Astronomy 111 — Practice Final Exam

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Name: ______________________________

If this were a real exam, you would be reminded of the **Exam rules** here: "You may consult *only* one page of formulas and constants and a calculator while taking this test. You may *not* consult any books, digital resources, or each other. All of your work must be written on the attached pages, using the reverse sides if necessary. The final answers, and any formulas you use or derive, must be indicated clearly (answers must be circled or boxed). You will have two hours and thirty minutes to complete the exam. Good luck!"

Your results will improve if you take this practice test under realistic test-like conditions: in one sitting, with your already-prepared cheat sheet at hand, and with the will to resist peeking at the solutions until you are finished. Also, as usual:

- First, work on the problems you find the easiest. Come back later to the more difficult or less familiar material. Do not get stuck.
- The amount of space left for each problem is not necessarily an indication of the amount of writing it takes to solve it.
- You must show your work or explain your answer to receive full credit.
- Numerical answers are incomplete without units and should not be written with more significant figures than they deserve.
- Remember, you can earn partial credit for being on the right track. Be sure to show enough of your reasoning that we can figure out what you are thinking.

$$R_\odot = 6.96 \times 10^{10} \text{ cm} \qquad R_\oplus = 6.378 \times 10^{8} \text{ cm}$$
$$M_\odot = 1.989 \times 10^{33} \text{ g} \qquad M_\oplus = 5.972 \times 10^{27} \text{ g}$$
$$L_\odot = 3.827 \times 10^{33} \text{ erg/s} \qquad G = 6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2}$$
$$T_\odot = 5772 \text{ K} \qquad c = 3 \times 10^{10} \text{ cm/s}$$
$$1 \text{ AU} = 1.496 \times 10^{13} \text{ cm} \qquad k = 1.38 \times 10^{-16} \text{ erg/K}$$
$$1 \text{ pc} = 206,265 \text{ AU} \qquad \sigma = 5.6704 \times 10^{-5} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ K}^{-4}$$

1. Please write in complete sentences, and feel free to use equations and/or sketches to help explain your thoughts.

   (a) (5 points) On Venus, the dominant high-altitude wind systems blow straight from the equatorial regions to the polar regions, and the low-altitude winds blow in the opposite direction. Briefly explain why this is.

   Solution: Venus rotates slowly enough that its atmospheric circulation is dominated by one Hadley cell per hemisphere. A Hadley cell is a circulation pattern extending from a latitude where the heating is large (the equator) to one where it is cold (the poles); air heats and rises where it is hot, and cools and sinks where it is cold. Thus, the equatorial air rises, flows toward the poles, sinks when it arrives, and flows back toward the equator.

   (b) (5 points) Within ten minutes or so, what will be the sidereal time at midnight tonight?

   Solution: The Winter Solstice is 12/21, so the sidereal time at midnight is within minutes of 6:00:00 for the next few nights.

   More precisely, the sidereal time at midnight tonight (12/17) is

$$\mathrm{ST} = 6^{\mathrm{h}} - \left(\frac{4}{365.25}\right)(24^{\mathrm{h}})$$

$$\boxed{\mathrm{ST} = 5.73^{\mathrm{h}} = 5^{\mathrm{h}}44^{\mathrm{m}}13.80^{\mathrm{s}}}$$

(c) (5 points) Suppose that Saturn's rings were made only of very small particles and that Saturn had no moons at all. Describe the differences this would make in the appearance of the rings.

> **Solution:** The rings would be smooth and fairly uniform: there would be no gaps, spiral density waves, or bending waves, which are all structures created by the moons.

(d) (5 points) Rank these bodies' surfaces in order of age, and briefly explain your answer: the Moon, Io, and Enceladus.

> **Solution:** Based on crater density, the Moon is the oldest (heavily cratered highlands), followed by Enceladus (which has few craters), followed by Io (which has no craters).

