1. Can gravitational energy alone supply a star's luminosity?

   (a) A star with mass M and radius R has a uniform density ρ. What is its gravitational potential energy in terms of M and R?

   Solution: Imagine brining an infinitesimal mass dm from infinity to a mass $M(r)$. The change in gravitational potential energy is equal to the work done on this mass:

$$\begin{aligned} dU &= -W = +\int_\infty^r dF(r')dr' \\ &= \int_\infty^r \frac{G\ dm}{r'^2} M \frac{r^3}{R^3} dr' \\ &= \frac{GM\ dm}{R^3} r^3 \int_\infty^r \frac{dr'}{r'^2} = \frac{GM\ dm}{R^3} r^3 \left[-\frac{1}{r'}\right]_\infty^r \\ &= -\frac{GM\ dm}{R^3} r^2 \end{aligned}$$

   Then let $dm = \rho dV = \left(\frac{3M}{4\pi R^3}\right) 4\pi r^2 dr$ so that

$$U = \int dU = -\frac{GM^2}{R^6} \int_0^R r^4 dr = -\frac{3}{5}\frac{GM^2}{R}$$

   (b) The star shrinks in radius at a small but constant rate $dR/dt = -v$ where v is a positive number with units of velocity. At what rate dU/dt does the star's gravitational potential energy change?

   Solution: Use the chain rule:

$$\frac{dU}{dt} = \frac{dU}{dR}\frac{dR}{dt} = -\frac{3}{5}\frac{GM^2}{R^2} v$$

   Since all the constants in this expression are positive the potential energy is decreasing (i.e., getting more negative).

   (c) Normally, the change in potential energy would be accompanied by a change in thermal energy. But suppose that the star stays at constant temperature (constant thermal energy) during this collapse and radiates the energy away that would normally add to its thermal energy. Suppose also that this radiation is the only radiation emitted by the star. What is the star's luminosity?

   Solution: By energy conservation $dK/dt = -dU/dt$. But if the star radiates the energy balance away instead of converting it to heat the luminosity is

$$L = -\frac{dU}{dt} = \frac{3}{5}\frac{GM^2}{R^2} v$$

   (d) Suppose the Sun derived its luminosity in this fashion. At what speed would it need to shrink to produce the presently observed luminosity? How long in years would it continue to shine? How long in years would it take for the luminosity to double? Do you think this process can be ruled out as the source of the Sun's power?

   Solution:

$$v = \frac{5}{3}\frac{L_\odot R_\odot^2}{GM_\odot^2} = 1.17 \times 10^{-4}\ \text{cm/s}$$

This does not seem like much (it would be hard to measure the associated Doppler shift) but this could only go on for

$$\Delta t = \frac{R_\odot}{v} = 6 \times 10^{14}\ \text{s} \approx 1.9 \times 10^7\ \text{yr}$$

The luminosity would not be constant either because of the dependence of L on R. It will have doubled by the time the radius shrinks to $R_\odot/\sqrt{2}$, which takes

$$\Delta t_2 = \frac{R_\odot - R_\odot/\sqrt{2}}{v} = (6 \times 10^{14}\ \text{s}) \left(1 - \frac{1}{\sqrt{2}}\right) \approx 5.6 \times 10^6\ \text{yr}$$

The Solar system is 4.567 billion years old and life has existed on Earth for at least 3 billion years. It seems unlikely that the Sun's luminosity has changed much over billions of years, let alone millions, so this process is unlikely to produce the Sun's luminosity.

2. **The CNO Bi-Cycle**: The complete CNO cycle of nuclear reactions is

$$^{12}_{6}\text{C} + {}^{1}_{1}\text{H} \rightarrow {}^{13}_{7}\text{N} + {}^{0}_{0}\gamma$$

$$^{13}_{7}\text{N} \rightarrow {}^{13}_{6}\text{C} + \underline{\quad\quad} + \underline{\quad\quad}$$

$$^{13}_{6}\text{C} + \underline{\quad\quad} \rightarrow {}^{14}_{7}\text{N} + \underline{\quad\quad}$$

$$^{14}_{7}\text{N} + {}^{1}_{1}\text{H} \rightarrow {}^{15}_{8}\text{O} + \underline{\quad\quad}$$

$$^{15}_{8}\text{O} \rightarrow {}^{15}_{7}\text{N} + {}^{0}_{1}e^{+} + \underline{\quad\quad}$$

$$^{15}_{7}\text{N} + \underline{\quad\quad} \rightarrow {}^{12}_{6}\text{C} + \underline{\quad\quad} \qquad \text{99.96\% of the time}$$

$$\rightarrow {}^{16}_{8}\text{O} + {}^{0}_{0}\gamma \qquad \text{0.04\% of the time}$$

$$^{16}_{8}\text{O} + {}^{1}_{1}\text{H} \rightarrow {}^{17}_{9}\text{F} + {}^{0}_{0}\gamma$$

$$^{17}_{9}\text{F} \rightarrow \underline{\quad\quad} + \underline{\quad\quad} + {}^{0}_{0}\nu_e$$

$$\underline{\quad\quad} + {}^{1}_{1}\text{H} \rightarrow \underline{\quad\quad} + {}^{4}_{2}\text{He}$$

(a) Fill in the blanks in the table of reactions with the missing names of the reacting particles.

