Astronomy 111 — Practice Midterm #2

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Name: ____

If this were a real exam, you would be reminded of the **Exam rules** here: "You may consult *only* one page of formulas and constants and a calculator while taking this test. You may *not* consult any books, digital resources, or each other. All of your work must be written on the attached pages, using the reverse sides if necessary. The final answers, and any formulas you use or derive, must be indicated clearly (answers must be circled or boxed). You will have one hour and fifteen minutes to complete the exam. Good luck!"

Your results will improve if you take this practice test under realistic test-like conditions: in one sitting, with your already-prepared cheat sheet at hand, and with the will to resist peeking at the solutions until you are finished. Also, as usual:

- First, work on the problems you find the easiest. Come back later to the more difficult or less familiar material. Do not get stuck.
- The amount of space left for each problem is not necessarily an indication of the amount of writing it takes to solve it.
- You must show your work to receive full credit.
- Numerical answers are incomplete without units and should not be written with more significant figures than they deserve.
- If you need a physical or astronomical constant that is not on your equation sheet, do not give up: estimate its value. If your estimate is reasonable, you will not lose any credit.
- Remember, you can earn partial credit for being on the right track. Be sure to show enough of your reasoning that we can figure out what you are thinking.

$$
\begin{aligned}
R_\odot &= 6.96 \times 10^{10} \text{ cm} & R_\oplus &= 6.378 \times 10^{8} \text{ cm} \\
M_\odot &= 1.989 \times 10^{33} \text{ g} & M_\oplus &= 5.972 \times 10^{27} \text{ g} \\
L_\odot &= 3.827 \times 10^{33} \text{ erg/s} & G &= 6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2} \\
T_\odot &= 5772 \text{ K} & c &= 3 \times 10^{10} \text{ cm/s} \\
1 \text{ AU} &= 1.496 \times 10^{13} \text{ cm} & k &= 1.38 \times 10^{-16} \text{ erg/K} \\
1 \text{ pc} &= 206,265 \text{ AU} & \sigma &= 5.6704 \times 10^{-5} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ K}^{-4}
\end{aligned}
$$

Planet	**Mass [g]**	**Radius [cm]**
Mercury	3.301×10^{26}	2.441×10^{8}
Venus	4.867×10^{27}	6.052×10^{8}
Mars	6.417×10^{26}	3.396×10^{8}
Jupiter	1.898×10^{30}	7.149×10^{9}
Saturn	5.683×10^{29}	6.027×10^{9}
Uranus	8.681×10^{28}	2.556×10^{9}
Neptune	1.024×10^{29}	2.476×10^{9}

1. Please write in complete sentences, and feel free to use equations and/or sketches to help explain your thoughts.

   (a) (5 points) Earth is not a blackbody; its UV-visible albedo is 0.37, and its infrared emissivity is 1.0. As a result, by what factor is its temperature different from that of a blackbody?

   Solution: Assuming uniform surface temperature,

$$
\begin{aligned}
T &= \left(\frac{1-A}{\varepsilon} \frac{L}{16\pi\sigma r^2} \right)^{1/4} \\
&= \left(\frac{1-A}{\varepsilon} \right)^{1/4} T_{bb} = \left(\frac{1-0.37}{1} \right)^{1/4} T_{bb} \\
&\boxed{T = 0.89 T_{bb}}
\end{aligned}
$$

   (b) (5 points) Why do we know Europa has liquid-water oceans under its smooth ice surface?

   Solution: Europa repels Jupiter's magnetic field in much the same way as an electrically-conducting surface the size of the moon would. Both solid ice and silicate rock are poor electrical conductors, but liquid salt water has the necessary electrical conductivity. Thus, we infer that Europa has a subsurface ocean under all of its surface ice.

   This is consistent with the pack-ice appearance of and lack of craters on Europa's surface, as well as the tidal heating it experiences. None of these conclusively show, though, that there is liquid water under the surface now; only the magnetic-field measurements do.

(c) (5 points) A small main-belt asteroid has a mass $M_A = 10^{18}$ g and a radius $R_A = 6 \times 10^5$ cm. Calculate the asteroid's bulk density, describe the sorts of rocks and minerals that are likely to make it up, and describe its structure.