2. *The habitable zone*

   (a) (10 points) Choose a suitable Bond albedo and emissivity for a habitable planet and calculate the radii (in AU) of the inner and outer edge of the habitable zone for rapid rotators around a star with luminosity $L = 1L_\odot$.

   Solution: The parameters of Earth are a pretty good start, so $A_b = 0.31$ and $\varepsilon = 1$. The inner radius of the habitable zone corresponds to a surface temperature of 373 K (the boiling point of water); the outer edge corresponds to 273 K. For a rapidly rotating planet,

$$r = \sqrt{\frac{1 - A_b}{\varepsilon} \frac{L}{4\pi^2 \sigma T^4}}$$

$$r_{\text{inner}} = \sqrt{\frac{1 - 0.31}{1} \frac{3.827 \times 10^{33} \text{ erg/s}}{4\pi^2 (5.6704 \times 10^{-5} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ K}^{-4})(373 \text{ K})^4}} = \boxed{0.52 \text{ AU}}$$

$$r_{\text{outer}} = \sqrt{\frac{1 - 0.31}{1} \frac{3.827 \times 10^{33} \text{ erg/s}}{4\pi^2 (5.6704 \times 10^{-5} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ K}^{-4})(273 \text{ K})^4}} = \boxed{0.98 \text{ AU}}$$

   (b) (5 points) According to your results from part a, what is the maximum orbital eccentricity for a planet which always lies in the habitable zone?

   Solution: In terms of semimajor axis and eccentricity, the perihelion and aphelion are given by

$$r_p = a(1 - \varepsilon) \qquad r_a = a(1 + \varepsilon) \tag{1}$$

   Dividing these two eliminates a and enables us to solve for ε:

$$\frac{r_p}{r_a} = \frac{1 - \varepsilon}{1 + \varepsilon}$$

$$\varepsilon = \frac{1 - \frac{r_p}{r_a}}{1 + \frac{r_p}{r_a}} = \frac{0.98 \text{ AU} - 0.52 \text{ AU}}{0.98 \text{ AU} + 0.52 \text{ AU}}$$

$$\boxed{\varepsilon = 0.30}$$

(c) **(5 points extra credit)** Show that this maximum orbital eccentricity for the habitable zone depends only upon the temperatures at the inner and outer boundary of the habitable zone, not upon whether the body is uniform in temperature, slowly rotating, or rapidly rotating.

Solution: Note that the relationship between the radius and the surface temperature is always of the form

$$r = \sqrt{\frac{1 - A_b}{\varepsilon} \frac{L}{C\sigma T^4}} \tag{2}$$

where $C = 16\pi$ for uniform temperature, 4π for the substellar point of a slow rotator, and $4\pi^2$ for the equator of a fast rotator. Everything cancels out but the temperature in the ratio:

$$\begin{aligned} \frac{r_p}{r_a} &= \sqrt{\frac{1 - A_b}{\varepsilon} \frac{L}{C\sigma T_p^4}} \sqrt{\frac{\varepsilon}{1 - A_b} \frac{C\sigma T_a^4}{L}} \\ &= \frac{T_a^2}{T_p^2} \end{aligned}$$

Therefore,

$$\varepsilon = \frac{1 - \frac{r_p}{r_a}}{1 + \frac{r_p}{r_a}}$$

$$\boxed{\varepsilon = \frac{1 - \frac{T_a^2}{T_p^2}}{1 + \frac{T_a^2}{T_p^2}}}$$

3. A meteorite from Antarctica is analyzed for the relative abundances of ^{87}Rb and ^{87}Sr, with results for a couple of different silicate minerals within the meteorite as follows:

Mineral	^{87}Rb/^{86}Sr	^{87}Sr/^{86}Sr
A	0.03653	0.70113
B	0.14560	0.70799

The decay rate of ^{87}Rb is $\lambda = 1.39 \times 10^{-11}\ \mathrm{yr}^{-1}$.

(a) (5 points) How old is the meteorite (in years)?

