Astronomy 111 — Problem Set #8

Prof. Douglass

Due November 4, 2025 at the beginning of lecture

1. *Isothermal versus adiabatic.* 300 cm^3 of an ideal diatomic gas at a pressure of 1 atmosphere and a temperature of 300 K (i.e. room temperature) is contained by a cylinder with a piston top. A camshaft turns and drives the piston further into the cylinder, reducing the volume to 100 cm^3.

   (a) What is the final pressure if this compression takes place isothermally? What is the final pressure and temperature if the compression takes place adiabatically?

   Solution: Isothermal processes have a constant temperature, so $\Delta T = 0$. Since $PV = NkT$ for an ideal gas, this means that

   $$P_i V_i = P_f V_f$$
   $$P_f = P_i \frac{V_i}{V_f} = (1\text{ atm})\left(\frac{300\text{ cm}^3}{100\text{ cm}^3}\right)$$
   $$\boxed{P_f = 3\text{ atm}}$$

   Adiabatic processes do not exchange heat with their exterior, so $\Delta Q = 0$ and

   $$P_i V_i^\gamma = P_f V_f^\gamma$$
   $$P_f = P_i \left(\frac{V_i}{V_f}\right)^\gamma = (1\text{ atm})\left(\frac{300\text{ cm}^3}{100\text{ cm}^3}\right)^{7/5}$$
   $$\boxed{P_f = 4.7\text{ atm}}$$

   (b) Pistons in auto-engine cylinders perform this sort of compression hundreds to thousands of times per minute. Is auto-engine compression more likely to be isothermal or adiabatic?

   Solution: If it were isothermal, then all the heat generated during one compression cycle would need to be carried away as it is generated. If it were adiabatic, then there would be no heat transfer during a compression cycle. In general, isothermal processes are slow (since this heat dispersal takes time), while adiabatic processes are fast (the system has no time to disperse the generated heat). Therefore, auto-engine compression is most likely an adiabatic process.

   (c) Which sort of compression would lead to a more efficient engine: isothermal or adiabatic?

   Solution: It takes less work to push against a lower pressure, so the auto-engine compression would ideally be an isothermal process.

2. *An adiabatic atmosphere.* A certain planet has surface gravity $g = \frac{GM}{R^2}$ and an atmosphere with mean molecular mass μ, adiabatic index γ, and vertical extent much smaller than the planetary radius R, so

that it may be accurately treated in one-dimension. The surface pressure and temperature are P_0 and T_0, respectively. Suppose that the dependence of temperature on elevation z is determined by convection (i.e. by the adiabatic temperature gradient), and the dependence of pressure by hydrostatic equilibrium. Derive a formula for the pressure P as a function of elevation.

Solution: We can solve the adiabatic temperature gradient for the temperature as a function of elevation.

$$\begin{aligned}\frac{dT}{dz} &= -\frac{\gamma-1}{\gamma}\frac{\mu g}{k} \\ \int_{T_0}^{T(z)} dT' &= -\frac{\gamma-1}{\gamma}\frac{\mu g}{k}\int_0^z dz' \\ T(z)-T_0 &= -\frac{\gamma-1}{\gamma}\frac{\mu g}{k}z \\ T(z) &= T_0-\frac{\gamma-1}{\gamma}\frac{\mu g}{k}z\end{aligned}$$

Armed with the temperature as a function of elevation, we can integrate the pressure gradient in hydrostatic equilibrium with the idea gas law to derive the pressure of the atmosphere as a function of the elevation.

$$\begin{aligned}\frac{dP}{dz} &= -\rho g = -\frac{\mu P}{kT}g \\ &= -\frac{\mu g P}{k}\left(T_0-\frac{\gamma-1}{\gamma}\frac{\mu g}{k}z\right)^{-1} \\ &= -P\left(\frac{kT_0}{\mu g}-\frac{\gamma-1}{\gamma}z\right)^{-1} \\ \int_{P_0}^{P(z)}\frac{dP'}{P'} &= -\int_0^z\left(\frac{kT_0}{\mu g}-\frac{\gamma-1}{\gamma}z'\right)^{-1}dz'\end{aligned}$$