> **Solution:**
>
> $$\begin{aligned} \rho &= \frac{M_A}{V_A} \\ &= \frac{3M_A}{4\pi R_A^3} = \frac{3 \times 10^{18}\ \text{g}}{4\pi (6 \times 10^5\ \text{cm})^3} \\ &\boxed{\rho = 1.11\ \text{g/cm}^3} \end{aligned}$$
>
> This is much less than the density of the silicate minerals that comprise most of the asteroids (the mafic minerals): those are around 3 g/cm^3. Thus, this asteroid is a very porous collection of silicate rocks and probably not very solid: it is likely to be a "rubble pile."

(d) (5 points) Besides mass and distance from the Sun, what is the biggest difference between Uranus and Neptune on one hand, and Jupiter and Saturn on the other?

> **Solution:** Uranus and Neptune are ice giants: their icy-rocky cores comprise a much larger fraction of their mass than is the case for the gas-giant planets Jupiter and Saturn.

2. A Hot Jupiter is a planet with a mass the same as Jupiter's ($M_J = 1.8986 \times 10^{30}$ g) and a radius 1.22 times that of Jupiter's ($R_J = 71,492$ km) that orbits extremely close to its star. Suppose one of these is in orbit around a Sun-like star.

   (a) (10 points) What is the minimum orbital radius of this Hot Jupiter? Give your answer in units of AU.

   Solution: The minimum orbital radius of the planet is its Roche limit:

$$\begin{aligned} a_{\text{Roche}} &\cong 2.46 R_{\text{planet}} \left(\frac{\rho_{\text{planet}}}{\rho}\right)^{1/3} \\ &\cong 2.46 R_{\odot} \left(\frac{M_{\odot}}{\frac{4}{3}\pi R_{\odot}^3} \frac{\frac{4}{3}\pi R^3}{M}\right)^{1/3} \\ &\cong 2.46 \left(\frac{M_{\odot}}{M_J}\right)^{1/3} (1.22 R_J) = 2.46 \left(\frac{1.99 \times 10^{33}\ \text{g}}{1.8986 \times 10^{30}\ \text{g}}\right)^{1/3} (1.22)(7.1492 \times 10^{9}\ \text{cm}) \end{aligned}$$

$$\boxed{a_{\text{Roche}} \cong 2.18 \times 10^{11}\ \text{cm} = 0.015\ \text{AU}}$$

   (b) (5 points) What would happen to a typical Hot Jupiter if it were placed in an orbit smaller than this?

   Solution: The planet's self gravity would be weaker than the tidal forces from the Sun, so the Hot Jupiter would get pulled apart. This would result in some sort of ring or disk forming around the Sun.

3. (10 points) Show that in a thin, uniform-temperature atmosphere on a terrestrial planet, the pressure depends upon the altitude z according to

$$P(z) = P_0 e^{-z/H} \tag{1}$$

where P_0 is the pressure at the surface ($z = 0$). Be sure to define the expression for the scale height H in terms of the planet's surface properties (temperature, gravitational acceleration, etc.).

Solution: The equation for hydrostatic equilibrium in one dimension for the atmosphere is

$$\frac{dP}{dz} = -\rho g = -\frac{\mu P}{kT} g \tag{2}$$

Let $\boxed{H \equiv \frac{kT}{\mu g}}$. Then

$$\begin{aligned}
\frac{dP}{dz} &= -\frac{P}{H} \\
\int_{P_0}^{P(z)} \frac{dP'}{P'} &= -\frac{1}{H} \int_0^z dz' \\
\ln\left(\frac{P(z)}{P_0}\right) &= -\frac{z}{H} \\
\frac{P(z)}{P_0} &= e^{-z/H} \\
&\boxed{P(z) = P_0 e^{-z/H}}
\end{aligned}$$

4. *Limit of differentiation*

(a) (10 points) Derive a formula for the temperature T as a function of radius r inside a uniform, spherical Solar System body which has a radius R, surface temperature T_s, density ρ, radioactive heating power per unit mass Λ, and thermal conductivity κ_T.

Solution: The Poisson equation for heating and heat conduction within a spherically symmetric body is

$$\begin{aligned}
\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) &= -\frac{\rho\Lambda}{\kappa_T} \\
\int_0^r \frac{d}{dr'}\left(r'^2\frac{dT}{dr'}\right)dr' &= -\frac{\rho\Lambda}{\kappa_T}\int_0^r r'^2\,dr' \\
r^2\frac{dT}{dr} &= -\frac{\rho\Lambda}{3\kappa_T}r^3 \\
\int_{T_0}^{T(r)} dT &= -\frac{\rho\Lambda}{3\kappa_T}\int_0^r r'\,dr' \\
T(r) - T_0 &= -\frac{\rho\Lambda}{6\kappa_T}r^2
\end{aligned}$$

