1. The Planck formula for the intensity (energy per unit time, per unit area, per unit wavelength, per unit solid angle) of a blackbody is

$$B_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

Approximations to the Planck formula for very long or very short wavelengths often turn out to be useful. Derive those expressions here as an exercise in making approximations.

(a) Carefully consider the Planck formula. To what quantity should we compare the wavelength to determine whether or not the wavelength is large or small?

Solution: Look at the dimensionless variable $x = \lambda kT/hc$. This appears in the exponential in the denominator of the Planck function. When $x \gg 1$, $1/x \ll 1$, and $e^{1/x}$ may be replaced with the first-order approximation. So, evidently, we need to consider whether x may be large or small compared to 1, which is the same as λ being large or small compared to hc/kT.

(b) To what form does the Planck function reduce for very long wavelengths? This is called the **Rayleigh-Jeans** limit of the blackbody function.

Solution: Long wavelengths means $\lambda \gg hc/kT$, or $x \gg 1$, or $1/x \ll 1$. In this case we can make the first-order approximation to the exponential:

$$\frac{1}{e^{hc/\lambda kT} - 1} = \frac{1}{e^{1/x} - 1} \approx \frac{1}{(1 + 1/x) - 1} = x = \frac{\lambda kT}{hc}$$

$$\boxed{B_\lambda(\lambda, T) \approx \frac{2hc^2}{\lambda^5} \frac{\lambda kT}{hc} = \frac{2ckT}{\lambda^4}}$$

(c) To what form does the Planck function reduce for very short wavelengths? This is called the **Wien** limit of the blackbody function.

Solution: For short wavelengths $1/x \gg 1$, so $e^{1/x} \gg 1$ as well:

$$\frac{1}{e^{hc/\lambda kT} - 1} = \frac{1}{e^{1/x} - 1} \approx \frac{1}{e^{1/x}} = e^{-1/x} = e^{-hc/\lambda kT}$$

$$\boxed{B_\lambda(\lambda, T) \approx \frac{2hc^2}{\lambda^5} e^{-hc/\lambda kT}}$$

2. The absolute bolometric magnitude of the Sun is $M_\odot = 4.75$, and the luminosity of the Sun is $L_\odot = 3.827 \times 10^{33}$ erg/s. Show that the apparent bolometric magnitude m and absolute bolometric magnitude M of a star with luminosity L that lies a distance r away from the Solar System are related by

$$M = m - 5 \log \left(\frac{r}{10 \text{ pc}} \right) = 4.75 - 2.5 \log \left(\frac{L}{L_\odot} \right)$$

The difference between the apparent and absolute magnitudes, $m - M = 5 \log (r/10 \text{ pc})$, is called the *distance modulus.*

Solution: The absolute magnitude is the magnitude that the star would have if it were 10 pc away, so

$$\begin{aligned}
M - m &= 2.5 \log\left(\frac{f(r)}{f(10\text{ pc})}\right) = 2.5\log\left(\frac{L}{4\pi r^2}\frac{4\pi(10\text{ pc})^2}{L}\right) \\
&= 2.5\log\left[\left(\frac{10\text{ pc}}{r}\right)^2\right] = 5\log\left(\frac{10\text{ pc}}{r}\right) = -5\log\left(\frac{r}{10\text{ pc}}\right) \\
M &= m - 5\log\left(\frac{r}{10\text{ pc}}\right)
\end{aligned}$$

For absolute magnitudes M and $M_\odot$ of the star and the Sun, we have

$$\begin{aligned}
M - M_\odot &= 2.5\log\left(\frac{f_\odot(10\text{ pc})}{f(10\text{ pc})}\right) = 2.5\log\left(\frac{L_\odot}{4\pi(10\text{ pc})^2}\frac{4\pi(10\text{ pc})^2}{L}\right) \\
&= 2.5\log\left(\frac{L_\odot}{L}\right) = -2.5\log\left(\frac{L}{L_\odot}\right) \\
M &= M_\odot - 2.5\log\left(\frac{L}{L_\odot}\right) = 4.75 - 2.5\log\left(\frac{L}{L_\odot}\right)
\end{aligned}$$

3. A variable star changes in brightness by a factor of 4. What is the change in magnitude?

Solution: Suppose that its flux at time t_2 is larger by a factor of 4 than its flux at time t_1:

$$f_2 = 4f_1$$

Then

$$\begin{aligned}
m_1 - m_2 &= 2.5\log\left(\frac{f_2}{f_1}\right) = 2.5\log\left(\frac{4f_1}{f_1}\right) \\
m_1 - m_2 &= 1.5
\end{aligned}$$

It changes by 1.5 magnitudes.

4. Suppose there is a type of star we can identify independent of distance, and we know that all examples of the star have the same absolute magnitude M. Then a measurement of the apparent magnitude m of one such star can be used to infer its distance. Suppose further that the apparent magnitude can be determined to an accuracy of ± 0.01 magnitudes. What is the accuracy to which distances to the stars can be determined?