In order to integrate the righthand side of this equation, we need to use u-substitution. Let $u = \frac{kT_0}{\mu g}-\frac{\gamma-1}{\gamma}z'$, from which we see that $du = -\frac{\gamma-1}{\gamma}dz'$. Then,

$$\begin{aligned}\ln P(z)-\ln P_0 &= \frac{\gamma}{\gamma-1}\int_{kT_0/\mu g}^{kT_0/\mu g-(\gamma-1)z/\gamma}\frac{du}{u} \\ \ln\left(\frac{P(z)}{P_0}\right) &= \frac{\gamma}{\gamma-1}\ln\left(1-\frac{\gamma-1}{\gamma}\frac{\mu g}{kT_0}z\right) \\ \frac{P(z)}{P_0} &= \left(1-\frac{\gamma-1}{\gamma}\frac{\mu g}{kT_0}z\right)^{\gamma/(\gamma-1)} \\ \Aboxed{P(z) &= P_0\left(1-\frac{\gamma-1}{\gamma}\frac{\mu g}{kT_0}z\right)^{\gamma/(\gamma-1)}}\end{aligned}$$

3. The Galileo Probe reported a pressure of $P_0 = 1$ bar and a temperature of $T_0 = 166.1$ K just before it dropped beneath the cloud tops and disappeared from view. It kept on transmitting pressure and temperature readings until the pressure was about 22 bars. By making reasonable assumptions about the composition and temperature structure of Jupiter's atmosphere, calculate the distance the probe

traveled between the cloud tops and the end of operations. (You might find the mass of molecular hydrogen, $m_{H_2} = 3.32 \times 10^{-24}$ g, useful.)

Solution:

Jupiter's atmosphere is mostly hydrogen ($\mu \approx m_{H_2} = 3.32 \times 10^{-24}$ g) which is diatomic ($\gamma = \frac{7}{5}$). We are concerned with altitudes close to Jupiter's "surface" at 1 bar, so

$$g = \frac{GM_J}{R_J^2} = \frac{(6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2})(1.8982 \times 10^{30} \text{ g})}{(7.1492 \times 10^9 \text{ cm})^2}$$
$$g = 2479 \text{ cm/s}^2$$

This journey of the Galileo probe occurred only in the planet's troposphere, which is conductive. Thus, we can use the equation that we derived in Question 2

$$P(z) = P_0 \left(1 - \frac{\gamma - 1}{\gamma} \frac{\mu g}{kT_0} z\right)^{\gamma/(\gamma-1)}$$
$$z(P) = \frac{\gamma}{\gamma - 1} \frac{kT_0}{\mu g} \left(1 - \left(\frac{P(z)}{P_0}\right)^{(\gamma-1)/\gamma}\right)$$
$$z(22 \text{ bars}) = \frac{7/5}{7/5 - 1} \frac{(1.38 \times 10^{-16} \text{ erg/K})(166.1 \text{ K})}{(3.32 \times 10^{-24} \text{ g})(2479 \text{ cm/s}^2)} \left(1 - \left(\frac{22 \text{ bars}}{1 \text{ bar}}\right)^{(7/5-1)/(7/5)}\right)$$
$$\boxed{z = 138 \text{ km}}$$

4. *The amplitude of ocean tides.* Consider a coordinate system centered on Earth with the x-axis pointing through the center of the Moon. The Moon (mass M) lies a distance r away from the Earth (radius $R \ll r$); a test particle (mass m) lies in between them at position x on the axis close to the surface of the Earth. Use the steps outlined below to calculate the size of the tides on Earth's oceans, using the Moon as the primary agent of these tides.

   (a) Show that (or explain why) the tidal force on the test particle is $F_t = 3GMmx/r^3$; that is, the force is in the $+x$-direction for positive x.