At $r = R$, $T(R) = T_s$, so

$$\begin{aligned}
T_s - T_0 &= -\frac{\rho\Lambda}{6\kappa_T}R^2 \\
T_0 &= T_s + \frac{\rho\Lambda}{6\kappa_T}R^2
\end{aligned}$$

Therefore,

$$\boxed{T(r) = T_s + \frac{\rho\Lambda}{6\kappa_T}(R^2 - r^2)}$$

(b) (5 points) Suppose that $\rho = 3.35$ g/cm^3 and $\kappa_T = 4.79 \times 10^5$ erg s^{-1} g^{-1} K^{-1}, and that the radioactive heating rate is as it was when the Solar System was brand new, $\Lambda = 2.233 \times 10^{-3}$ erg s^{-1} g^{-1}. What is the maximum radius for a Solar System object with a core temperature that never exceeds that on the surface by more than 2000 K so that it does not differentiate?

Solution:

$$T(0) = \frac{\rho\Lambda}{6\kappa_T} R^2 + T_s$$

$$R = \sqrt{\frac{6\kappa_T}{\rho\Lambda}(T(0) - T_s)} = \sqrt{\frac{6(4.79 \times 10^5 \text{ erg s}^{-1} \text{ g}^{-1} \text{ K}^{-1})}{(3.35 \text{ g/cm}^3)(2.233 \times 10^{-3} \text{ erg s}^{-1} \text{ g}^{-1})}(2000 \text{ K})}$$

$$\boxed{R = 8.77 \times 10^5 \text{ cm}}$$

5. (10 points) A cylindrical "planetoid" with radius R and length H rotates rapidly with its axis perpendicular to the Solar System and revolves at an orbital radius r. Its thermal conductivity is very poor, so its surface temperature is determined just by Solar heating and blackbody radiation. Take its albedo to be A and its emissivity to be ε.

   Derive a formula for the temperature over the planetoid's entire surface.

Solution: The power being absorbed by the surface is due to the sunlight incident on the side of the planetoid facing the Sun.

$$P_{\text{in}} = \frac{(1-A)L_\odot}{4\pi r^2}(2RH) \tag{3}$$

The power emitted by the surface is due to blackbody radiation.

$$P_{\text{out}} = \varepsilon\sigma T_s^4 2\pi RH \tag{4}$$

Due to energy conservation, these two quantities must be equal.

$$P_{\text{in}} = P_{\text{out}}$$

$$(1-A)\frac{L_\odot}{4\pi r^2}2RH = \varepsilon\sigma T_s^4 2\pi RH$$

$$\boxed{T_s = \left(\frac{1-A}{\varepsilon}\frac{L_\odot}{4\pi^2\sigma r^2}\right)^{1/4}}$$

Since the circular ends of the planetoid do not receive any sunlight, their temperature $T_s = 0$.

6. **Late giant-planet formation and the ultimate fate of Earth.** Someday, a couple billion years from now, the Sun will become a red giant star, and stay that way for about 2 Myr before shedding its outer layers and giving rise to a white dwarf from the inner ones.

Suppose, plausibly, that when this happens, half of the Sun's mass expands into a uniform-density, pressure-supported, nonrotating sphere mostly made of atomic hydrogen gas that is just barely large enough to engulf the Earth in its orbit at $r = 1$ AU. The rest of the Sun's mass stays at the center, in a much smaller core.

(a) (5 points) What is the density of the gas in which the Earth is engulfed?

Solution:

$$\begin{aligned}\rho &= \frac{M}{V} \\ &= \frac{M}{\frac{4}{3}\pi R^3} = \frac{3 \cdot 0.5(1.989 \times 10^{33}\ \text{g})}{4\pi(1.496 \times 10^{13}\ \text{cm})^3}\end{aligned}$$

$$\boxed{\rho = 7.09 \times 10^{-8}\ \text{g/cm}^3}$$

(b) (10 points) What is our Earth's Keplerian orbital speed v, what is its orbital angular momentum L, and what headwind speed does our Earth, with mass $M_\oplus$, face in the Sun's red-giant atmosphere?

