1. Suppose we somehow know that the mass density with a star of mass M and radius R decreases linearly from the center to the surface of the star and vanishes at the surface, i.e.,

$$\rho(r) = \rho_c \left(1 - \frac{r}{R} \right)$$

(a) What is the density ρ_c at the center of the star in terms of M and R?

Solution: We can find this out by calculating M from the form given for the density:

$$M = \int \rho(r) dV = \rho_c \int \left(1 - \frac{r}{R} \right) r^2 \sin \theta \, dr \, d\theta \, d\phi$$

$$= 4\pi \rho_c \int_0^R \left(r^2 - \frac{r^3}{R} \right) dr$$

$$= 4\pi \rho_c \left[\frac{r^3}{3} - \frac{r^4}{4R} \right]_0^R = \frac{\pi \rho_c R^3}{3}$$

$$\rho_c = \frac{3M}{\pi R^3}$$

(b) Show that the pressure at the center of the star is

$$P_c = \frac{5}{4\pi} \frac{GM^2}{R^4}$$

Solution: Begin by integrating the radial pressure gradient:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\int_{P(0)}^0 dP = -G \int_0^R \frac{M(r)\rho(r)}{r^2} dr$$

$$P(0) = P_c = \int_0^R \frac{GM(r)\rho(r)}{r^2} dr$$

Before solving this we have to integrate the radial profile of the density to get an expression for $M(r)$:

$$M(r) = \int_0^r \rho(r') \, dV = \rho_c \int \left(1 - \frac{r'}{R} \right) r'^2 \sin \theta \, dr' \, d\theta \, d\phi$$

$$= \frac{12M}{R^3} \int_0^r \left(r'^2 - \frac{r'^3}{R} \right) dr'$$

$$= 12M \left[\frac{1}{3} \left(\frac{r}{R} \right)^3 - \frac{1}{4} \left(\frac{r}{R} \right)^4 \right]$$

$$M(r) = M \left[4 \left(\frac{r}{R} \right)^3 - 3 \left(\frac{r}{R} \right)^4 \right]$$

Now use $M(r)$ to solve for $P_c = P(0)$:

$$P_c = \int_0^R \frac{GM(r)\rho(r)}{r^2}\,dr$$

$$= \frac{3GM^2}{\pi R^3}\int_0^R \frac{1}{r^2}\left[4\left(\frac{r}{R}\right)^3 - 3\left(\frac{r}{R}\right)^4\right]\left(1 - \frac{r}{R}\right)dr$$

$$= \frac{3GM^2}{\pi R^6}\int_0^R \left(4r - 7\frac{r^2}{R} + 3\frac{r^3}{R^2}\right)dr$$

$$= \frac{3GM^2}{\pi R^6}\left[2r^2 - \frac{7}{3}\frac{r^3}{R} + \frac{3}{4}\frac{r^4}{R^2}\right]_0^R$$

$$P_c = \frac{5}{4\pi}\frac{GM^2}{R^4}$$

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2. Suppose that the mass density of a star of radius R increases quadratically with distance from the center,

$$\rho(r) = \rho_c\left[1 - \left(\frac{r}{R}\right)^2\right]$$

(a) Find the mass M of the star in terms of ρ_c and R.

Solution:

$$M = \int \rho(r)\,dV = \rho_c\int_0^R\left[1 - \left(\frac{r}{R}\right)^2\right]r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 4\pi\rho_c\int_0^R\left(r^2 - \frac{r^4}{R^2}\right)dr$$

$$= 4\pi\rho_c\left[\frac{r^3}{3} - \frac{r^5}{5R^2}\right]_0^R$$

$$= \frac{8\pi\rho_c R^3}{15}$$

(b) Find the average density of the star in terms of ρ_c.

Solution:

$$\bar{\rho} = \frac{M}{V} = \frac{8\pi\rho_c R^3}{15}\frac{3}{4\pi R^3} = \frac{2}{5}\rho_c$$

(c) Show that the central pressure of the star is

$$P_c = \frac{15}{16\pi}\frac{GM^2}{R^4}$$

Solution: First generate an expression for $M(r)$ by integrating the radial density profile be-

tween 0 and r:

$$M(r) = \int \rho(r') \, dV = \rho_c \int \left[1 - \left(\frac{r'}{R}\right)^2 \right] r'^2 \sin\theta \, dr' \, d\theta \, d\phi$$

$$= 4\pi\rho_c \int_0^r \left(r'^2 - \frac{r'^4}{R^2} \right) dr'$$

$$= 4\pi\rho_c \left[\frac{r'^3}{3} - \frac{r'^5}{5R^2} \right]_0^r$$

$$= \frac{M}{2} \left[5\left(\frac{r}{R}\right)^3 - 3\left(\frac{r}{R}\right)^5 \right]$$