   Solution: The force exerted by the Moon on the test particle is

   $$F(x) = \frac{GMm}{(r-x)^2} - m\omega^2(r - x)$$

   At the Earth, the gravitational force and the centripetal acceleration are equal, so the angular frequency of the orbit is

   $$\frac{GM}{r^2} = \omega^2 r$$
   $$\omega^2 = \frac{GM}{r^3}$$

   Therefore, $F(x) = \frac{GMm}{(r-x)^2} - \frac{GMm}{r^3}(r - x)$.

The tidal force on the test particle due to the Moon close to the origin (where $x \ll r$) is

$$F_t = x\left[\frac{dF_m}{dx}\right]_{x=0}$$
$$= x\left[\frac{2GMm}{(r-x)^3} + \frac{GMm}{r^3}\right]_{x=0}$$
$$\boxed{F_t = \frac{3GMmx}{r^3}}$$

(Because $x \ll r$, $r - x \cong r$, and the force will increase linearly with small x.)

(b) Calculate the work, $\int F(x)\,dx$, that the tidal force would do in moving the test particle from the origin to position x in the absence of Earth.

Solution:

$$W_t = \int_0^x F_t(x')\,dx'$$
$$= \int_0^x \frac{3GMm}{r^3}x'\,dx'$$
$$\boxed{W_t = \frac{3GMm}{2r^3}x^2}$$

(c) Calculate the work required to move the test particle from the surface of the Earth to position x in the absence of the Moon. Because the Moon's tidal force is what is responsible for moving this test particle to this position above the Earth's surface, set this work equal to that done by the Moon's tidal force and solve for $x - R$. (Hint: Assume that $x - R \ll R$; the position x is very near to the Earth's surface.) $x - R$ would then be the height above the surface that the tidal force could elevate a test particle, or the amplitude of the lunar tide.

Solution: If $h = x - R \ll R$, then the work required to move a mass m a distance h above the Earth's surface is

$$W = mgh$$
$$\frac{3GMm}{2r^3}x^2 = mg(x-R)$$
$$0 = \frac{3GM}{2r^3}x^2 - gx + gR$$
$$x = \frac{g \pm \sqrt{g^2 - 4\left(\frac{3GM}{2r^3}\right)gR}}{\frac{3GM}{r^3}}$$
$$= \frac{r^3 g}{3GM}\left(1 \pm \sqrt{1 - \frac{6GMR}{gr^3}}\right)$$
$$\boxed{x - R_\oplus = \frac{r^3 g}{3GM}\left(1 \pm \sqrt{1 - \frac{6GMR_\oplus}{gr^3}}\right) - R_\oplus}$$

(d) Calculate the value of $x - R$. Look up the average height of lunar ocean tides online for comparison (and give the URL of your reference). How does your calculation compare to the real value?

Solution:

$$x - R_\oplus = \frac{(3.844 \times 10^{10}\ \mathrm{cm})^3 (980.7\ \mathrm{cm/s^2})}{3(6.674 \times 10^{-8}\ \mathrm{dyn\ cm^2\ g^{-2}})(7.342 \times 10^{23}\ \mathrm{g})} \left(1 - \sqrt{1 - \frac{6(6.674 \times 10^{-8}\ \mathrm{dyn\ cm^2\ g^{-2}})(7.342 \times 10^{23}\ \mathrm{g})(6.378 \times 10^{8}\ \mathrm{cm})}{(980.7\ \mathrm{cm/s^2})(3.844 \times 10^{8}\ \mathrm{cm})^3}}\right) - 6.378 \times 10^{8}\ \mathrm{cm}$$

$$\boxed{x - R_\oplus = 28.4\ \mathrm{cm}}$$

According to the Earth tide Wikipedia article, the vertical amplitude of the lunar tide is 38.5 cm (the value labeled M_2 in the table at `https://en.wikipedia.org/wiki/Earth_tide`). This is fairly close to the estimate that we calculated.