1. Suppose that two stars with masses m_1 and m_2 are in circular orbits about their common center of mass with radii r_1 and r_2, respectively. The orbital period is P.

   (a) Show that $r_1 = m_2 r/(m_1+m_2)$ and $r_2 = m_1 r/(m_1+m_2)$, where $r = r_1+r_2$ is the distance between the two stars.

   Solution: By the definition of the center of mass,

   $$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

   Combining this with the definition $r = r_1 + r_2$ gives

   $$\begin{aligned} r_1 &= \frac{m_2}{m_1} r_2 = \frac{m_2}{m_1}(r - r_1) \\ r_1\left(1 + \frac{m_2}{m_1}\right) &= \frac{m_2}{m_1} r \\ r_1 &= \frac{m_2}{m_1\left(1 + m_2/m_1\right)}\, r \\ \boxed{r_1 = \frac{m_2}{m_1 + m_2}\, r} \end{aligned}$$

   and

   $$\begin{aligned} r_2 &= r - r_1 \\ &= \left(1 - \frac{m_2}{m_1 + m2}\right) r \\ &= \left(\frac{m_1 + m_2}{m_1 + m_2} - \frac{m_2}{m_1 + m_2}\right) r \\ \boxed{r_2 = \frac{m_1}{m_1 + m_2}\, r} \end{aligned}$$

   (b) The speeds of the stars in orbit are constant and given by $v_1 = \Omega r_1$ and $v_2 = \Omega r_2$, respectively, where $\Omega = 2\pi/P$ is the angular speed of revolution of either star about the center of mass. Show that Newton's second law, applied to either star, leads to

   $$m_1 + m_2 = \frac{\Omega^2 r^3}{G}$$

   This should look familiar: it is Kepler's third law, in a slightly different form than usual.

   Solution:

   $$\begin{aligned} F &= ma \\ G\frac{m_1 m_2}{r^2} &= m_1 \frac{v_1^2}{r_1} = m_1 \Omega^2 r_1 = m_1 \Omega^2 \frac{m_2}{m_1 + m_2}\, r \\ m_1 + m_2 &= \frac{\Omega^2 r^3}{G} \end{aligned}$$

   (c) Show that if the orbital axis (the line through the center of mass, perpendicular to the plane of the orbits) is inclined by an angle i with respect to our line of sight, then the maximum Doppler velocities that will be observed for the two stars relative to the Doppler velocity of their center of mass are $v_{1r} = \Omega r_1 \sin i$ and $v_{2r} = \Omega r_2 \sin i$, respectively.

   Solution: If the orbital axis is inclined by an angle i with respect to the line of sight, then all vectors in the orbital plane that are also in the plane of the axis and line of sight are inclined with respect to the line of sight by the angle $\pi/2 - i$. The maximum and minimum radial velocities fit this description, so the Doppler velocity amplitudes for the stars are

   $$v_{1r} = r_1 \Omega \cos\left(\frac{\pi}{2} - i\right) = r_1 \Omega \sin i$$

   and similarly

   $$v_{2r} = r_2 \Omega \sin i$$

(d) Eliminate r_1 and r_2 from the previous expressions to show that

$$r = \frac{v_{1r} + v_{2r}}{\Omega \sin i} \qquad \text{and} \qquad \frac{m_2}{m_1} = \frac{v_{1r}}{v_{2r}}$$

Note that the right-hand side of each expression contains only observables. With these, and the results of part b, measurements can be used to obtain the masses and separations of the stars.

Solution: Adding the radial velocities just obtained gives

$$v_{1r} + v_{2r} = (r_1 + r_2)\Omega \sin i = r\Omega \sin i$$
$$r = \frac{v_{1r} + v_{2r}}{\Omega \sin i}$$

Dividing the radial velocities gives

$$\frac{v_{1r}}{v_{2r}} = \frac{r_1}{r_2} = \frac{m_2 r}{m_1 + m_2} \frac{m_1 + m_2}{m_1 r} = \frac{m_2}{m_1}$$

(e) Suppose that a certain binary is eclipsing, and you have measured the period to be 11 days and the radial velocities to be $v_{1r} = 75$ km/s and $v_{2r} = 100$ km/s. What are the masses (in $M_\odot$) and separation (in $R_\odot$) of the two stars?

Solution: If the binary is eclipsing we know that $i = 90°$. Therefore, using the results of parts d and b gives

$$r = \frac{v_{1r} + v_{2r}}{\Omega \sin i} = \frac{(75 + 100) \times 10^5 \text{ cm/s}}{2\pi} (11 \text{ day} \times 86400 \text{ s/day})$$
$$\boxed{r = 2.65 \times 10^{12} \text{ cm} = 38 R_\odot}$$

$$m_1 + m_2 = \frac{\Omega^2 r^3}{G} = \frac{4\pi^2}{(11 \times 86400 \text{ s})^2} \frac{(2.65 \times 10^{12} \text{ cm})^3}{6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2}}$$
$$m_1 + m_2 = 1.22 \times 10^{34} \text{ g} = 6.11 M_\odot$$

Combining $m_1 + m_2$ with the expression

$$\frac{v_{1r}}{v_{2r}} = \frac{m_2}{m_1} = 0.75$$

allows us to solve for m_1 and m_2:

$$\boxed{m_1 = \frac{6.11 M_\odot}{1.75} = 3.49 M_\odot} \qquad \text{and} \qquad \boxed{m_2 = 6.11 M_\odot - m_1 = 2.62 M_\odot}$$

2. *Single-line spectroscopic binaries and the "mass function."* Consider the binary star system from the previous problem.