Then insert $M(r)$ into the integral of the equation for hydrostatic equilibrium:

$$P_c = \int_0^R \frac{GM(r)\rho(r)}{r^2} dr$$

$$= \frac{GM\rho_c}{2} \int_0^R \frac{1}{r^2} \left[5\left(\frac{r}{R}\right)^3 - 3\left(\frac{r}{R}\right)^5 \right] \left[1 - \left(\frac{r}{R}\right)^2 \right] dr$$

$$= \frac{GM\rho_c}{2} \int_0^R \left(\frac{r}{3} - \frac{8r^3}{15R^2} + \frac{r^5}{5R^4} \right) dr = \frac{GM\rho_c}{2} \left[\frac{r^2}{6} - \frac{2r^4}{15R^2} + \frac{r^6}{30R^4} \right]_0^R$$

$$P_c = \frac{15}{16\pi} \frac{GM^2}{R^4}$$

3. Can gravitational energy alone supply a star's luminosity?

   (a) A star with mass M and radius R has a uniform density ρ. What is its gravitational potential energy in terms of M and R?

   Solution: Imagine brining an infinitesimal mass dm from infinity to a mass $M(r)$. The change in gravitational potential energy is equal to the work done on this mass:

   $$dU = -W = + \int_\infty^r dF(r')dr'$$

   $$= \int_\infty^r \frac{G \, dm}{r'^2} M \frac{r^3}{R^3} dr'$$

   $$= \frac{GM \, dm}{R^3} r^3 \int_\infty^r \frac{dr'}{r'^2} = \frac{GM \, dm}{R^3} r^3 \left[-\frac{1}{r'} \right]_\infty^r$$

   $$= -\frac{GM \, dm}{R^3} r^2$$

   Then let $dm = \rho dV = \left(\frac{3M}{4\pi R^3}\right) 4\pi r^2 dr$ so that

   $$U = \int dU = -\frac{GM^2}{R^6} \int_0^R r^4 dr = -\frac{3}{5} \frac{GM^2}{R}$$

   (b) The star shrinks in radius at a small but constant rate $dR/dt = -v$ where v is a positive number with units of velocity. At what rate dU/dt does the star's gravitational potential energy change?

> **Solution:** Use the chain rule:
>
> $$\frac{dU}{dt} = \frac{dU}{dR}\frac{dR}{dt} = -\frac{3}{5}\frac{GM^2}{R^2}v$$
>
> Since all the constants in this expression are positive the potential energy is decreasing (i.e., getting more negative).

(c) Normally, the change in potential energy would be accompanied by a change in thermal energy. But suppose that the star stays at constant temperature (constant thermal energy) during this collapse and radiates the energy away that would normally add to its thermal energy. Suppose also that this radiation is the only radiation emitted by the star. What is the star's luminosity?

> **Solution:** By energy conservation $dK/dt = -dU/dt$. But if the star radiates the energy balance away instead of converting it to heat the luminosity is
>
> $$L = -\frac{dU}{dt} = \frac{3}{5}\frac{GM^2}{R^2}v$$

(d) Suppose the Sun derived its luminosity in this fashion. At what speed would it need to shrink to produce the presently observed luminosity? How long in years would it continue to shine? How long in years would it take for the luminosity to double? Do you think this process can be ruled out as the source of the Sun's power?

> **Solution:**
>
> $$v = \frac{5}{3}\frac{L_\odot R_\odot^2}{GM_\odot^2} = 1.17 \times 10^{-4} \text{ cm/s}$$
>
> This does not seem like much (it would be hard to measure the associated Doppler shift) but this could only go on for
>
> $$\Delta t = \frac{R_\odot}{v} = 6 \times 10^{14} \text{ s} \approx 1.9 \times 10^7 \text{ yr}$$
>
> The luminosity would not be constant either because of the dependence of L on R. It will have doubled by the time the radius shrinks to $R_\odot/\sqrt{2}$, which takes
>
> $$\Delta t_2 = \frac{R_\odot - R_\odot/\sqrt{2}}{v} = (6 \times 10^{14} \text{ s})\left(1 - \frac{1}{\sqrt{2}}\right) \approx 5.6 \times 10^6 \text{ yr}$$
>
> The Solar system is 4.567 billion years old and life has existed on Earth for at least 3 billion years. It seems unlikely that the Sun's luminosity has changed much over billions of years, let alone millions, so this process is unlikely to produce the Sun's luminosity.

4. At the center of the Sun, the mass density is $\rho_c = 1.52 \times 10^5$ kg/m^3 and the mean opacity is $\kappa = 0.12$ m^2/kg. (The opacity is a measure of how opaque a material is — it is often a function of temperature, density, and chemical composition.) What is the mean free path for a photon at the Sun's center?

> **Solution:**

From dimensional analysis, we see that

$$\ell_c = \frac{1}{\kappa \rho_c} = \frac{1}{(0.12 \text{ m}^2/\text{kg})(1.52 \times 10^5 \text{ kg/m}^3)}$$

$$\ell_c = 5.5 \times 10^{-3} \text{ cm}$$