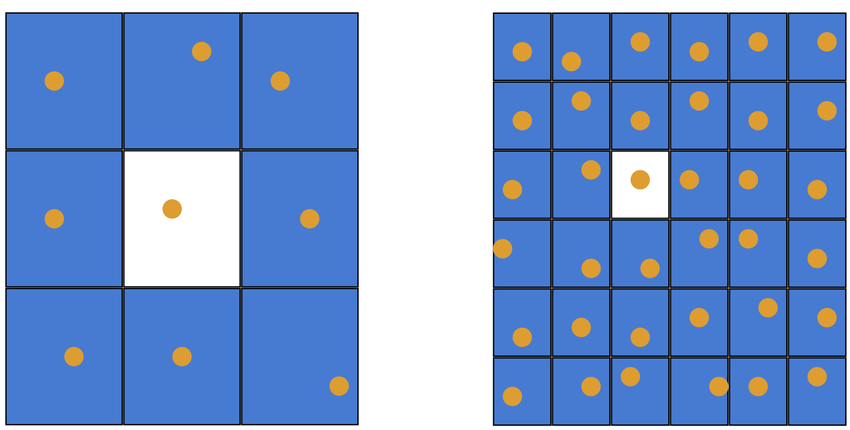
- The size of an atom, in the case of electrons (about 10^{-8} cm in diameter)
- A much smaller space for protons and neutrons (about 10^{-11} cm in diameter)
- Generally, the more massive the particle is, the smaller the confinement space required to make it exhibit wave properties.

ELEMENTARY PARTICLE MASSES

In a reference frame in which the particle is at rest,

$$m_e = 9.1094\times10^{-28} \text{ g (electron)}$$
$$m_p = 1.6726\times10^{-24} \text{ g (proton)}$$
$$m_n = 1.6705\times10^{-24} \text{ g (neutron)}$$

- To reveal their wave properties, electrons need to be confined to atomic dimensions (about 10^{-8} cm), and neutrons and protons need to be confined to a space a factor of about 1836 smaller (about 10^{-11} cm). This factor is equal to the ratio of the particles' masses to that of an electron.
- Photons – particles of light – have a rest mass of 0. (This goes with them having no rest frame; they always appear to travel at the speed of light.) Therefore, photons do not need to be confined to reveal their wave properties.



CONFINEMENT OF ELEMENTARY PARTICLES

Particles like electrons, protons, and neutrons can be confined to a small space by being surrounded by other particles of the same type, very nearby.

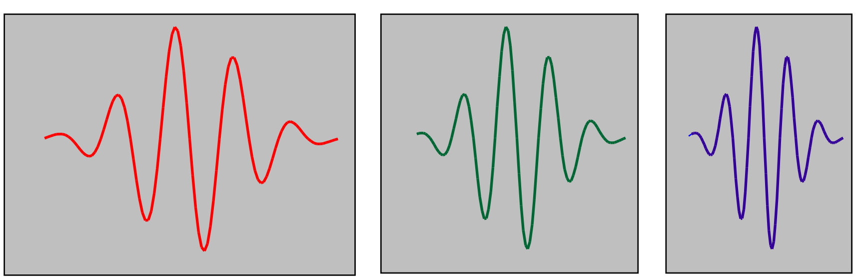
CONFINEMENT OF ELEMENTARY PARTICLES

This confinement has to do not only with the electric repulsion they may experience – there is an additional **quantum mechanical repulsion** of electrons by each other, which sets in at very small distances, such that wave properties are displayed.

- If the separation is small enough that this quantum repulsion is bigger than the electric repulsion, the electrons are said to be **degenerate**.
- Note for those who have taken physics or chemistry before: you may know this quantum repulsion as the **Pauli exclusion principle**.
- Protons and neutrons can each confine each other in a similar fashion. Because electrons are less massive, though, they become degenerate with less confinement (a space roughly 1800 times larger, as we have seen).
- Photons do not do this; the Pauli exclusion principle does not apply to light.

IMPLICATIONS OF CONFINEMENT ARISING FROM THE WAVE PROPERTIES OF ELEMENTARY PARTICLES

If an electron is confined to a smaller space, its wavelength decreases.



Just as is the case for light, a shorter wavelength means a larger energy for each confined electron.

IMPLICATIONS OF CONFINEMENT ARISING FROM THE WAVE PROPERTIES OF ELEMENTARY PARTICLES

With this increase in energy, each electron exerts itself harder on the walls of its "cell;" this is the same as an increase in pressure. So:

1. Squeeze a lot of matter from a very small space into an even smaller space…
2. Electrons are more tightly confined…
3. The electrons have more energy and exert more pressure against their confinement.

This extra pressure from the increase in wave energy under very tight confinement is **degeneracy pressure**, first described by British physicist Ralph Fowler in 1926.

Sir Ralph Fowler (AIP)

IMPLICATIONS OF CONFINEMENT ARISING FROM THE WAVE PROPERTIES OF ELEMENTARY PARTICLES

- Another equivalent way to view the wave-particle duality-induced extra resistance to compression is to invoke the **Heisenberg uncertainty principle**:

  The more precisely the position of an elementary particle is determined along some dimension, the less precisely its momentum (mass times velocity) along that same direction is determined.

- In other words: confining a bunch of elementary particles each to a very small distance (thus determining each position precisely) leads to a very large variation in their momenta and speed.
- Confinement to a smaller space → increase speed of particles on average → increase the force they exert on their "cell walls" (degeneracy pressure).

LET'S QUICKLY CHECK…

Deduce, from what you have just heard, which of these statements is **false**:

A. Degeneracy pressure can hold ordinary objects together.
B. A degenerate object made entirely of neutrons would be smaller than an object of the same mass made entirely of electrons.
C. The more tightly confined the electrons are, the larger their degeneracy pressure is.
D. If I add mass to a degenerate object, it should get smaller in diameter (not bigger).
E. The more tightly confined the electrons are, the larger their momentum is likely to be.

ELECTRON DEGENERACY PRESSURE & THE PREVENTION OF BLACK HOLES

- Most stars are stable because their weight is held up by gas pressure. Do stars exist that are held up by electron degeneracy pressure, rather than gas pressure?
  - Yes: **white dwarfs**
- How are such stars made?
  - From normal stars at the end of life, when they have run out of fuel, cannot generate gas pressure, and collapse under their own weight.
- Can electron degeneracy pressure balance gravity for all compact stars, preventing them from collapsing so far that they acquire horizons and black holes?
  - **Not entirely**, as we will see next time.