   (Hint: Recall that for any function $f(x)$, a *small* interval Δx in the independent variable corresponds to an interval
   $$\Delta f = \frac{df}{dx}\Delta x$$
   in the value of the function.)

Solution: First solve the definition of distance modulus for distance:

$$
\begin{aligned}
m - M &= 5\log\left(r/10\text{ pc}\right) \\
r &= (10\text{ pc})\ 10^{(m-M)/5}
\end{aligned}
$$

Then differentiate r with respect to m:

$$
\begin{aligned}
\frac{dr}{dm} &= \frac{d}{dm}(10\text{ pc})\ 10^{(m-M)/5} \\
&= \frac{d}{dm}(10\text{ pc})\ e^{(m-M)\ln 10/5} \\
&= (10\text{ pc})\ e^{(m-M)\ln 10/5}\ \frac{d}{dm}\left(\frac{m-M}{5}\ln 10\right) \\
&= (10\text{ pc})e^{(m-M)\ln 10/5}\left(\frac{\ln 10}{5}\right) \\
\frac{dr}{dm} &= r\frac{\ln 10}{5}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Delta r &= \frac{dr}{dm}\Delta m = r\frac{\ln 10}{5}\Delta m, \text{ or} \\
\frac{\Delta r}{r} &= \frac{\Delta m \ln 10}{5} = \frac{(0.01)(2.303)}{5} \\
&\boxed{\frac{\Delta r}{r} = 0.005 = 0.5\%}
\end{aligned}
$$

In reality, the technique is not nearly this accurate because corrections due to *extinction* add considerably larger uncertainties. We will discuss extinction later in the course.

5. (a) Show that, for a *blackbody* which occupies a small solid angle $\Delta\Omega$, at low temperatures the relation between temperature and $B - V$ color is something like

$$
T \approx \frac{7400\text{ K}}{(B-V) + 1.4}
$$

(Hint: Use the Wien approximation and assume the properties of the B and V filters given in Lecture 3.)

Solution:

$$
\begin{aligned}
B - V &= 2.5\log\left(\frac{f_V}{f_B}\right) = 2.5\log\left(\frac{B_\lambda(\lambda_V,T)\Delta\lambda_V\Delta\Omega}{B_\lambda(\lambda_B,T)\Delta\lambda_B\Delta\Omega}\right) \\
&= 2.5\log\left[\left(\frac{\lambda_B}{\lambda_V}\right)^5 \frac{e^{hc/\lambda_B kT}-1}{e^{hc/\lambda_V kT}-1}\frac{\Delta\lambda_V}{\Delta\lambda_B}\right]
\end{aligned}
$$

If the temperature is low, so that $T \ll hc/k\lambda$ for both wavelengths, then

$$
e^{hc/\lambda kT} \gg 1
$$

and the Wien approximation applies. Therefore,

$$\begin{aligned}
B - V &\approx 2.5\log\left[\left(\frac{\lambda_B}{\lambda_V}\right)^5 e^{hc/\lambda_B kT - hc/\lambda_V kT}\frac{\Delta\lambda_V}{\Delta\lambda_B}\right] \\
&= 2.5\log\left[\left(\frac{\lambda_B}{\lambda_V}\right)^5 \frac{\Delta\lambda_V}{\Delta\lambda_B}\right] + 2.5\log\left(e^{hc/\lambda_B kT - hc/\lambda_V kT}\right) \\
&= 2.5\log\left[\left(\frac{\lambda_B}{\lambda_V}\right)^5 \frac{\Delta\lambda_V}{\Delta\lambda_B}\right] + 2.5\log e\ln\left(e^{hc/\lambda_B kT - hc/\lambda_V kT}\right) \\
&= 2.5\log\left[\left(\frac{\lambda_B}{\lambda_V}\right)^5 \frac{\Delta\lambda_V}{\Delta\lambda_B}\right] + 2.5\log e\frac{hc}{kT}\left(\frac{1}{\lambda_B} - \frac{1}{\lambda_V}\right) \\
&= 2.5\log\left[\left(\frac{430\text{ nm}}{540\text{ nm}}\right)^5 \frac{90\text{ nm}}{100\text{ nm}}\right] \\
&\quad + 2.5\log e\frac{(6.6261\times 10^{-27}\text{ erg s})(3\times 10^{10}\text{ cm/s})}{(1.38\times 10^{-16}\text{ erg/K})T}\left(\frac{1}{430\text{ nm}} - \frac{1}{540\text{ nm}}\right) \\
&= -1.351 + \frac{7400\text{ K}}{T} \\
T &= \frac{7400\text{ K}}{(B-V)+1.4}
\end{aligned}$$

(b) This relation is a very poor approximation at high temperatures. Why?

Solution: If the temperature T is too high, then the $e^{hc/\lambda kT} \gg 1$ approximation fails. Since T appears in an exponential, making it very sensitive to the value of $hc/\lambda kT$, it fails pretty badly.