Solution: The age is given by

$$t = \frac{1}{\lambda} \ln\left(\frac{D_A - D_B}{N_A - N_B} + 1\right) = \frac{1}{1.39 \times 10^{-11}\ \mathrm{yr}^{-1}} \ln\left(\frac{0.70113 - 0.70799}{0.03653 - 0.14560} + 1\right)$$

$$\boxed{t = 4.39 \times 10^{9}\ \mathrm{yr}}$$

(b) (5 points) Is this meteorite typical? Briefly discuss its origin.

Solution: Most meteorites are closer to 4.6 Gyr old; 4.4 Gyr is closer to the age of the oldest rocks on the Moon. Thus we would not be able to rule out an origin on the Moon — like a rock ejected from the Moon during a relatively recent meteorite impact — instead of a more pristine body from further out in the Solar System.

4. Consider a planet with radius R in which the density decreases linearly from the center to the edge:

$$\rho(r) = \rho_0 \left(1 - \frac{r}{R}\right)$$

(a) (15 points) Show that its total mass is given by

$$M = \frac{\pi \rho_0 R^3}{3}$$

Solution: The mass of the planet is

$$\begin{aligned} dm &= \rho r^2 \sin\theta \, dr \, d\theta \, d\phi \\ M = \int dm &= \rho_0 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^R r^2 \left(1 - \frac{r}{R}\right) \, dr \\ &= \rho_0 \cdot 2\pi \cdot 2 \cdot \left(\frac{R^3}{3} - \frac{R^3}{4}\right) \\ &\boxed{M = \frac{\pi \rho_0 R^3}{3}} \end{aligned}$$

(b) (15 points) Show that its moment of inertia is given by

$$I = \frac{4}{15} M R^2$$

Solution: From the equation for the mass, we see that

$$\rho_0 = \frac{3M}{\pi R^3}$$

The planet's moment of inertia is

$$\begin{aligned}
I &= \int r_\perp^2 \, dm \\
&= \rho_0 \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta \, d\theta \int_0^R r^4 \left(1 - \frac{r}{R}\right) dr \\
&= \frac{8\pi}{3} \rho_0 \left(\frac{R^5}{5} - \frac{R^5}{6}\right) \\
&= \frac{8\pi \rho_0 R^5}{90} \\
&\boxed{I = \frac{4}{15} M R^2}
\end{aligned}$$

(c) (15 points) Suppose that the planet has a surface temperature T_s, and that its contents have radioactive heating power per unit mass Λ and thermal conductivity κ_T. Derive a formula for the temperature as a function of radius for the interior of the planet.

Solution: We can solve for the interior temperature of the planet by integrating the Poisson equation.

$$\begin{aligned}
\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) &= -\frac{\rho_0\Lambda}{\kappa_T}\left(1-\frac{r}{R}\right) \\
\int_0^r d\left(r'^2\frac{dT}{dr'}\right)\,dr' &= -\frac{\rho_0\Lambda}{\kappa_T}\int_0^r r'^2\left(1-\frac{r'}{R}\right)\,dr' \\
r^2\frac{dT}{dr} &= -\frac{\rho_0\Lambda}{\kappa_T}\left(\frac{r^3}{3}-\frac{r^4}{4R}\right) \\
\int_{T(0)}^{T(r)} dT &= -\frac{\rho_0\Lambda}{\kappa_T}\int_0^r\left(\frac{r'}{3}-\frac{r'^2}{4R}\right)\,dr' \\
T(r)-T(0) &= -\frac{\rho_0\Lambda}{\kappa_T}\left(\frac{r^2}{6}-\frac{r^3}{12R}\right)
\end{aligned}$$

We know the temperature at the surface ($r = R$), for which

$$\begin{aligned}
T_s = T(R) &= -\frac{\rho_0\Lambda}{\kappa_T}\left(\frac{R^2}{6}-\frac{R^3}{12R}\right)+T(0) \\
T(0) &= T_s+\frac{\rho_0\Lambda}{12\kappa_T}R^2
\end{aligned}$$