Solution: Since the uniform-density envelope just barely extends past Earth's orbit, $1M_\odot$ still lies within Earth's orbit (to a good approximation), so

$$\begin{aligned}F &= ma \\ -\frac{GM_\odot M_\oplus}{r^2} &= -M_\oplus \frac{v^2}{r} \\ v &= \sqrt{\frac{GM_\odot}{r}} = \sqrt{\frac{(6.674 \times 10^{-8}\ \text{dyn cm}^2\ \text{g}^{-2})(1.989 \times 10^{33}\ \text{g})}{1.496 \times 10^{13}\ \text{cm}}}\end{aligned}$$

$$\boxed{v = 2.98 \times 10^6\ \text{cm/s}}$$

$$\begin{aligned}L &= I\omega \\ &= (M_\oplus r^2)\left(\frac{v}{r}\right) \\ &= M_\oplus vr = (5.972 \times 10^{27}\ \text{g})(2.98 \times 10^6\ \text{cm/s})(1.496 \times 10^{13}\ \text{cm})\end{aligned}$$

$$\boxed{L = 2.66 \times 10^47\ \text{g cm}^2\text{/s}}$$

Since the expanded Solar atmosphere is not rotating, the **headwind speed is the same as the Keplerian orbital speed**.

(c) (10 points) Show that the path of our Earth's Hill sphere through the Solar atmosphere has volume $V = 2\pi^2 r^3 \left(\frac{M_\oplus}{3M_\odot}\right)^{2/3}$, and calculate the mass M that Earth would have if it were suddenly to accrete all this mass. Express your answer in Jupiter masses.

Solution:

$$\begin{aligned} V &= 2\pi r S_{\text{Hillsphere}} \\ &= 2\pi r(\pi r_H^2) \end{aligned}$$

$$\boxed{V = 2\pi^2 r^3 \left(\frac{M_\oplus}{3M_\odot}\right)^{2/3}}$$

If it were to accrete all this mass, the Earth's mass would become

$$\begin{aligned} M &= M_\oplus + \rho V \\ &= M_\oplus + \left(\frac{3 \cdot 0.5 M_\odot}{4\pi r^3}\right)\left(2\pi^2 r^3 \left(\frac{M_\oplus}{3M_\odot}\right)^{2/3}\right) \\ &= M_\oplus + \frac{\pi}{4}(3M_\odot M_\oplus^2)^{1/3} = (5.972 \times 10^{27}\ \text{g}) + \frac{\pi}{4}(3(1.989 \times 10^{33}\ \text{g})(5.972 \times 10^{27}\ \text{g})^2)^{1/3} \end{aligned}$$

$$\boxed{M = 4.74 \times 10^{29}\ \text{g} = 0.25 M_J}$$

Just a bit less than Saturn's mass.

(d) (10 points) Does the Earth stay in orbit, or does it drift into the Sun's core? Calculate the rate at which the headwind robs the planet of angular momentum as the planet accretes the gas that falls within its Hill sphere, and estimate the time it would take for the headwind to drag Earth into the center of the red-giant Sun.

Solution: The rate of loss of angular momentum is equal to the torque exerted on the planet from the gas that it is accreting:

$$\begin{aligned}
\tau &= \frac{dM}{dt} v_{HW} \\
&= (f_{\text{gas}} S) v_{HW} \\
&= (\rho v)(\pi r_H^2) v \\
&= \pi r_H^2 \rho v^2 \\
&= \frac{G}{8r^2} (3 M_\odot^4 M_\oplus^2)^{1/3}
\end{aligned}$$

The torque vector points opposite the angular momentum vector, because the headwind points opposite the Earth's momentum, so it drains Earth's angular momentum, according to the usual timescale relation:

$$\begin{aligned}
t_{\text{drag}} &= \left| \frac{L}{\frac{dL}{dt}} \right| \\
&= \frac{L}{\tau} \\
&= \frac{8r^2 M_\oplus v r}{G} (3 M_\odot^4 M_\oplus^2)^{-1/3} \\
&= \frac{8r^3 v}{G} \left(\frac{M_\oplus}{3 M_\odot^4} \right)^{1/3} = \frac{8(1.496 \times 10^{13} \text{ cm})(2.98 \times 10^6 \text{ cm/s})}{6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2}} \left(\frac{5.972 \times 10^{27} \text{ g}}{3(1.989 \times 10^{33} \text{ g})^4} \right)^{1/3}
\end{aligned}$$

$$\boxed{t_{\text{drag}} = 6.01 \times 10^{18} \text{ s} = 1.91 \times 10^{11} \text{ yr}}$$

This is much longer than 2 Myr, the time that the Sun will last as a red giant, so the Earth will stay in orbit and become a Saturn-sized gas giant orbiting the eventual white dwarf.