(b) One part of the bi-cycle is indicated in the reaction table. Find another catalytic cycle among the reactions and label it like the first part.

(c) What are the *overall* reactions associated with each of the two catalytic cycles? How does the energy released in each overall reaction compare with that released in the pp chain?

Solution: The solution is given below. The new cycle, which we may call the "NO cycle," uses $^{14}_{7}\text{N}$ as a catalyst. Both cycles add up to the same overall reaction:

$$4\left({}^{1}_{1}\text{H}\right) \rightarrow {}^{4}_{2}\text{He} + 2\left({}^{0}_{1}e\right) + 2\left({}^{0}_{0}\nu_e\right) + 3\left({}^{0}_{0}\gamma\right)$$

Apart from the number of (massless) photons this is also the same as the pp chain, and hence involves the same energy release.

$$^{12}_{6}\text{C} + {}^{1}_{1}\text{H} \rightarrow {}^{13}_{7}\text{N} + {}^{0}_{0}\gamma$$

$$^{13}_{7}\text{N} \rightarrow {}^{13}_{6}\text{C} + {}^{0}_{1}e^{+} + {}^{0}_{0}\nu_e$$

$$^{13}_{6}\text{C} + {}^{1}_{1}\text{H} \rightarrow {}^{14}_{7}\text{N} + {}^{0}_{0}\gamma$$

$$^{14}_{7}\text{N} + {}^{1}_{1}\text{H} \rightarrow {}^{15}_{8}\text{O} + {}^{0}_{0}\gamma$$

$$^{15}_{8}\text{O} \rightarrow {}^{15}_{7}\text{N} + {}^{0}_{1}e^{+} + {}^{0}_{0}\nu_e$$

$$^{15}_{7}\text{N} + {}^{1}_{1}\text{H} \rightarrow {}^{12}_{6}\text{C} + {}^{4}_{2}\text{He} \qquad \text{99.96\% of the time}$$

$$\rightarrow {}^{16}_{8}\text{O} + {}^{0}_{0}\gamma \qquad \text{0.04\% of the time}$$

$$^{16}_{8}\text{O} + {}^{1}_{1}\text{H} \rightarrow {}^{17}_{9}\text{F} + {}^{0}_{0}\gamma$$

$$^{17}_{9}\text{F} \rightarrow {}^{17}_{8}\text{O} + {}^{0}_{1}e^{+} + {}^{0}_{0}\nu_e$$

$$^{17}_{8}\text{O} + {}^{1}_{1}\text{H} \rightarrow {}^{14}_{7}\text{N} + {}^{4}_{2}\text{He}$$

3. Suppose we have Z protons and we have to distribute them into two nuclei, one with Z_1 protons and the other with Z_2 protons, such that $Z = Z_1 + Z_2$.

(a) What arrangements give the *maximum* and *minimum* Coulomb repulsion between the two nuclei?

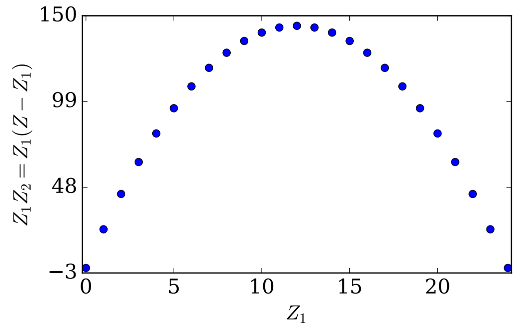
Solution: For a given distance between the nuclei the repulsive force is proportional to the product of the two nuclear charges $Z_1 Z_2 q_p^2 = Z_1(Z - Z_1)q_p^2$, so the extrema in the force correspond to the extrema of $Z_1 Z_2 = Z_1(Z - Z_1)$. If we pretend the atomic number is a continuous variable then this function of Z_1 is a parabola that opens downward. We can then determine the extremum in the usual way:

$$\frac{d}{dZ_1}\left(Z_1(Z - Z_1)\right) = Z - 2Z_1 = 0 \text{ at the extremal values of } Z_1;$$

$$Z_{1,\text{ext}} = \frac{Z}{2}$$

$$\text{But } \frac{d^2}{dZ_1^2}\left(Z_1(Z - Z_1)\right) = -2 < 0$$

which is the condition for a maximum. Thus the maximum involves equally charged nuclei with $Z_1 = Z_2 = Z/2$ for $Z_1 Z_2 = Z^2/4$. The function $Z_1 Z_2$ decreases as Z_1 changes in either direction from $Z/2$. Formally, there is no miminmum except for the trivial case where $Z_1 = 0$. The smallest the product gets is that produced by $Z_1 = 1$, $Z_2 = Z - 1 = Z_1 Z_2$. The product for $Z = 24$ is plotted below.



(b) What does this tell you about the types of fusion most likely to take place in stars?

Solution: It suggests that fusion reactions involving heavy nuclei and light nuclei (or single protons) are faster than those involving two heavy nuclei, and thus that heavy elements tend to be produced by building up one baryon at a time rather than via direct fusion of two multi-baryon nuclei.