Therefore,

$$T(r) = -\frac{\rho_0\Lambda}{\kappa_T}\left(\frac{r^2}{6}-\frac{r^3}{12R}\right)+T_s+\frac{\rho_0\Lambda}{12\kappa_T}R^2$$

$$\boxed{T(r) = T_s+\frac{\rho_0\Lambda}{12\kappa_T}\left(R^2-2r^2+\frac{r^3}{R}\right)}$$

5. Over the course of many years, Jupiter (period $P_J = 11.856523$ yrs) perturbs the orbit of a main-belt asteroid that was originally in a 2:1 mean-motion resonance with the giant planet. Miraculously, it avoids collisions as its orbital eccentricity grows, until its aphelion is on its original circular orbit and its perihelion is on Earth's orbit.

   (a) (10 points) What is the orbit's eccentricity?

Solution: We need to first work out the original radius, that for which $P = \frac{P_J}{2} = 5.931$ yrs. We get this from Kepler's Third law:

$$P^2 = \frac{4\pi^2}{GM} r^3$$

$$r = \left(\frac{GM}{4\pi^2} P^2\right)^{1/3} = \left(\frac{(6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2})(1.989 \times 10^{33} \text{ g})}{4\pi^2} (5.931 \text{ yr})^2\right)^{1/3}$$

$$r = 3.276 \text{ AU} = r_a$$

Since the perihelion $r_p = r_\oplus = 1$ AU, the eccentricity is

$$\frac{r_a}{r_p} = \frac{1+\varepsilon}{1-\varepsilon}$$

$$\varepsilon = \frac{\frac{r_a}{r_p} - 1}{\frac{r_a}{r_p} + 1} = \frac{3.276 - 1}{3.276 + 1}$$

$$\boxed{\varepsilon = 0.532}$$

(b) (10 points) The asteroid has mass 10^{15} g. How much momentum (in g cm s^{-1}) did Jupiter transfer to (or from) the asteroid to change it to its new orbit?

Solution: We derived a formula for this once, but it may be easier to do it again than to remember it. The speed in the initial circular orbit around the Sun is

$$v_i = \sqrt{\frac{GM_\odot}{r_a}}$$

According to the vis viva equation, the speed at aphelion in the new orbit is

$$\begin{aligned} v_a &= \sqrt{GM_\odot\left(\frac{2}{r_a} - \frac{1}{a}\right)} \\ &= \sqrt{GM_\odot\left(\frac{2}{r_a} - \frac{1+\varepsilon}{r_a}\right)} \\ v_a &= \sqrt{\frac{GM_\odot}{r_a}(1-\varepsilon)} \end{aligned}$$

The difference in the magnitude of the momentum is then

$$\begin{aligned} \Delta p &= mv_a - mv_i \\ &= m\sqrt{\frac{GM_\odot}{r_a}}(1-\sqrt{1-\varepsilon}) = (1\times 10^{15}\ \text{g})\sqrt{\frac{(6.674\times 10^{-8}\ \text{dyn cm}^2\ \text{g}^{-2})(1.989\times 10^{33}\ \text{g})}{(3.276\ \text{AU})(1.496\times 10^{13}\ \text{cm/AU})}}(1-\sqrt{1-0.532}) \end{aligned}$$

$$\boxed{\Delta p = -5.2\times 10^{20}\ \text{g cm/s}}$$

(c) (10 points) How fast (in km/s) would the asteroid and Earth be going with respect to each other if they collide?