(a) From the equations you derived in the previous problem, show that the sum of the stellar masses is given in terms of the radial velocities by

$$m_1 + m_2 = \frac{P}{2\pi G} \frac{(v_{1r} + v_{2r})^3}{\sin^3 i}$$

Solution:

$$\begin{aligned}
m_1 + m_2 &= \frac{\Omega^2 r^3}{G} \\
&= \frac{\Omega^2}{G}\left(\frac{v_{1r}+v_{2r}}{\Omega \sin i}\right)^3 = \frac{1}{\Omega G}\frac{(v_{1r}+v_{2r})^3}{\sin^3 i} \\
m_1 + m_2 &= \frac{P}{2\pi G}\frac{(v_{1r}+v_{2r})^3}{\sin^3 i}
\end{aligned}$$

(b) Next, show that

$$\frac{m_2^3}{(m_1+m_2)^2}\sin^3 i = \frac{P}{2\pi G}v_{1r}^3 \tag{1}$$

In the right hand side, only the observable quantities P and v_{1r} appear. This quantity, $f(m_1, m_2) = Pv_{1r}^3/2\pi G$, is called the *mass function*.

Solution:

$$\begin{aligned}
m_1 + m_2 &= \frac{P}{2\pi G}\frac{(v_{1r}+v_{2r})^3}{\sin^3 i} \\
&= \frac{P}{2\pi G}\frac{1}{\sin^3 i}\left(v_{1r} + v_{1r}\frac{m_1}{m_2}\right)^3 \\
&= \frac{P}{2\pi G}\frac{v_{1r}^3}{\sin^3 i}\left(\frac{m_1+m_2}{m_2}\right)^3 \\
\frac{m_2^3}{(m_1+m_2)^2}\sin^3 i &= \frac{P}{2\pi G}v_{1r}^3 \equiv f(m_1, m_2)
\end{aligned}$$

(c) Show that the left hand side of Eqn. 1 is always less than m_2; in other words, m_2 is always greater than the mass function.

Note: Equation 1 is useful for single-line spectroscopic binaries: those in which one of the stars (1) is much brighter than the other, so that only its spectral lines are seen. In particular, the lower limit to the mass of the unseen companion, m_2, obtained in this manner has been useful in the identification of black holes in binary stellar systems, as we will see in a couple of weeks.

Solution:

$$f(m_1, m_2) = \frac{m_2^3}{(m_1+m_2)^2}\sin^3 i < \frac{m_2^3}{(m_1+m_2)^2} = m_2\frac{1}{(1+m_1/m_2)^2}$$

But $1/(1+m_1/m_2)^2 < 1$ since m_1/m_2 is a positive number. Therefore, $f(m_1, m_2) < m_2$.

3. At the center of the Sun, the mass density is $\rho_c = 1.52 \times 10^5$ kg/m^3 and the mean opacity is $\kappa = 0.12$ m^2/kg. (The opacity is a measure of how opaque a material is — it is often a function of temperature, density, and chemical composition.) What is the mean free path for a photon at the Sun's center?

Solution:

From dimensional analysis, we see that

$$\ell_c = \frac{1}{\kappa\rho_c} = \frac{1}{(0.12\ \mathrm{m^2/kg})(1.52 \times 10^5\ \mathrm{kg/m^3})}$$
$$\ell_c = 5.5 \times 10^{-3}\ \mathrm{cm}$$

4. Suppose we somehow know that the mass density with a star of mass M and radius R decreases linearly from the center to the surface of the star and vanishes at the surface, i.e.,

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right)$$

(a) What is the density ρ_c at the center of the star in terms of M and R?