Solution: If the asteroid collided with Earth, it would be at its perihelion. According to the vis viva equation, its speed at perihelion is

$$\begin{aligned} v_p &= \sqrt{GM_\odot \left(\frac{2}{r_p} - \frac{1}{a}\right)} \\ &= \sqrt{GM_\odot \left(\frac{2}{r_p} - \frac{1+\varepsilon}{r_a}\right)} \end{aligned}$$

The Earth is traveling at a speed

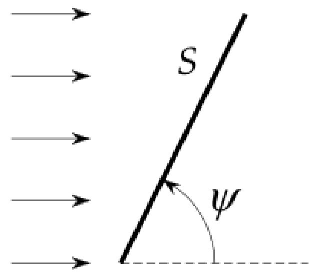
$$v_\oplus = \sqrt{\frac{GM_\odot}{r_p}}$$

Therefore,

$$\begin{aligned} \Delta v &= v_p - v_\oplus \\ &= \sqrt{GM_\odot \left(\frac{2}{r_p} - \frac{1+\varepsilon}{r_a}\right)} - \sqrt{\frac{GM_\odot}{r_p}} \end{aligned}$$

$$\boxed{\Delta v = 7.1 \text{ km/s}}$$

6. A wafer-like dust grain lies a distance r from a star with luminosity L. Its two flat surfaces have area S, and each surface has the same Bond albedo A_b and emissivity ε. The plane of one surface lies at an angle Ψ with respect to the incident starlight. It is thin enough and high enough in conductivity to be uniform in temperature.



(a) (10 points) Derive a formula for its temperature.

Solution: If the surface is inclined at an angle Ψ with respect to sunlight, then it casts a shadow with area $S \sin\Psi$ as can be seen in the above figure. The power absorbed from the starlight is then

$$P_{\text{in}} = (1 - A_b) f S \sin\Psi = \frac{(1 - A_b) L S \sin\Psi}{4\pi r^2}$$

The radiated power is

$$P_{\text{out}} = \varepsilon \sigma T^4 (2S)$$

if the grain has a temperature T throughout and we neglect radiation from the edges. Energy conservation then tells us that

$$\frac{(1 - A_b) L S \sin\Psi}{4\pi r^2} = 2\varepsilon\sigma T^4 S$$

$$\boxed{T = \left(\frac{(1 - A_b)}{\varepsilon} \frac{L \sin\Psi}{8\pi\sigma r^2} \right)^{1/4} = T_0 (\sin\Psi)^{1/4}}$$

where T_0 is the temperature for $\sin\Psi = 1$ (i.e. $\Psi = 90°$).

(b) (5 points) At what angle does the wafer have to be tilted for its temperature to be half of its value for normally-incident starlight ($\Psi = 90°$)?

Solution:

$$T = \frac{T_0}{2} = T_0(\sin\Psi)^{1/4}$$

$$\Psi = \sin^{-1}\left(\frac{1}{2^4}\right)$$

$$\boxed{\Psi = 3.6°}$$

7. Venus and Mars both have atmospheres made mostly of CO_2, which has molecular weight $\mu = 3.71 \times 10^{-23}$ g and adiabatic index $\gamma = \frac{9}{7}$. Other properties of these two planets include:

	Venus	Mars
Mass [g]	4.87×10^{27}	6.42×10^{26}
Radius [cm]	6.05×10^{8}	3.40×10^{8}
Surface temperature [K]	373	210

(a) (5 points) What is the gravitational acceleration (in cm/s^2) at the surface of each of these two planets?

Solution:

$$g = \frac{GM}{R^2}$$

$$g_V = \frac{(6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2})(4.87 \times 10^{27} \text{ g})}{(6.05 \times 10^{8} \text{ cm})^2} = \boxed{887 \text{ cm/s}^2}$$

$$g_M = \frac{(6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2})(6.42 \times 10^{26} \text{ g})}{(3.40 \times 10^{8} \text{ cm})^2} = \boxed{370 \text{ cm/s}^2}$$

(b) (5 points) If the atmospheres were isothermal, what would be the pressure scale height (in cm) near the surface of each planet?

Solution:

$$H = \frac{kT}{\mu g}$$

$$H_V = \frac{(1.38 \times 10^{-16} \text{ erg/K})(373 \text{ K})}{(3.71 \times 10^{-23} \text{ g})(887 \text{ cm/s}^2)} = \boxed{1.57 \times 10^{6} \text{ cm}}$$

$$H_M = \frac{(1.38 \times 10^{-16} \text{ erg/K})(210 \text{ K})}{(3.71 \times 10^{-23} \text{ g})(370 \text{ cm/s}^2)} = \boxed{1.07 \times 10^{6} \text{ cm}}$$

(c) (10 points) If the atmospheres were adiabatic, how much cooler would it be at one isothermal pressure scale-height above the surface? (Give your answer in K.)

Solution:

$$\begin{aligned}
\frac{dT}{dz} &= \frac{\gamma - 1}{\gamma}\frac{\mu g}{k} \\
\Delta T &= \frac{dT}{dz}\Delta z = \frac{\gamma - 1}{\gamma}\frac{\mu g}{k}\frac{kT}{\mu g} \\
&= \frac{\gamma - 1}{\gamma} T \\
\Delta T_V &= \frac{\frac{9}{7} - 1}{\frac{9}{7}} 373 \text{ K} = \boxed{164 \text{ K}} \\
\Delta T_M &= \frac{\frac{9}{7} - 1}{\frac{9}{7}} 210 \text{ K} = \boxed{46.7 \text{ K}}
\end{aligned}$$

8. In a young, sedimented protoplanetary disk, a KBO-size object ($R_0 = 100$ km $= 10^7$ cm) has just formed in a region where the disk's gas density is $\rho_g = 8.7 \times 10^{-12}$ g/cm^3, the dust density is $\rho_d = 7.0 \times 10^{-11}$ g/cm^3, and the headwind speed is $v_{HW} = 1.0 \times 10^4$ cm/s. (These are the values you would get in our usual model of the Solar nebula at a radius of 1 AU.)

(a) (10 points) What is the minimum time (in millions of years) for this planetesimal to grow to Earth size, $R(t) = R_\oplus$, by dust accretion?

Solution: The minimum time corresponds to the fastest growth mechanism, where the planetesimal's escape speed is much greater than the speed at which accretable material arrives:

$$\frac{dR}{dt} = \frac{2\pi\rho_d R^2}{3v_{HW}}$$

We have three choices. Either integrate this expression and solve for the time t:

$$\begin{aligned} \int_{R_0}^{R_\oplus} \frac{dR'}{R'^2} &= \frac{2\pi\rho_d}{3v_{HW}} \int_0^t dt' \\ -\frac{1}{R_\oplus} + \frac{1}{R_0} &= \frac{2\pi\rho_d}{3v_{HW}} t \\ t &= \frac{3v_{HW}}{2\pi\rho_d}\left(\frac{1}{R_0} - \frac{1}{R_\oplus}\right) \\ &\boxed{t = 1.0 \times 10^{14}\ \text{s} = 3.2\ \text{Myr}} \end{aligned}$$

Or we can take the expression we have derived from this procedure for $R(t)$, which many of you would have on your equation sheet:

$$R(t) = \frac{R_0}{1 - \frac{2\pi G\rho_d R_0}{3v_{HW}} t}$$

where $R(t) = R_\oplus$. Solving this for t would give you the same answer as above. Or, consider that $R_\oplus$ is so much larger than the original radius of the planetesimal that $R_\oplus$ would be reached very close to the time for runaway accrection, which is

$$t_{\text{runaway}} = \frac{3v_{HW}}{2\pi G\rho_d R_0} = 3.3\ \text{Myr}$$

(b) (5 points) What is the minimum time (in millions of years) for the resulting Earth-size planetesimal to grow to gas-giant planetary size by accretion of gas? (Assume that the disk's dust is all used up in the first phase.)

Solution: Here, you are implicitly asked to calculate the runaway accretion time:

$$t_{\text{runaway}} = \frac{3v_{HW}}{2\pi G\rho_g R_\oplus} = 0.41\ \text{Myr}$$