Solution: We can find this out by calculating M from the form given for the density:

$$\begin{aligned}
M &= \int \rho(r) dV = \rho_c \int \left(1 - \frac{r}{R}\right) r^2 \sin\theta \, dr \, d\theta \, d\phi \\
&= 4\pi\rho_c \int_0^R \left(r^2 - \frac{r^3}{R}\right) dr \\
&= 4\pi\rho_c \left[\frac{r^3}{3} - \frac{r^4}{4R}\right]_0^R = \frac{\pi\rho_c R^3}{3} \\
\rho_c &= \frac{3M}{\pi R^3}
\end{aligned}$$

(b) Show that the pressure at the center of the star is

$$P_c = \frac{5}{4\pi}\frac{GM^2}{R^4}$$

Solution: Begin by integrating the radial pressure gradient:

$$\begin{aligned}
\frac{dP}{dr} &= -\frac{GM(r)\rho(r)}{r^2} \\
\int_{P(0)}^0 dP &= -G\int_0^R \frac{M(r)\rho(r)}{r^2}\, dr \\
P(0) = P_c &= \int_0^R \frac{GM(r)\rho(r)}{r^2} dr
\end{aligned}$$

Before solving this we have to integrate the radial profile of the density to get an expression for

$M(r)$:

$$
\begin{aligned}
M(r) &= \int_0^r \rho(r')\,dV = \rho_c \int \left(1 - \frac{r'}{R}\right) r'^2 \sin\theta\,dr'\,d\theta\,d\phi \\
&= \frac{12M}{R^3}\int_0^r \left(r'^2 - \frac{r'^3}{R}\right) dr' \\
&= 12M\left[\frac{1}{3}\left(\frac{r}{R}\right)^3 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right] \\
M(r) &= M\left[4\left(\frac{r}{R}\right)^3 - 3\left(\frac{r}{R}\right)^4\right]
\end{aligned}
$$

Now use $M(r)$ to solve for $P_c = P(0)$:

$$
\begin{aligned}
P_c &= \int_0^R \frac{GM(r)\rho(r)}{r^2} dr \\
&= \frac{3GM^2}{\pi R^3}\int_0^R \frac{1}{r^2}\left[4\left(\frac{r}{R}\right)^3 - 3\left(\frac{r}{R}\right)^4\right]\left(1 - \frac{r}{R}\right) dr \\
&= \frac{3GM^2}{\pi R^6}\int_0^R \left(4r - 7\frac{r^2}{R} + 3\frac{r^3}{R^2}\right) dr \\
&= \frac{3GM^2}{\pi R^6}\left[2r^2 - \frac{7}{3}\frac{r^3}{R} + \frac{3}{4}\frac{r^4}{R^2}\right]_0^R \\
P_c &= \frac{5}{4\pi}\frac{GM^2}{R^4}
\end{aligned}
$$

5. Suppose that the mass density of a star of radius R increases quadratically with distance from the center,

$$
\rho(r) = \rho_c\left[1 - \left(\frac{r}{R}\right)^2\right]
$$

(a) Find the mass M of the star in terms of ρ_c and R.

Solution:

$$
\begin{aligned}
M &= \int \rho(r)\,dV = \rho_c \int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right] r^2 \sin\theta\,dr\,d\theta\,d\phi \\
&= 4\pi\rho_c \int_0^R \left(r^2 - \frac{r^4}{R^2}\right) dr \\
&= 4\pi\rho_c \left[\frac{r^3}{3} - \frac{r^5}{5R^2}\right]_0^R \\
&= \frac{8\pi\rho_c R^3}{15}
\end{aligned}
$$

(b) Find the average density of the star in terms of ρ_c.

Solution:

$$\bar{\rho} = \frac{M}{V} = \frac{8\pi\rho_c R^3}{15}\frac{3}{4\pi R^3} = \frac{2}{5}\rho_c$$

(c) Show that the central pressure of the star is

$$P_c = \frac{15}{16\pi}\frac{GM^2}{R^4}$$

Solution: First generate an expression for $M(r)$ by integrating the radial density profile between 0 and r:

$$\begin{aligned} M(r) &= \int \rho(r')\,dV = \rho_c \int \left[1 - \left(\frac{r'}{R}\right)^2\right] r'^2 \sin\theta\,dr'\,d\theta\,d\phi \\ &= 4\pi\rho_c \int_0^r \left(r'^2 - \frac{r'^4}{R^2}\right) dr' \\ &= 4\pi\rho_c \left[\frac{r'^3}{3} - \frac{r'^5}{5R^2}\right]_0^r \\ &= \frac{M}{2}\left[5\left(\frac{r}{R}\right)^3 - 3\left(\frac{r}{R}\right)^5\right] \end{aligned}$$

Then insert $M(r)$ into the integral of the equation for hydrostatic equilibrium:

$$\begin{aligned} P_c &= \int_0^R \frac{GM(r)\rho(r)}{r^2} dr \\ &= \frac{GM\rho_c}{2}\int_0^R \frac{1}{r^2}\left[5\left(\frac{r}{R}\right)^3 - 3\left(\frac{r}{R}\right)^5\right]\left[1 - \left(\frac{r}{R}\right)^2\right] dr \\ &= \frac{GM\rho_c}{2}\int_0^R \left(\frac{r}{3} - \frac{8r^3}{15R^2} + \frac{r^5}{5R^4}\right) dr = \frac{GM\rho_c}{2}\left[\frac{r^2}{6} - \frac{2r^4}{15R^2} + \frac{r^6}{30R^4}\right]_0^R \\ P_c &= \frac{15}{16\pi}\frac{GM^2}{R^4} \end{aligned}$$